Hydrodynamic Effective Field Theories for Topological Insulators via Functional Bosonization

Shinsei Ryu
Univ. of Illinois, Urbana-Champaign

AtMa Chan, Taylor Hughes, Eduardo Fradkin (UIUC)

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Objectives:

-- Goal: developing "hydrodynamic" theory of topological insulators (as opposed to effective field theory of response)

[Cho-Moore (11), Vishwanath-Senthil (12) etc.]

-- Motivations:

- QH droplet can be understood as an incompressible liquid: Corresponding field theory: Chern-Simons theory

- A clue for stability of topological insulator phases in the presence of weak interactions

- A clue for the case where topological states arise from strong interactions "fractional topological insulator"

Effective "hydrodynamic" field theory of QHE

Example: composite particle theories: [Zhang-Hansson-Kivelson, Jain, ...]

$$\mathcal{L} = \Psi^\dagger \partial_\tau \Psi + \Psi^\dagger \frac{1}{2m} (\partial + ieA_{uni})^2 \Psi + \cdots$$

Electron

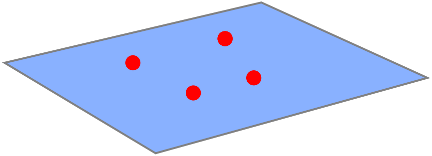
-- Flux attachement:

Composite boson

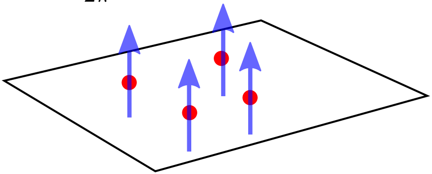
Statistical gauge field

$$\mathcal{L} = \Phi^\dagger \partial_\tau \Phi + \Phi^\dagger \frac{1}{2m} (\partial + ieA_{uni} + i\alpha_{CS})^2 \Phi + V(\Phi) + \frac{1}{4\pi} \alpha_{CS} d\alpha_{CS} + \cdots$$

Ψ



$$\Phi^\dagger \Phi = \frac{-1}{2\pi} \nabla \times \alpha_{CS}$$



Effective "hydrodynamic" field theory of QHE

-- Boson-vortex duality: (0-form $\longleftrightarrow$ 1-form)

$$(A^{uni}_\mu + \alpha_\mu) j^\mu_{el} \longrightarrow \frac{1}{2\pi} a_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (A^{uni} + \alpha)_\lambda$$

dual gauge field

-- Integrating over statistical gauge field:

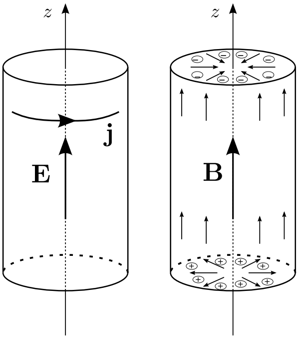
$$\mathcal{L} = a d\alpha_{CS} + \frac{1}{4\pi} \alpha_{CS} d\alpha_{CS} \longrightarrow \mathcal{L} = \frac{-1}{4\pi} a d a$$

Response theory of 3+1 d topological insulators

-- Topological insulators: undeformable to atomic limit (topologically trivial state) under some symmetry conditions

-- Characterized by anomalous ("topological") response
E.g. magnetoelectric effect in 3D TR symmetric TI

$$\boldsymbol{M} = (e^2/hc)\boldsymbol{E}$$
$$\boldsymbol{P} = (e^2/hc)\boldsymbol{B}$$



-- Response theory described by topological terms
E.g. axion term

$$\frac{1}{8\pi^2}\int d^4x\,\theta\epsilon^{\mu\nu\rho\sigma}\partial_\mu A^{\rm ex}_\nu \partial_\rho A^{\rm ex}_\sigma$$

Functional bosonization recipe

[Luther, Damgaard-Nielsen-Sollacher, Fradkin-Schaposnik, Burgess-Lutken-Quevedo, Banerjee (incomplete list)...]]

-- Microscopic fermionic system:

$$Z[A^{\mathrm{ex}}] = \int \mathcal{D}\left[\bar{\psi}, \psi\right] \exp\left(iK_F[\bar{\psi}, \psi, A^{\mathrm{ex}}]\right)$$

-- Interested only in conserved quantities:

$$\begin{aligned} &\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle \\ &= \frac{1}{i} \frac{\delta}{\delta A^{\mathrm{ex}}_{\mu_1}(x_1)} \frac{1}{i} \frac{\delta}{\delta A^{\mathrm{ex}}_{\mu_2}(x_2)} \cdots \ln Z[A^{\mathrm{ex}}]. \end{aligned}$$

-- Making use of U(1) gauge invariance:

$$Z[A^{\mathrm{ex}} + a] = Z[A^{\mathrm{ex}}]. \qquad Z[A^{\mathrm{ex}}] = \int \mathcal{D}[a]_{\mathrm{pure}} Z[A^{\mathrm{ex}} + a]$$

Functional bosonization recipe

-- Hubbard-Stratnovich the pure gauge condition: $f_{\mu\nu}[a] = \partial_\mu a_\nu - \partial_\nu a_\mu = 0$

$$Z[A^{\mathrm{ex}}] = \int \mathcal{D}[a,b] Z[A^{\mathrm{ex}} + a] \times \exp\Big(-\frac{i}{2} \int d^D x \, b_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} f_{\alpha\beta}[a] \Big)$$

-- Shift a --> a + A^{ex}

$$Z[A^{\mathrm{ex}}] = \int \mathcal{D}[a,b] Z[a] \times \exp\Big(-\frac{i}{2} \int d^D x \, b_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} \left(f_{\alpha\beta}[a] - f_{\alpha\beta}[A^{\mathrm{ex}}] \right) \Big)$$

-- Theory in terms of three fields:

$$a_\mu, \quad b_{\mu\nu\dots} \quad A^{\mathrm{ex}}_\mu$$

Bosonization rule:

$$j^\mu(x) \quad \Leftrightarrow \quad \epsilon^{\mu\nu\lambda\rho\cdots} \partial_\nu b_{\lambda\rho\dots}(x)$$

functional bz in D=1+1

-- Applied to D=1+1d massive fermions ("1d topological insulator"):
functional integral can be done exactly
reproduces the bose-fermi correspondense.

$$j^\mu = \epsilon^{\mu\nu}\partial_\nu b$$

-- Applied to D=1+1d topological insulator:
1+1 d "BF" theory + "axion" term:

$$\ln Z[a] = \frac{i\theta}{2\pi}\int d^2x\, \epsilon^{\mu\nu}\partial_\mu a_\nu + \cdots$$

$$\mathcal{L} = -b\epsilon^{\mu\nu}\partial_\mu(a_\nu - A^{\mathrm{ex}}_\nu) + \frac{\theta}{2\pi}\epsilon^{\mu\nu}\partial_\mu a_\nu + \cdots$$

functional bz in D=2+1

-- Effective field theory of trivial insulator:
BF theory w/o Chern-Simons term:

$$\mathcal{L} = -\frac{2\mathrm{k}}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A^{\mathrm{ex}}_\lambda) \qquad \text{with k = 1}$$

-- Functional bz derivation of dual appoach to (band) insulators

[Lee-Kivelson (03)] [Shindou-Imura-Ogata (06)]

-- Theory is almost empty:
no ground state degeneracy, no fermion, gapped edge state

-- C.f. dual theory of BCS SC: [Hansson-Ognesyan-Sondhi (04)]
BF theory at level 2: (k = 2)

$$\int_S \delta\rho \, dS = e \int_S (\nabla \times \boldsymbol{a}) \cdot d\boldsymbol{S}$$

$$e \oint_{\partial S} \nabla\Theta \cdot d\boldsymbol{l} = \int_S \delta\rho dS$$

functional bosonization in D=2+1

-- Effective field theory of Chern insulator
BF theory with Chern-Simons:

$$\ln Z[a] = \frac{i\mathsf{Ch}}{4\pi}\int d^3x\, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \cdots$$

Ch=Chern number

$$\mathcal{L} = -b_\mu \epsilon^{\mu\nu\lambda}\partial_\nu (a_\lambda - A^{\mathrm{ex}}_\lambda) + \frac{\mathsf{Ch}}{4\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

-- Functional bz derivation; alternative to composite particle theories

[Schaposnik (95), Schaposnik-Fradkin (95), Berci-Oxman (00), Shizuya (01)]

-- Theory is less empty:
existence of fermions, gapless edge state,
but no ground state degeneracy

-- Equally applicable to QHE in continuum and Chern insulators on lattices.

Functional bosonization in D=3+1

-- 3+1d topological insulator with chiral symmetry ("class AIII")
characterized by an integer topological invariant
(physical realization: superconductor with conserved Sz)

$$\nu_3 = \int_{\mathrm{BZ}} \frac{d^3k}{24\pi^2} \varepsilon^{\mu\nu\rho} \mathrm{Tr}\left[(q^{-1}\partial_\mu q)(q^{-1}\partial_\nu q)(q^{-1}\partial_\rho q) \right]$$

-- A microscopic lattice mode: [Hosur-Ryu-Vishwanath (10)]

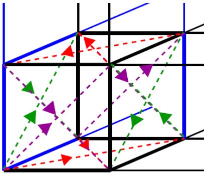
Kogut-Susskind staggered fermion
+ diagonal hopping

$$H = \sum_{r,r'} \psi^\dagger(r) t_{r,r'} \psi(r') + h.c.$$

$$\mathcal{C}\psi(t,r)\mathcal{C}^{-1} = (-1)^r \psi^\dagger(t,r),$$

$$(-1)^r = \begin{cases} +1, & r \in \mathrm{A\ sublattice} \\ -1, & r \in \mathrm{B\ sublattice} \end{cases}$$

$$\mathcal{T}\psi(t,r)\mathcal{T}^{-1} = \psi(-t,r), \quad \mathcal{T}i\mathcal{T}^{-1} = -i,$$



Effective field theory in D=3+1

-- Z[a]: $$\ln Z[a] = -\frac{1}{8\pi}\left[\frac{4\pi}{g^2} f^{\mu\nu}[a] f_{\mu\nu}[a] + \frac{i\theta}{4\pi}\epsilon^{\mu\nu\lambda\rho} f_{\mu\nu}[a] f_{\lambda\rho}[a]\right]$$

-- Effective field theory BF theory with Axion term:

$$\mathcal{L} = -b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda(a_\rho - A^{\rm ex}_\rho) + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda a_\rho - \frac{1}{4\pi^2 g^2} f_{\mu\nu} f^{\mu\nu} + \cdots$$

-- Reproduces the axion resonse: $$\frac{1}{8\pi^2}\int d^4x\, \theta \epsilon^{\mu\nu\rho\sigma}\partial_\mu A^{\rm ex}_\nu \partial_\rho A^{\rm ex}_\sigma$$

-- Axion term "attaches" monopole to electron:

$$j^\mu = \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu\left(\theta\partial_\lambda A^{\rm ex}_\rho\right)$$

-- Comparison with Cho-Moore story: See also [Vishwanath-Senthil (12)]

$$\mathcal{L} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho} a_\mu \partial_\nu b_{\lambda\rho} + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho} A^{\rm ex}_\mu \partial_\nu b_{\lambda\rho}$$
$$+ C\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda A^{\rm ex}_\rho,$$

Gauge transformation: $$b_{\mu\nu} \to b_{\mu\nu} + \frac{\theta}{16\pi^2}\left(\partial_\mu a_\nu - \partial_\nu a_\mu\right)$$

EM duality (S-duality)

-- Maxwell theory $\mathcal{L} = \frac{\alpha}{4} f^{\mu\nu}[a] f_{\mu\nu}[a] + \frac{i\beta}{4} f_{\mu\nu}[a]\epsilon^{\mu\nu\lambda\rho} f_{\lambda\rho}[a]$

-- Introduce monopole gauge field (u) and aux field (v) [Witten 1995]

$$\begin{aligned}\mathcal{L} = &\frac{i}{2} f_{\mu\nu}[v]\epsilon^{\mu\nu\lambda\rho} u_{\lambda\rho} \\ &+ \frac{\alpha}{4}(f^{\mu\nu}[a] - u^{\mu\nu})(f_{\mu\nu}[a] - u_{\mu\nu}) \\ &+ \frac{i\beta}{4}(f_{\mu\nu}[a] - u_{\mu\nu})\epsilon^{\mu\nu\lambda\rho}(f_{\lambda\rho}[a] - u_{\lambda\rho})\end{aligned}$$

monopole gauge transf.

$$a_\mu \to a_\mu + \xi_\mu,$$
$$u_{\mu\nu} \to u_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

-- Gauge away a: $a_\mu \to a_\mu + (-a_\mu) \quad u_{\mu\nu} \to u_{\mu\nu} - \partial_\mu a_\nu + \partial_\nu a_\mu$

$$\mathcal{L} = \frac{i}{2} f_{\mu\nu}[v]\epsilon^{\mu\nu\lambda\rho} u_{\lambda\rho} + \frac{\alpha}{4} u^{\mu\nu} u_{\mu\nu} + \frac{i\beta}{4}\epsilon^{\mu\nu\lambda\rho} u_{\mu\nu} u_{\lambda\rho}$$

-- Integrate over u:

$$\mathcal{L} = \frac{\mathsf{q}}{2} f_{\mu\nu}[v] f_{\mu\nu}[v] + \frac{i\mathsf{p}}{4} f_{\mu\nu}[v]\epsilon_{\mu\nu\lambda\rho} f_{\mu\nu}[v]$$

-- Duality:

$$a_\mu \leftrightarrow v_\mu, \quad \frac{\alpha}{4} \leftrightarrow \frac{\alpha}{2(\alpha^2 + 2\beta^2)}, \quad \frac{\beta}{4} \leftrightarrow \frac{-\beta}{2(\alpha^2 + 2\beta^2)}$$

Integrating over "statistical" gauge field

-- BF-Maxwell-Axion theory

-- Introduce monopole gauge field (u) and aux field (v)

$$
\begin{aligned}
\mathcal{L} = & -\frac{i}{2} b_{\mu\nu} \epsilon^{\mu\nu\lambda\rho} (f_{\lambda\rho}[a] - u_{\lambda\rho}) + \frac{i}{2} f_{\mu\nu}[v] \epsilon^{\mu\nu\lambda\rho} u_{\lambda\rho} \\
& + \frac{\alpha}{4} (f^{\mu\nu}[a] - u^{\mu\nu})(f_{\mu\nu}[a] - u_{\mu\nu}) \\
& + \frac{i\beta}{4} (f_{\mu\nu}[a] - u_{\mu\nu}) \epsilon^{\mu\nu\lambda\rho} (f_{\lambda\rho}[a] - u_{\lambda\rho}). \qquad (3.21)
\end{aligned}
$$

-- Gauge away statistical gauge field (a)

Integrate over u:

$$
\begin{aligned}
\mathcal{L} = & \frac{i}{2} b_{\mu\nu} \epsilon_{\mu\nu\lambda\rho} f_{\lambda\rho}[A^{\mathrm{ex}}] \\
& - \frac{i}{2} (f_{\mu\nu}[v] + b_{\mu\nu}) \epsilon_{\mu\nu\lambda\rho} U^{\mathrm{ex}}_{\lambda\rho} \\
& + \frac{\mathsf{q}}{2} \left(f_{\mu\nu}[v] + b_{\mu\nu}\right) \left(f_{\mu\nu}[v] + b_{\mu\nu}\right) \\
& + \frac{i\mathsf{p}}{4} \left(f_{\mu\nu}[v] + b_{\mu\nu}\right) \epsilon_{\mu\nu\lambda\rho} \left(f_{\lambda\rho}[v] + b_{\lambda\rho}\right)
\end{aligned}
$$

-- Can gauge away v: "Higgs" or "Julia-Toulouse"

$$
b_{\mu\nu} \ \to \ b_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \qquad v_\mu \ \to \ v_\mu - \xi_\mu
$$

Theory is written solely in terms of hydrodynamic gauge field, b.

Julia-Toulouse approach to defect condensation

[Julia-Toulouse (79)
Quevedo-Trugenberger (97)]

-- Theory in "Coulomb" phase

$$S = \int \frac{(-1)^{h-1}}{e^2} d\phi_{h-1} \wedge (\star d\phi_{h-1})$$

e.g.

QED: $$S = \int \frac{1}{e^2} dA \wedge (\star dA)$$

KT: $$S = \int \frac{1}{e^2} d\phi \wedge (\star d\phi)$$

-- Theory after defect condensation:

$$S = \int \frac{(-1)^h}{\Lambda^2} \Omega_{h+1} \wedge \star \Omega_{h+1} + \frac{(-1)^{h-1}}{e^2} (\omega_h - d\phi_{h-1}) \wedge \star (\omega_h - d\phi_{h-1})$$

Here:

$\Omega_{h+1} := d\omega_h$

Λ : scale associated to condensation

-- Basic idea behind:

$$\int_{S_h} \omega_h, \quad \omega_h = d\phi_{h-1}$$

$$J_{d-h} = \star \Omega_{h+1} = \star (d\omega_h)$$

Dimensional reduction

-- Topological insulators with Z_2 topological invariants can be obtained from a higher dimensional system by dimensional reduction
E.g. QSHE and 3D time-reversal symmetric topological insulators

-- Effective field theory for response can be obtained by dimensional reduction [Qi-Hughes-Zhang (08)

-- "Hydrodynamic" theory can also be derived by dimensional reduction

$$\Phi(x_\mu) = \sum_{n_w=-\infty}^{+\infty} e^{i2\pi n_w w/L_w} \Phi(x_i, n_w)$$

-- Let's start from the "parent" theory in D=4+1d:

$$S = -\frac{1}{2\pi} \int d^5x\, b_{\mu\nu\lambda} \epsilon^{\mu\nu\lambda\rho\sigma} \partial_\rho (a_\sigma - A^{\mathrm{ex}}_\sigma) + \frac{\mathrm{Ch}_2}{24\pi^2} \int d^5x\, \epsilon^{\mu\nu\lambda\rho\sigma} a_\mu \partial_\nu a_\lambda \partial_\rho a_\sigma + \cdots$$

3D TR symmetric TI and QSHE

-- Hydrodynamic field theory for 3+1d TR symmetric TI:

bf coupling

$$\mathcal{L} = -\frac{\epsilon^{\mu\nu\lambda\rho}}{2\pi}(\phi - \theta^{\rm ex})\partial_\mu b_{\nu\lambda\rho} - \frac{\epsilon^{\mu\nu\lambda\rho}}{2\pi}(a_\mu - A^{\rm ex}_\mu)\partial_\nu u_{\lambda\rho} + \frac{\mathsf{Ch}_2}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\phi\partial_\mu a_\nu \partial_\lambda a_\rho + \cdots$$

axion term

-- Hydrodynamic field theory for 2+1d QSHE:

$$\mathcal{L} = -\frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\phi - \theta^{\rm ex})\partial_\mu g_{\nu\lambda} - \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\psi - \chi^{\rm ex})\partial_\mu u_{\nu\lambda} - \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(a_\mu - A^{\rm ex}_\mu)\partial_\nu v_\lambda + \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda}\phi\partial_\mu\psi\partial_\nu a_\lambda,$$

-- Quantized responses can be derived from these theories.

fractional TIs ?

-- Topological insulators beyond non-interacting systems:

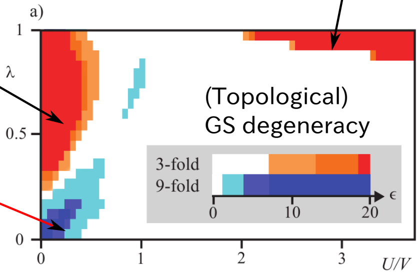
Recent numerics: Hubbard model with spin-orbit interactions on 3x4 cluster

$$H = H_0 + U \sum_i n_{i\uparrow} n_{i\downarrow} + V \sum_{\langle i,j \rangle}^{n.n} n_i n_j$$

T-broken (FQH state)

T-unbroken

fractional quantum spin Hall liquid



spin-orbit coupling

interaction U/V

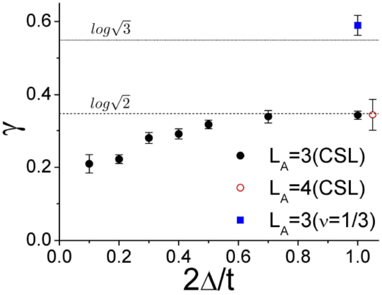
[Neupert, Santos, SR, Chamon, Mudry, Phys. Rev. B 84, 165107 (2011)]

Parton construction

-- A possible direction: parton construction

- A way to generate model wfns with topological order

- Edge theory: conformal embedding and coset construction



[Zhang-Grover-Vishwanath (11)]

Blok-Wen parton construction

-- Splitting an electron into (p+1) partons

$$e = \sum_i e_i = 1 \qquad \nu = \frac{1}{\mathrm{p}+1/\mathrm{m}} = \frac{\mathrm{m}}{\mathrm{mp}+1}$$

-- Functional Bz for free partons

- Constraint $j_\mu^{(i)} = j_\mu^{(j)}, \quad i,j = 1,\ldots,\mathrm{p}+1$

- Effective field theories for partons in their QH states

$$\mathcal{L} = \frac{-1}{2\pi}\sum_i b\epsilon\partial a^i + \frac{1}{4\pi}\sum_i^{\mathrm{p}+1}\epsilon a^i\partial a^i$$

$$\mathcal{L} = \frac{-(\mathrm{p}+1)}{2\pi} b\epsilon\partial b$$

CS theory at level p+1

- In 2d FQHE, this construction (parton + func bz) is equivalent to composite particle theories (at least at this level).

Blok-Wen parton construction for general TIs

-- Blok-Wen construction + hydro BF theories:
E.g.

$$\mathcal{L} = -\mathsf{k}\epsilon^{\mu\nu\lambda\rho} b_{\mu\nu}\partial_\lambda \alpha_\rho + \frac{\mathsf{k}\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu \alpha_\nu \partial_\lambda \alpha_\rho \\ - e\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} A^{\mathrm{ex}}_\mu .$$

k^3 ground state degeneracy on T^3

fractional magnetoelectric effect

$$\mathcal{L} = \frac{\theta}{8\pi^2 \mathsf{k}}\epsilon^{\mu\nu\lambda\rho}\partial_\mu A^{\mathrm{ex}}_\nu \partial_\lambda A^{\mathrm{ex}}_\rho$$

[See also Swingle-Barkeshli, McGreevy-Senthil, Maciejko-Qi-Karch-Zhang]

Summary

- Functional bz derivation of hydrodynamic field theory of topological insulators.

- Parton construction to get higher level k and fractional magnetoelectric effect

- Other issues:
  Topological superconductors ?
  Other approach than partons to raise the level k.

Functional bosonization in D=4+1

--Can repeat the derivation for all dimensions, and all symmetry classes, as far as U(1) current is conserved

For D=4+1, BF theory with 5D CS term at level 1:

$$\ln Z[a] = \frac{i\mathrm{Ch}_2}{24\pi^2}\int d^5x\, \epsilon^{\mu\nu\lambda\rho\sigma} a_\mu \partial_\nu a_\lambda \partial_\rho a_\sigma - \frac{1}{4\pi g^2}\partial_\mu a_\nu \partial^\mu a^\nu + \cdots ,$$

$$\mathcal{L} = -b_{\mu\nu\lambda}\epsilon^{\mu\nu\lambda\rho\sigma}\partial_\rho(a_\sigma - A^{\mathrm{ex}}_\sigma) + \frac{\mathrm{Ch}_2}{24\pi^2}\epsilon^{\mu\nu\lambda\rho\sigma} a_\mu \partial_\nu a_\lambda \partial_\rho a_\sigma + \cdots$$