Layer construction of three-dimensional topological states and String-String braiding statistics

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Outline

Part I
• 2D topological states and layer construction
• Generalization to 3D: a simplest example
• Layer construction of 3D topological states: general setting and examples
• Field theory description

Part II
• Some general results on string-string braiding.

• Ref: Chao-Ming Jian & XLQ, arXiv:1405.6688

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(arriving here tonight...)
Topologically ordered states

Topological ground state degeneracy; quasiparticles with fractional quantum numbers and fractional statistics.

- Integer quantum Hall
- Fractional quantum Hall

$E_{gap}$

$g = 0$
1 ground state

$g = 1$
m ground states
Key properties of topologically ordered states

• Quasiparticles have no knowledge about distance. Only topology matters.

• Fusion $a \times b = N_{ab}^c c$

• Braiding $a, b$ and spinning $a, b$ is equivalent to spinning $c$. Topological spin of particles $h_a$

\[
\begin{align*}
R_{ab}^c R_{ba}^c &= e^{i2\pi(h_a+h_b-h_c)}
\end{align*}
\]

\[
\begin{align*}
a &\quad = e^{i2\pi h_a}
\end{align*}
\]
Examples of topologically ordered states

1. Laughlin state $\Psi(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i|z_i|^2}$

Quasiparticles labeled by $q = 0, \frac{1}{m}, \frac{2}{m}, \ldots, 1 - \frac{1}{m}$

Fusion rule $q_1 + q_2$ braiding $R^{q_1+q_2}_{q_1 q_2} = \exp\left[i\pi \frac{q_1 q_2}{m}\right]$.

Spin $h_q = \frac{q^2}{2m}$

2. $Z_2$ gauge theory (toric code)

Quasiparticles include charge $e$, flux $m$ and their boundstate $\psi = e \times m$.

Nontrivial braiding $R^{\psi}_{e m} = i$

Goal of this work: understanding 3D topological states from 2D ones
Part I: Layer construction

- 2D topological states can be constructed from coupled 1D chains (Sondhi&Yang ‘01, Kane et al ‘02, Teo&Kane, ‘10)
- Weakly coupled chains as a controlled limit that can realize these topological states.
- Both integer and fractional quantum Hall states can be realized.
Layer construction of 2D topological states

- Example 1: integer quantum Hall (Sondhi & Yang ‘01)
- Electron tunneling between edge states of each strip:
  \[ \langle c_{nL}^+ c_{n+1,R} \rangle \neq 0, \]
- Electron tunneling can be equivalently viewed as exciton condensation
- Condensation of the exciton (particle-hole pair) leads to coherent tunneling between quasi-1D strips
- The strips are glued to a quantum Hall state
Layer construction of 2D topological states

- Example 2: Laughlin 1/3 state (Kane et al ’02)
- Electron tunneling between $\nu = 1/3$ edges of chiral Luttinger liquids $e = 3 \times \frac{e}{3}$,

$$c_{nL}^+ c_{n+1,R} = \left( e^{i(\phi_{nL} - \phi_{n+1,R})} \right)^3 ,$$

- Electron tunneling effectively generates coherent quasiparticle tunneling $\Rightarrow$ 2D topological order.
- The coherent tunneling can be understood as a “boson condensation” of the quasiparticle exciton with charge $\left( \frac{e}{3}, -\frac{e}{3} \right)$
Generalization of the layer construction to 3D

- **General principle:** Inter-layer coupling by boson condensation (Wang & Senthil ‘2013)

- **Abelian states:** Chern-Simons theory and $K$ matrix (Wen)

\[
\mathcal{L} = \frac{1}{4\pi} K^{IJ} a_{I\mu} \varepsilon^{\mu\nu\tau} \partial_\nu a_{J\tau} - l^I a_{I\mu} j_\mu
\]

- **Quasiparticles** labeled by integer vectors $l$

- **Equation of motion**

\[
j^\mu l^I = \frac{1}{2\pi} K^{IJ} \varepsilon^{\mu\nu\tau} \partial_\nu a_{J\tau}
\]

- **A quasiparticle carries flux**

\[
\nabla \times a^I = 2\pi (K^{-1} l)^I
\]
Examples of K-matrix theory

- Mutual statistics of $l_1, l_2$ given by $\theta_{12} = 2\pi l_1^T K^{-1} l_2$
- Local particles given by $\lambda = Kl$ (bosons or fermions)

- Examples:
  - Laughlin $1/m$ state $K = m$. Quasiparticle braiding $\theta_{12} = \frac{2\pi q_1 q_2}{m}$. Local particle (electron) $q = m$

- $Z_N$ gauge theory $K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$
  - Charge $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Quasiparticle braiding $\theta_{em} = 2\pi [K^{-1}]_{12} = \frac{2\pi}{N}$
General setting of the layer construction

- $L$ layers of 2D Abelian states, each with a $K$ matrix
- Find quasiparticles $p_i, q_i$ in each layer, so that the bound state are bosonic and mutually bosonic.
- In 2D language,
  - Requirements
    \[ p_i^T K^{-1} p_j + q_i^T K^{-1} q_j = 0, \]
    \[ p_i^T K^{-1} q_j = 0. \]
- Number of condensed particles: $i = 1, 2, \ldots, N$ when $\text{dim } K = 2N$.
- This is an “almost complete” set of null vectors. (Haldane ‘95, Levin ‘13, Barkeshli et al ‘13) There may be remaining particles, responsible for the topological order.
- With open boundary, $q_i$ at top surface is always deconfined.
Example 1: 3D $\mathbb{Z}_p$ gauge theories

- Starting from layers of 2D $\mathbb{Z}_p$ gauge theories
  \[ K = \begin{bmatrix} 0 & p \\ p & 0 \end{bmatrix}, \quad \mathcal{L} = \frac{p}{2\pi} a_\mu \epsilon^{\mu\nu\tau} \partial_\nu b_{\tau} + a_\mu j^\mu_e + b_\mu j^\mu_m, \]

- Coupling the neighbor layers by condensation of $\binom{e}{-e}$ pair

- $p = \binom{1}{0}, q = \binom{-1}{0}$

- Particles with nontrivial braiding with the condensed particle are confined.

- Particles different by a condensed particle are identified

- Deconfined particles: $e$ in 3D, and $m$ string (flux tube) $\Rightarrow$ 3D $\mathbb{Z}_p$ gauge theory
Example 2: Surface and bulk topological order

- $Z_p$ toric code with tri-layer coupling
- A variation of the construction in Wang&Senthil ’13
- $p \neq 3n$
  All bulk particles are confined. purely 2D topological order
- Surface central charge $c = 4$ for $p = 3n - 1$. ($p = 2$: surface theory of a 3D bosonic TI Vishwanath&Senthil ’13)
- $p = 3n$
  Bulk deconfined particles coexisting with surface particles. $Z_3$ bulk topological order
- Surface central charge $c = 2$
General criteria of surface-only topological order

- $p_i, q_i$ expand all quasiparticles in a layer $\Rightarrow$ condensation leads to surface-only topological order. Surface particles are $q_i$ at top surface, $p_i$ at bottom surface.

- Surface $K$ matrix $K_S = \left[q_i^T K^{-1} q_j\right]^{-1}$

- The same topological order at the side surfaces

- Bulk has nontrivial particle when $\{p_i\} \cap \{q_i\} \neq \emptyset$

- Relation to Walker-Wang model (K Walker & Z Wang, '12):
  - modular tensor category $\Rightarrow$ Surface-only topological order
  - Pre-modular tensor category $\Rightarrow$ Bulk nontrivial topological order
Example 3: String-String braiding

- $Z_{4n}$ toric code theories with 4-layer coupling
- Condensed particles: hybridization of the red and blue layers
- Bulk deconfined particles: 2 point particles, 2 strings
- String-particle braiding
- String-string braiding phase $\omega_{em} = \frac{2\pi L}{4n}$ proportional to the number of layers
String-String braiding and dislocations

• Strings wrapping around z direction have braiding proportional to system size
• Contractible strings have trivial braiding
• The more fundamental process of string braiding can be defined at presence of an edge dislocation
• Braiding at presence of the dislocation

$$\omega_{em}^d = \frac{2\pi}{4n} b_z$$, proportional to the Burgers vector $$b_z$$
Topological field theory description

- A generalized BF theory can be written down to characterize the string-particle braiding and string-string braiding

\[ \mathcal{L} = \frac{Q_{IJ}}{2\pi} \varepsilon_{\mu\nu\sigma\tau} b_{\mu\nu}^{I} \partial_{\sigma} a_{\tau}^{J} + \frac{\Theta}{8\pi^2} R_{IJ} \varepsilon_{\mu\nu\sigma\tau} \partial_{\mu} a_{\nu}^{I} \partial_{\lambda} a_{\sigma}^{J} + j_{I}^{\mu} a_{I}^{\mu} \]

- \( j_{I}^{\mu} \): particle current; \( J_{\mu\nu}^{I} \): string current
- \( Q_{IJ} \): string-particle braiding
- \( R_{IJ} \): string-string braiding when strings are linked with \( \Theta \) vortex loop.
- Difference from BF theory for TI (Cho&Moore ’11, Vishwanath&Senthil ’12, Keyserlingk et al ‘13): \( \Theta \) is a dynamical field
- Winding number \( 2\pi n \) of \( \Theta \) \( \Rightarrow \) Chern-Simons term of \( a \) with \( K = nR. \Rightarrow \) String braiding \( \omega_{IJ}^{n} = 2\pi n(Q^{-1}TRQ^{-1})_{IJ} \)
Topological field theory description

- Ordinary $\mathbb{Z}_p$ gauge theory: $Q = p, R = 0$

- Example 3: $Q = \begin{pmatrix} 2n & 0 \\ 0 & 2 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- General structure of string braiding: two strings braid nontrivially only if they are not contractible.

- Consistent with other recent works on 3-string braiding (Wang&Levin 1403.7435, Jiang et al 1404.1062, Wang&Wen 1404.7854, Moradi&Wen 1404.4618)

- The dislocation is described by a $\Theta$ vortex string, which is an extrinsic defect.

- Intrinsic 3-string braiding can possibly be realized by deconfinement of the dislocations.
Part II: General results on string-string braiding

• General structure of 3D topologically ordered states are not understood yet.

• In 2D, we know the braiding phase $R_{ab}^d$ is not arbitrary. There are some identities satisfied by braiding and fusion, such as the hexagon identity.

\[
\begin{align*}
\text{a} \quad \text{b} \quad \text{c} & = \quad \text{a} \quad \text{b} \quad \text{c} \\
& = \quad \text{a} \quad \text{b} \quad \text{c}
\end{align*}
\]

• In 3D, some similar identities may exist as a property of the general structure of topologically ordered states.
General results on string-string braiding

• **Wang&Levin 1403.7435** proposed an identity of the 3-string braiding in twisted $\mathbb{Z}_p$ gauge theories

$$ p \left( \omega^c_{ab} + \omega^a_{bc} + \omega^b_{ca} \right) = 0 \pmod{2\pi}, $$

• Here we give a more general proof to a stronger identity

$$ \omega^c_{ab} + \omega^a_{bc} + \omega^b_{ca} = 0 \pmod{2\pi} $$

with the general conditions

1) Strings can fuse and split without additional phase;
2) Strings are Abelian;
3) Strings are not marked.
Step 1 of the proof: $\omega^c_{ab} = \Omega^c_{ab}$

String braiding $\omega^c_{ab}$

String-particle braiding $\Omega^c_{ab}$ between link of a, b and string c
Step 2 of the proof: $\tilde{\omega}_{ab}^c = \Omega_{ab}^c$

“linked” string braiding $\tilde{\omega}_{ab}^c$, for 3 mutually-linked strings

String-particle braiding $\Omega_{ab}^c$ between link of $a, b$ and string $c$
Step 3 of the proof: $\tilde{\omega}^c_{ab} + \tilde{\omega}^a_{bc} + \tilde{\omega}^b_{ca} = 0$

- $\tilde{\omega}^c_{ab}$: $2\pi$ rotation of $a$ and $b$ around $c$
- $\rightarrow \tilde{\omega}^c_{ab} + \tilde{\omega}^a_{bc} + \tilde{\omega}^b_{ca} \simeq$ global $4\pi$ rotation $\simeq$ trivial
String braiding identities

- Using this proof we obtain three identities
  \[ \tilde{\omega}_a^c + \tilde{\omega}_b^c + \tilde{\omega}_c^b = 0 \]
  \[ \omega_a^c + \omega_b^c + \omega_c^b = 0 \]
  \[ \Omega_a^c + \Omega_b^c + \Omega_c^b = 0 \]

- A new feature of 3D topological order that is qualitatively distinct from 2D case

- Open question: In general, is it always possible to require the strings to be *unmarked*, i.e., translation invariant along the string direction?
A non-Abelian example of string-string braiding

• Little is known about non-Abelian strings.
• However, an example can be found in 3D topological superconductors

\[
\Delta_L e^{i\theta_L}
\]

left, \( C = 1 \)

SC pairing

Majorana mass

\[
\Delta_R e^{i\theta_R}
\]

right, \( C = -1 \)

Superconducting pairing

\[
\int d^3 x \left( \Delta e^{i\theta_L} \psi_L^+ \sigma_y \psi_L^+ + \Delta e^{i\theta_R} \psi_R^+ \sigma_y \psi_R^+ \right)
\]

Weyl fermions

\[
H = d^3x \nu (\psi_L^+ \sigma \cdot p \psi_L - \psi_R^+ \sigma \cdot p \psi_R)
\]
A non-Abelian example of string-string braiding

- Chiral vortex strings: vortex loops of $\theta_L$ or $\theta_R$
- Each vortex string is an axion string, carrying a 1+1 Majorana-Weyl fermion (Callan&Harvey ’85, XLQ&Witten&Zhang ’12)
- Majorana zero modes carried by vortices with odd linking number.

- Non-Abelian braiding of $a, b$ similar to $(p + ip)$ vortices (Read&Green ‘2000) (see also M Sato, Physics Letters B 575 (2003) 126–130)
A non-Abelian example of string-string braiding

- Key difference from Abelian string: splitting/fusion of string is not adiabatic.
- Non-Abelian strings can fuse to Abelian strings.
- Braiding depends on the fusion channel.

\[ \sigma \]

2 zero modes on \( a, b \)

\[ \neq \]

no zero mode
Summary

• Layer construction provides an explicit approach to 3D topological states.
• Different types of 3D topological states can be generated, with surface-only topological order and/or bulk topological order
• String-string braiding can be induced in system with periodic boundary condition or dislocations
• General identity proved for Abelian string-string braiding
• Non-Abelian 3D topological order: An example can be found in topological superconductors. There are a lot of open questions for more general cases.