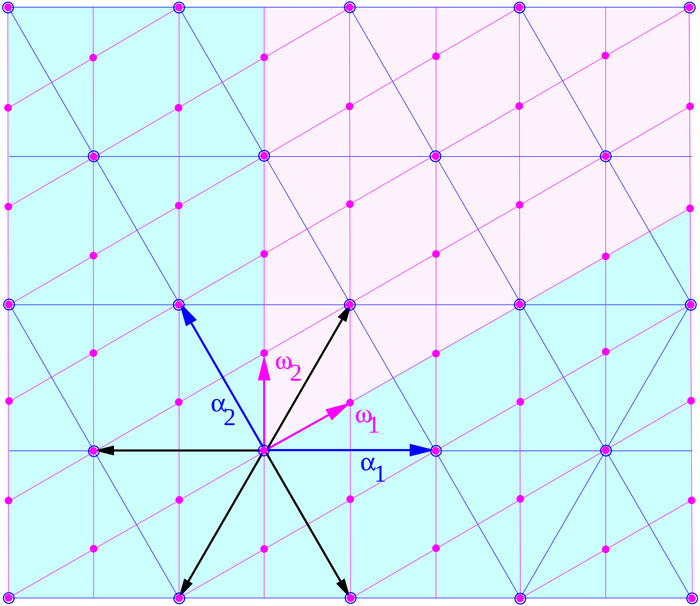


ω_2
α_2
ω_1
α_1

Berry Curvature, Spin, and Anomalous Velocity

Michael Stone

Institute for Condensed Matter Theory
University of Illinois

Talk based on:

Motivation

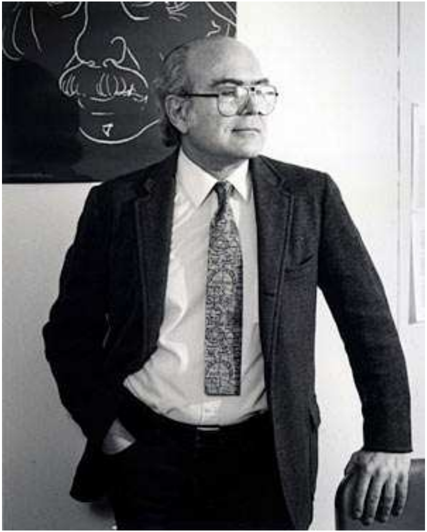
M.A.Stephanov, Y.Yin, *Chiral Kinetic Theory*, Phys. Rev. Lett. **109** 162001 (2012).

Our Work

MS, V.Dwivedi, *A Classical Version of the Non-Abelian Gauge Anomaly* Phys. Rev. **D88** 045012 (2013).
V.Dwivedi, MS, *Classical chiral kinetic theory and anomalies in even space-time dimensions*, J. Phys. A **47** 025401 (2014).
MS, V.Dwivedi, T.Zhou, *Berry Phase, Lorentz Covariance, and Anomalous Velocity for Dirac and Weyl Particles*, arXiv:1406.0354

Also important

J.Y.Chen, D.T.Son, M.A.Stephanov, H.U.Yee, Y.Yin, *Lorentz Invariance in Chiral Kinetic Theory*, arXiv:1404.5963



Bruno Zumino 1923-2014

Outline

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Anomalous Velocity

Luttinger, Blount, Niu, and others show that a Berry phase in the equations of motion of a Bloch quasiparticle ⇒ anomalous velocity:

$$\begin{aligned}\dot{\mathbf{k}} &= -\frac{\partial\varepsilon(\mathbf{k},\mathbf{x})}{\partial\mathbf{x}} + e(\dot{\mathbf{x}}\times\mathbf{B}),\\ \dot{\mathbf{x}} &= \frac{\partial\varepsilon(\mathbf{k},\mathbf{x})}{\partial\mathbf{k}} - (\dot{\mathbf{k}}\times\boldsymbol{\Omega}).\end{aligned}$$

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- Want to use for Dirac and Weyl particles
- Can we make these equations covariant?

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$$\dot{\mathbf{k}} = e(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}) \rightarrow \dot{k}_\mu = eF_{\mu\nu}\dot{x}^\nu, \quad \mu = 0,1,2,3. \quad \checkmark$$

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$$\dot{\mathbf{x}} = \mathbf{v}_\varepsilon - (\dot{\mathbf{k}}\times\boldsymbol{\Omega}) \to \dot{x}_i = v_{i,\varepsilon} + \Omega_{ij}\dot{k}^j, \quad i = 1,2,3. \quad ?$$

Covariant WKB for Dirac

Look for WKB solution of Dirac equation

$$(i\hbar\gamma^\mu(\partial_\mu + ieA_\mu/\hbar) - m)\,\psi = 0.$$

as

$$\psi(x) = a(x)e^{-i\varphi(x)/\hbar}, \quad a = a_0 + \hbar a_1 + \hbar^2 a_2 + \dots,$$

where

$$a_0(x) = u_\alpha(k(x))C^\alpha(x)$$

and $u_\alpha(k)$ (and later $v_\alpha(k)$) are solutions to

$$\begin{aligned}(\gamma^\mu k_\mu - m)u_\alpha(k) &= 0\\ (\gamma_\mu k^\mu + m)v_\alpha(k) &= 0\end{aligned}$$

covariantly normalized so that

$$\bar{u}_\alpha u_\beta = \delta_{\alpha\beta} = -\bar{v}_\alpha v_\beta$$

Spin Transport Equation

Plug WKB solution into Dirac. Find that

$$\left[\delta_{\alpha\beta}\left(V^{\mu}\frac{\partial}{\partial x^{\mu}}+\frac{1}{2}\frac{\partial V^{\mu}}{\partial x^{\mu}}\right)+\frac{ie}{2m}S^{\mu\nu}_{\alpha\beta}F_{\mu\nu}-i\,\mathfrak{a}_{\alpha\beta,\nu}\dot{k}^{\nu}\right]C^{\beta}(x)=0.$$

where

$$\frac{ie}{2m}S^{\mu\nu}_{\alpha\beta}F_{\mu\nu}$$

gives Larmor precession, and

$$\boxed{\mathfrak{a}_{\alpha\beta,\nu}=i\bar{u}_{\alpha}\frac{\partial u_{\beta}}{\partial k^{\nu}},\quad \nu=0,1,2,3}$$

is an unconventional, but covariant Berry connection.

Covariant Berry Curvature

Matrix-valued connection form

$$\mathfrak{a}_{\alpha\beta,\nu}\, dk^\nu = i\bar{u}_\alpha \frac{\partial u_\beta}{\partial k^\nu}\, dk^\nu.$$

Curvature form

$$\mathfrak{F} = d\mathfrak{a} - i\mathfrak{a}^2.$$

Use Dirac equation to find

$$\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2}(S_{\mu\nu})_{\alpha\beta}\, dk^\mu \wedge dk^\nu,$$

where

$$(S_{\mu\nu})_{\alpha\beta} = \bar{u}_\alpha \left(\frac{i}{4}[\gamma_\mu, \gamma_\nu]\right) u_\beta = i\bar{u}_\alpha \sigma_{\mu\nu} u_\beta.$$

Note that Dirac $\Rightarrow$ $k^\mu S_{\mu\nu} = 0$.

Pauli-Lubanski Tensor

Use mass-shell condition $E^2 \equiv k_0^2 = \mathbf{k}^2 + m^2$ to eliminate k_0 and find that

$$\boxed{\mathfrak{F}_{\alpha\beta} = \frac{1}{2m^2}\left(S_{ij} - \frac{k_i}{E}S_{0j} - S_{i0}\frac{k_j}{E}\right)_{\alpha\beta} dk^i \wedge dk^j,}$$

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Expression in parentheses is a skew-symmetric tensor generalization of the Pauli-Lubanski vector

Berry *versus* Llewellyn Thomas

Explicitly, in 3+1 dimensions we have

$$\mathfrak{F} = \frac{1}{2m^2\gamma}\left\{\frac{1}{2}\left(\boldsymbol{\sigma} + \frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{m^2(1+\gamma)}\right)\right\}\cdot(d\mathbf{k}\times d\mathbf{k}).$$

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$$\begin{aligned}
\mathfrak{a}_{\alpha\beta,i}\dot{k}^i &= \frac{1}{m^2(1+\gamma)}(\mathbf{k}\times\dot{\mathbf{k}})\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta} \\
&= \frac{\gamma^2}{1+\gamma}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta}, \quad \boldsymbol{\beta} = \mathbf{k}/E = \mathbf{k}/m\gamma \\
&= -\boldsymbol{\omega}_{\mathrm{Thomas}}\cdot\left(\frac{\boldsymbol{\sigma}}{2}\right)_{\alpha\beta}.
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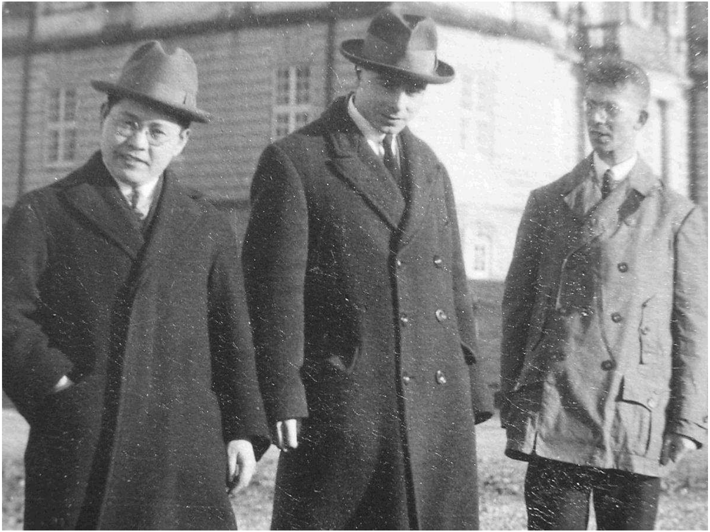
Covariant Berry-transport is Thomas precession

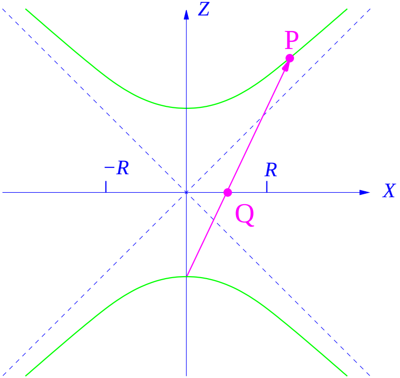
Nishina, Thomas, Hund



Yoshio Nishina, Llewellyn Thomas, Friedrich Hund

Thomas *versus* Lobachevsky

Thomas precession is parallel transport on the positive-energy mass-shell:



Embedding of three-dimensional Lobachevsky space into four-dimensional Minkowski space. The arrow shows the sterographic parametrization of the embedded space by the Poincaré ball $x_1^2 + x_2^2 + x_3^2 < R^2$.

Non-covariant WKB

With $u_\alpha^\dagger u_\beta = \delta_{\alpha\beta} = v_\alpha^\dagger v_\beta$, have

$$\left\{\delta_{\alpha\beta}\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{1}{2}\mathrm{div}\,\mathbf{v}\right) + N_{\alpha\beta}\right\} C^\beta(\mathbf{x},t) = 0,$$

with

$$N_{\alpha\beta} = -i\left(\frac{e}{m\gamma^2}\right)\mathbf{B}\cdot\left\{\frac{1}{2}\left(\boldsymbol{\sigma} + \frac{1}{m^2}\frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{\gamma+1}\right)_{\alpha\beta}\right\} - i\mathcal{A}_{\alpha\beta,i}\dot{k}^i,$$

$$\mathcal{A}_{\alpha\beta,i} = i u_\alpha^\dagger \frac{\partial u_\beta}{\partial k^i}, \quad i = 1,2,3$$

and

$$\mathcal{F}_{\alpha\beta} = -\frac{1}{2m^2\gamma^3}\left\{\left(\boldsymbol{\sigma} + \frac{1}{m^2}\frac{(\mathbf{k}\cdot\boldsymbol{\sigma})\mathbf{k}}{\gamma+1}\right)_{\alpha\beta}\right\}\cdot(d\mathbf{k}\times d\mathbf{k}).$$

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Berry curvature has opposite sign!

Outline

Classical action for spinning particle in GR

Let λ be a Lorentz transformation in Dirac representation, e_a^μ and e_μ^{*a} a frame and co-frame, and define

$$\begin{aligned} k_a &= \mathrm{tr}\,\{\kappa\lambda^{-1}\gamma_a\lambda\}, \quad \kappa = \kappa^a\gamma_a \\ S_{ab} &= \mathrm{tr}\{\Sigma\lambda^{-1}\sigma_{ab}\lambda\}, \quad \Sigma = \tfrac{1}{2}\Sigma^{ab}\sigma_{ab} \end{aligned}$$

where $[\kappa, \Sigma] = 0$, so that $k^a S_{ab} = 0$ (Tulczyjew-Dixon condition)
Action

$$S[x, \lambda] = \int \left\{k_a e_\mu^{*a} dx^\mu - \mathrm{tr}\{\Sigma\,\lambda^{-1}(d+\omega)\lambda\}\right\}.$$

where

$$\omega = \tfrac{1}{2}\sigma_{ab}\,\omega^{ab}{}_\mu dx^\mu$$

is spin connection one-form.

Mathisson-Papapetrou-Dixon equations

- Varying x^μ gives us

$$\boxed{\frac{Dk_c}{D\tau} + \tfrac{1}{2} S_{ab} R^{ab}{}_{cd} \dot{x}^d = 0}$$

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- Need additional condition such as $k^a S_{ab} =$ or $n^a S_{ab} = 0$ for closed system.

Anomalous velocity

Use $k^a S_{ab} = 0$ to get

$$-\frac{Dk^a}{D\tau} S_{ab} = k^2 \dot{x}_b - k_b(\dot{x} \cdot k).$$

or

$$\dot{x}_a = \frac{1}{m^2} \left(k_a(\dot{x} \cdot k) + S_{ac} \frac{Dk^c}{D\tau} \right).$$

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Chose "time" so that $\dot{x}^0 = 1$, then

$$1 = \frac{1}{m^2}\left\{(\dot{x} \cdot k)E + S_{0c}\frac{Dk^c}{Dt}\right\},$$

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Eliminate k^0, then

$$\boxed{\dot{x}_i = \frac{k_i}{E} + \frac{1}{m^2}\left(S_{ij} - S_{i0}\frac{k_j}{E} - \frac{k_i}{E}S_{0j}\right)\frac{Dk^j}{Dt}}$$

Meaning of conditions on spin tensor

- Lab frame energy centroid $X^i_{\rm L}$:

$$\left\{\int_{x^0=t} T^{00} d^3x\right\} X^i_{\rm L} = \int_{x^0=t} x^i T^{00} d^3x.$$

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- Angular momentum about x^μ_{A}:

$$M^{\mu\nu}_{\mathrm{A}} = \int_{x^0=t} \left\{(x^\mu - x^\mu_{\mathrm{A}})T^{\nu 0} - (x^\nu - x^\nu_{\mathrm{A}})T^{\mu 0}\right\} d^3x$$

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- Therefore

$$\begin{aligned} M_{\rm A}^{i0} &= \int_{x^0=t} \left\{(x^i - x_{\rm A}^i)T^{00} - (x^0 - x_{\rm A}^0)T^{i0}\right\} d^3x \\ &= (X_{\rm L}^i - x_{\rm A}^i)E. \end{aligned}$$

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- $k_a S^{ab} = 0$ implies that $x^\mu(\tau)$ in the M-P-D equation is trajectory of "centre of mass" — *i.e.* energy centroid in rest frame.
- Also see that Pauli-Lubansky "Berry curvature"

$$S_{\mu\nu} - S_{\mu 0}\frac{k_\nu}{E} - \frac{k_\mu}{E} S_{0\nu}$$

is angular momentum about lab-frame centroid.

Myron Mathisson explaining Spin

Outline

Massless case

When $m^2 = 0$ bad things happen!

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- Suppose that $k^2 = 0$, and S_{ab} satisfies $S_{ab}k^b = 0$, then

$$\tilde{S}_{ab} = S_{ab} + (k_a S_{pb} - k_b S_{pa})\Theta^p$$

still satisfies $\tilde{S}_{ab}k^b = 0$.

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A gauge invariance?

Wigner Translations

- Massless reference momentum $\kappa^a = (1, 0, 0, \dots, 0, 1)$.
- Little group: σ_{ab} with $0 < a, b, < d - 1$. Generate $\mathrm{SO}(d-2)$, together with "translations"

$$\pi_a = \kappa^b \sigma_{ba} \equiv \sigma_{0a} + \sigma_{(d-1)a}, \quad 0 < a < d - 1.$$

$$[\pi_a, \pi_b] = 0, \quad [\sigma_{ab}, \pi_c] = \eta_{bc}\pi_a - \eta_{ac}\pi_b.$$

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- Wigner says that the π_a must have no physical effect...
- ...but

$$\lambda \to \lambda \exp\left(\sum_{i=1}^{d-2} \theta^i \pi_i\right), \quad \text{in} \quad S_{ab} = \mathrm{tr}\{\Sigma \lambda^{-1} \sigma_{ab} \lambda\},$$

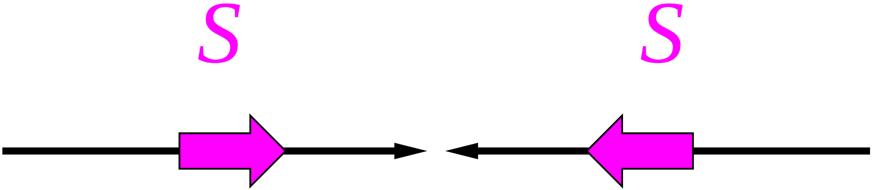
takes

$$S_{ab} \to S_{ab} + (k_a S_{pb} - k_b S_{pa})\Theta^p, \quad \Theta^p = \Lambda^p{}_i \theta^i.$$

Heisenberg, Wigner

Heisenberg and Eugene Wigner

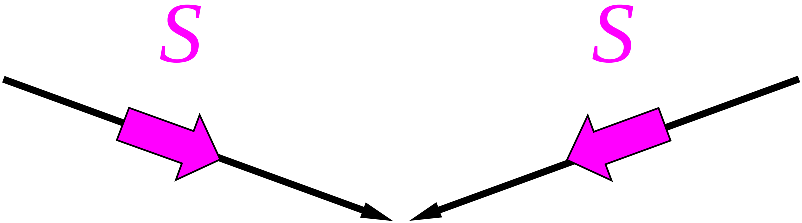
Physical Meaning of Wigner Translations



Head-on collision of massless spinning particles.

$L = 0,\ S = 0 \Rightarrow J = 0.$

Physical Meaning of Wigner Translations

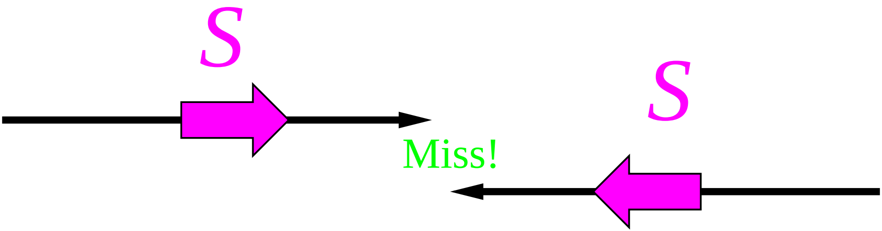


Run towards collision, top view.

$J = 0$, $S \neq 0$

$\Rightarrow L \neq 0$.

Physical Meaning of Wigner Translations



Boost towards collision, front view.
Miss by $\delta x = L/k$

Huh!

It's not that weird:

- Any interaction that occurs in one frame still occurs when viewed from another frame.
- Cross-sections depend on $J = L + S$.
- For massless particles, cannot separate L from S.
- Means that particle "position" is frame dependent.
- A serious problem for any covariant mechanics!

Show some Mathematica$^{\mathrm{TM}}$ plots to prove that frame dependence is a real phenomenon

Outline

Conclusions

- For massive particles, the Berry-phase equations of motion for relativistic spinning particles are the 3-dimensional reduction of 3+1 Lorentz covariant equations
- The Berry phase equations of motion for massless particles are not the $m \to 0$ limit of the massive-particle equations
- The Berry phase equations of motion for massless particles are not the 3-dimensional reduction of covariant equations
- The lack of covariance arises because the position ascribed to a massless particle is the lab-frame centroid, and is frame-dependent