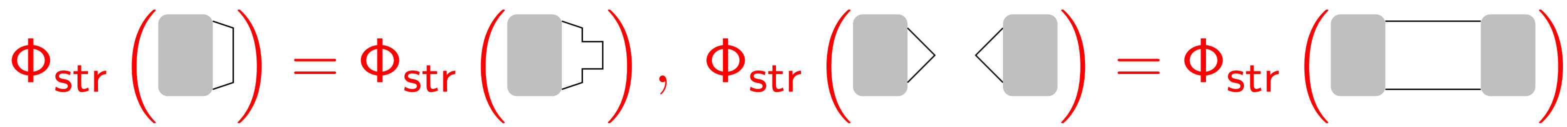
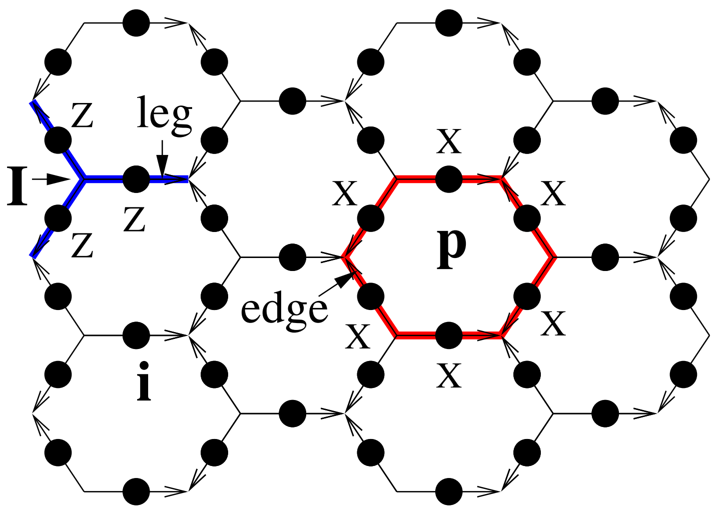
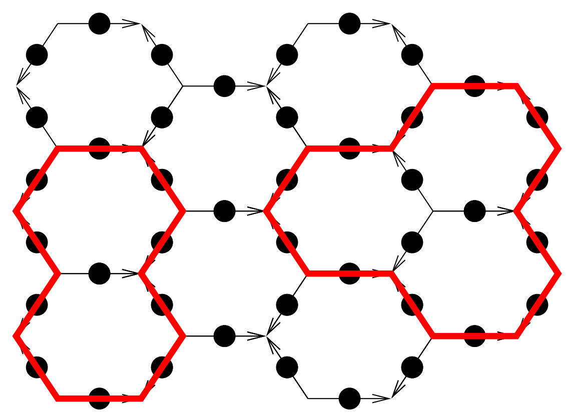
Toric-code model – Z_2 topological order, Z_2 gauge theory

Dancing rules:



- The Hamiltonian to enforce the dancing rules:



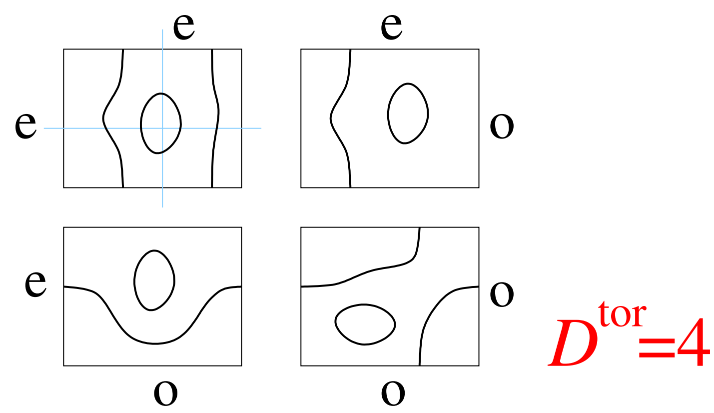


$$H = -U \sum_{\mathbf{I}} Q_{\mathbf{I}} - g \sum_{\mathbf{p}} F_{\mathbf{p}}, \quad Q_{\mathbf{I}} = \prod_{\text{legs of } \mathbf{I}} \sigma^z_{\mathbf{i}}, \quad F_{\mathbf{p}} = \prod_{\text{edges of } \mathbf{p}} \sigma^x_{\mathbf{i}}$$

- Ground state wave function $\Phi(X) = \text{const.}$

Stable ground state degeneracy and topo. order

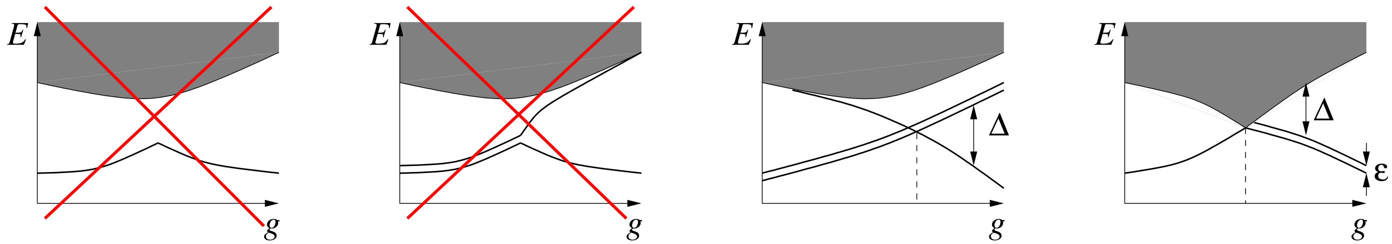
- The hamiltonian is a sum of commuting operators $[F_{\mathbf{p}}, F_{\mathbf{p}'}] = 0$, $[Q_{\mathbf{l}}, Q_{\mathbf{l}'}] = 0$, $[F_{\mathbf{p}}, Q_{\mathbf{l}}] = 0$. $F_{\mathbf{p}}^2 = Q_{\mathbf{l}}^2 = 1$
- Ground state $|\Psi_{\text{grnd}}\rangle$: $F_{\mathbf{p}}|\Psi_{\text{grnd}}\rangle = Q_{\mathbf{l}}|\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$, $E_{\text{grnd}} = -2UN_{\text{cell}} - gN_{\text{cell}}$
- Quasiparticle excitation energy gap $\Delta_p^Q = 2U$, $\Delta_p^F = 2g$



Ground state degeneracy

- Identities $\prod_{\mathbf{l}} Q_{\mathbf{l}} = 1$, $\prod_{\mathbf{p}} F_{\mathbf{p}} = 1$.
- Number of independent quantum numbers $F_{\mathbf{p}} = \pm 1$, $Q_{\mathbf{l}} = \pm 1$ on torus: $N_{\text{label}} = 2^{2N_{\text{cell}}} 2^{N_{\text{cell}}}/4$
  Number of states on torus: $N_{\text{label}} = 2^{2N_{\text{cell}}} 2^{N_{\text{cell}}}$
- H is a function of $F_{\mathbf{p}}, Q_{\mathbf{l}}$. The degeneracy of any eigenstates is 4.
- On genus g surface, ground state degeneracy $D_g = 4^g$
- The above degenerate ground states form a "code", which has a large "code distance" of order L (the linear size of the system).

Stable gapped ground state → gapped quantum phase



- Second-order transition point = gapless state
- First-order transition point = unstable gapped state

→ gapped quantum phase = stable gapped ground state

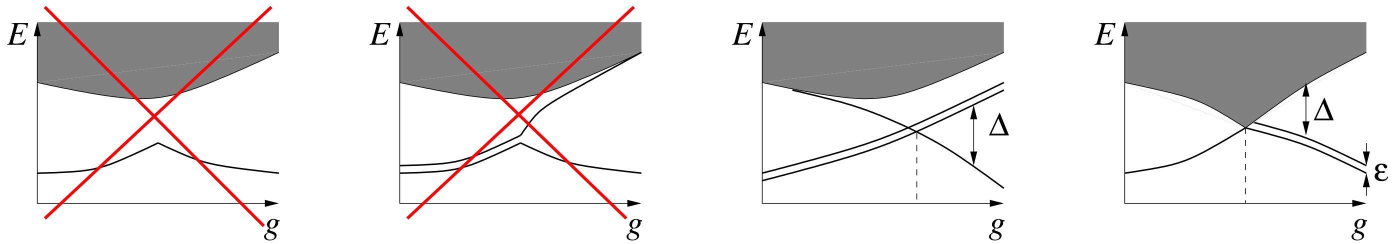
- **Stable** ground state degeneracy → Gapped quantum phase

However, for a long time, we thought that

- without symmetry, the stable ground state degeneracy always = 1
- with symmetry, the stable ground state degeneracy $\neq 1$,

→ symmetry breaking = emergence of ground state degeneracy which is stable against any perturbation that repect the symmetry.

Stable gapped ground state → gapped quantum phase



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→ gapped quantum phase = stable gapped ground state

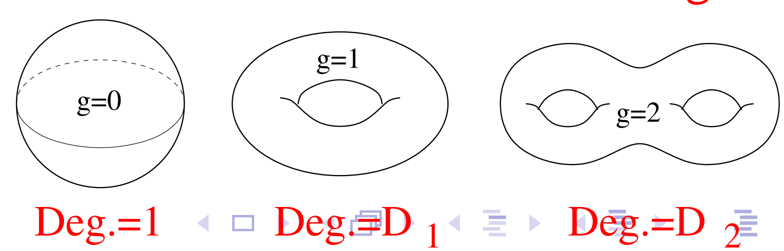
- **Stable** ground state degeneracy → Gapped quantum phase

However, for a long time, we thought that

- without symmetry, the stable ground state degeneracy always = 1
- with symmetry, the stable ground state degeneracy ≠ 1,
  → symmetry breaking = emergence of ground state degeneracy

which is stable against any perturbation that repect the symmetry.

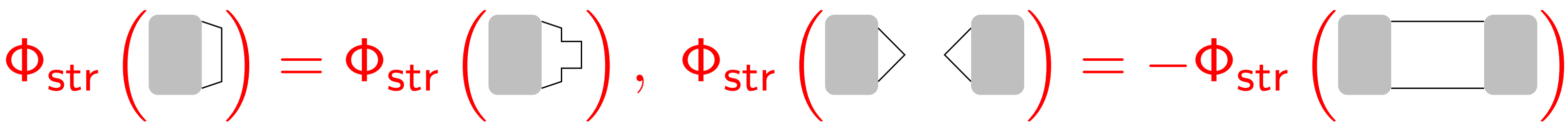
- The above **topology-dependent ground state degeneracy** D_g, is stable against any perturbations:
→ a new kind of order **topo. order**



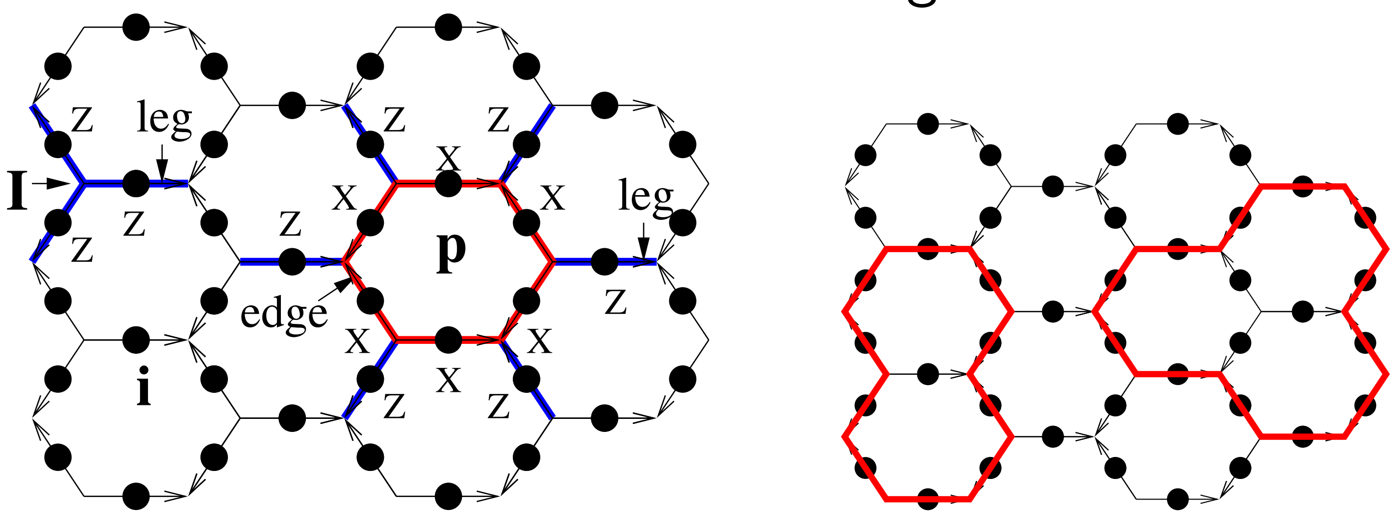
Wen 89; Wen-Niu 90

Double-semion theory

Dancing rules:



- The Hamiltonian to enforce the dancing rules:



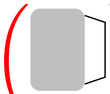
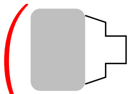
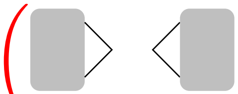
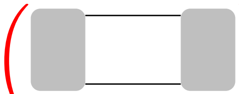
$H = -U\sum_{\mathbf{I}} Q_{\mathbf{I}} - \frac{g}{2}\sum_{\mathbf{p}}(F_{\mathbf{p}} + h.c.)$,

$Q_{\mathbf{I}} = \prod_{\text{legs of } \mathbf{I}} \sigma^z_{\mathbf{i}}, \quad F_{\mathbf{p}} = (\prod_{\text{edges of } \mathbf{p}} \sigma^x_{\mathbf{j}})(-\prod_{\text{legs of } \mathbf{p}} \mathrm{i}^{\frac{1-\sigma^z_{\mathbf{j}}}{2}})$

- Ground state wave function $\Phi(X) = (-)^{X_c}$, where X_c is the number of loops in the string configuration X.

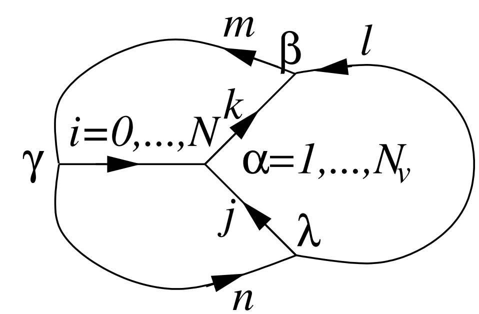
More general patterns of long-range entanglement

Generslize the Z_2/double-semion dancing rule:

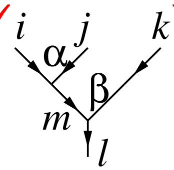
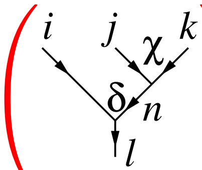
$\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$, $\Phi_{\text{str}}\Big($  $\Big) = \pm\Phi_{\text{str}}\Big($  $\Big)$

Graphic state:

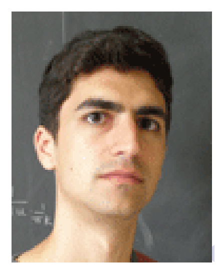
- More general wave functions are defined on graphs, with $N+1$ states on links and $N_v = N^{ij}_k$ states on vertices:



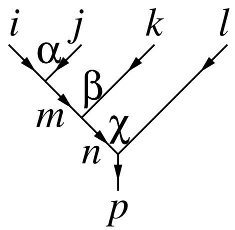
More general local rule: F-move Levin-Wen, 2005; Chen-Gu-Wen, 2010

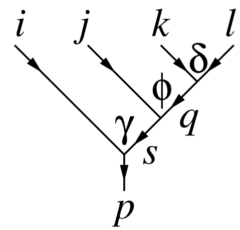
F-move: $\Phi\Big($  $\Big) = \sum_{n=0}^{N}\sum_{\chi=1}^{N_{kjn}}\sum_{\delta=1}^{N_{nil}} F^{ijk;m\alpha\beta}_{l;n\chi\delta}\,\Phi\Big($  $\Big)$

The matrix $F^{ijk}_l \to (F^{ijk}_l)^{m\alpha\beta}_{n\chi\delta}$ = local unitary transformation

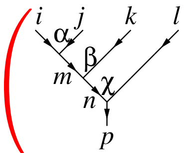
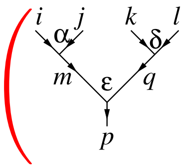
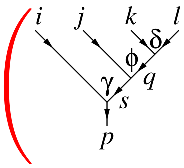
  

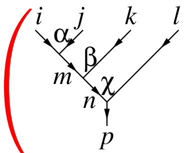
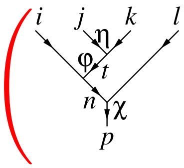
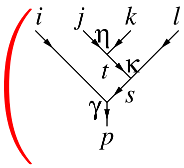
Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$: the pentagon identity

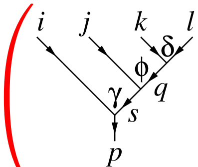


can be trans. to


through two different paths:

$\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon}\Phi\Big($  $\Big) = \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}\Phi\Big($  $\Big)$,

$\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi}\Phi\Big($  $\Big) = \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma}\Phi\Big($  $\Big)$

$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi}\Phi\Big($  $\Big)$.

The two paths should lead to the same LU trans.:

$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity.

Thier solution $N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta} \to$ **Unitary fusion category (UFC)**

$\to$ string-net states

A complete characterization of 2+1D topological order

- Both Z_2 topological order and double-semion topological order have the same degeneracy 4^g on genus g surfaces.

  Do they have the same topological order and belong to the same phase?

A complete characterization of 2+1D topological order

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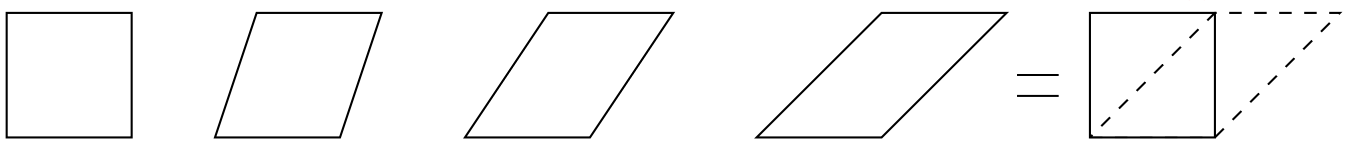
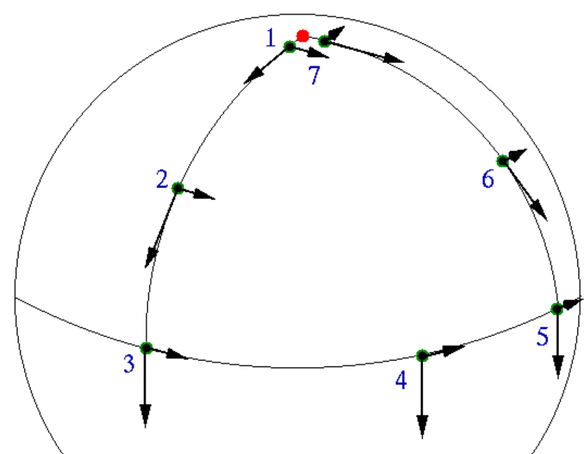
- **Non-Abelian geometric phases** Wilczek-Zee 84 of the degenerate ground state by deforming the torus: (1) $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$

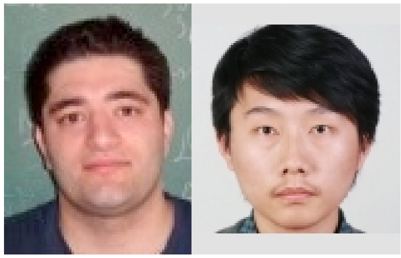
  $\hat{T}$:



  (2) 90° rotation $\hat{S}$: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$
- $\hat{T}, \hat{S}$ generate the MCG $SL(2,\mathbb{Z})$ of torus
- $T_{\alpha\beta}, S_{\alpha\beta}, c \to$ **complete characterization of topological order**
  - $D_{g=1} \to$ number quasiparticle types
  - Eigenvalues of $T_{\alpha\beta} \to$ quasiparticle fractional statistics

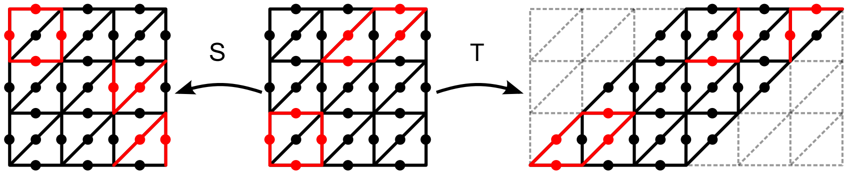


Measure topo. order: Universal wavefunction overlap



- Consider the ground states $|\Psi_\alpha\rangle$ on torus T^2, and two maps, $\hat{S} = 90^\circ$ rotation and $\hat{T} =$ Dehn twist. The non-Abelian geometric phases S, T via overlap

$$S_{\alpha\beta} e^{-f_S L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{S} | \Psi_\beta \rangle$$
$$T_{\alpha\beta} e^{-f_T L^2 + o(L^{-1})} = \langle \Psi_\alpha | \hat{T} | \Psi_\beta \rangle$$



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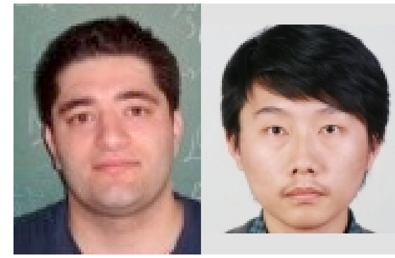

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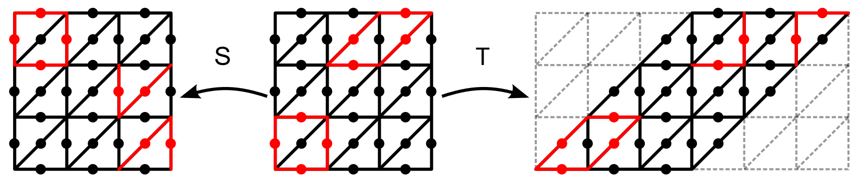
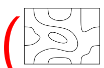
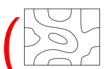
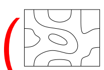
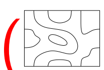


- For the first topo. order:

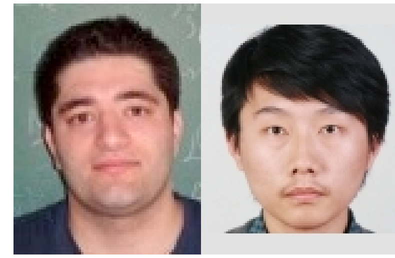
$\Psi_1($$) = g^{\text{string-length}}$

$\Psi_2($$) = (-)^{W_x} g^{\text{str-len}}$

$\Psi_3($$) = (-)^{W_y} g^{\text{str-len}}$

$\Psi_4($$) = (-)^{W_x + W_y} g^{\text{str-len}}$

Measure topo. order: Universal wavefunction overlap



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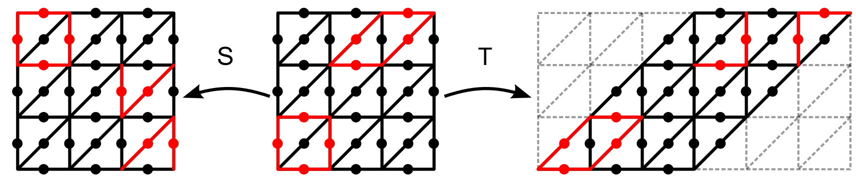

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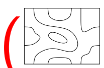
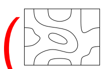
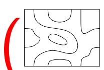
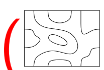
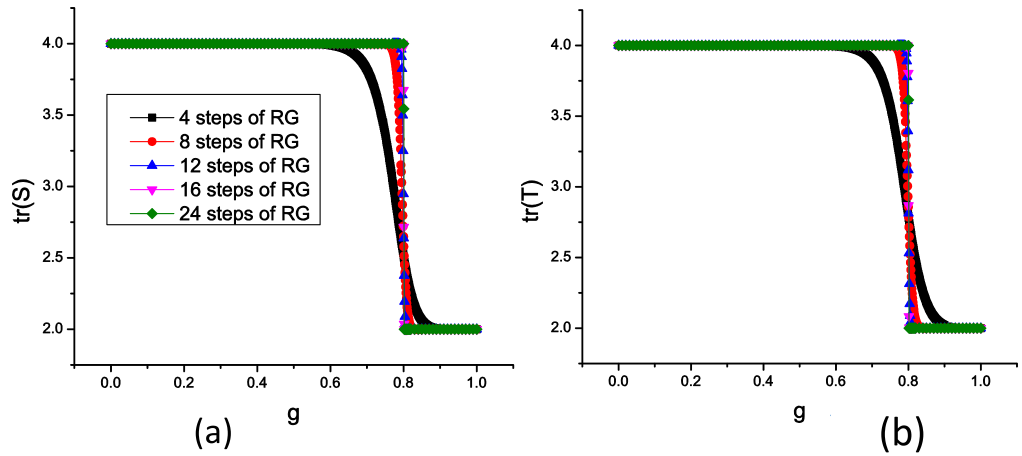
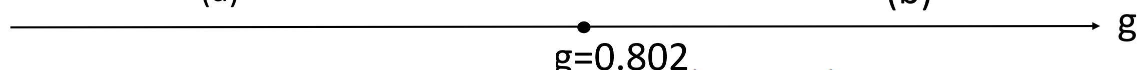
  $\Psi_1($$) = g^{\text{string-length}}$
  $\Psi_2($$) = (-)^{W_x} g^{\text{str-len}}$
  $\Psi_3($$) = (-)^{W_y} g^{\text{str-len}}$
  $\Psi_4($$) = (-)^{W_x + W_y} g^{\text{str-len}}$
- $g < 0.8$ small-loop phase
  $|\Psi_\alpha\rangle$ are the same state
- $g > 0.8$ large-loop phase
  $|\Psi_\alpha\rangle$ are four diff. states





$$S = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c)

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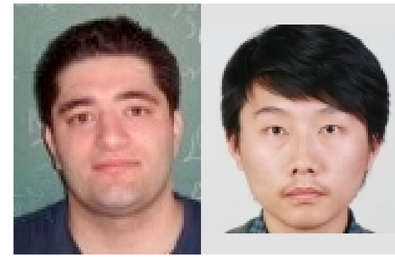
  The non-Abelian geometric phases S, T via overlap



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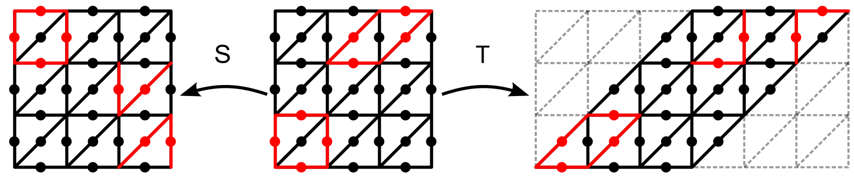
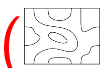
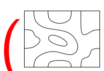
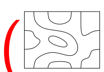
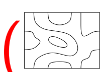
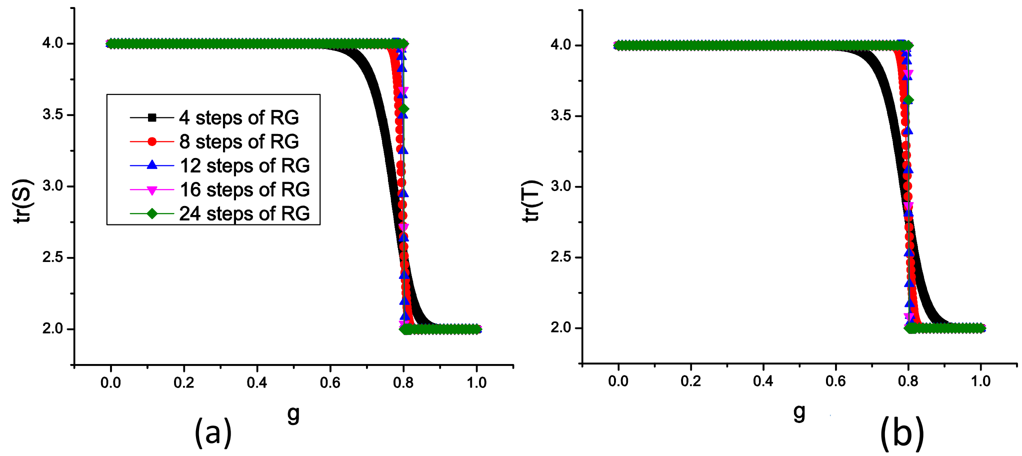


- For the first topo. order:
  $\Psi_1($$) = g^{\text{string-length}}$
  $\Psi_2($$) = (-)^{W_x} g^{\text{str-len}}$
  $\Psi_3($$) = (-)^{W_y} g^{\text{str-len}}$
  $\Psi_4($$) = (-)^{W_x+W_y} g^{\text{str-len}}$
- $g < 0.8$ small-loop phase
  $|\Psi_\alpha\rangle$ are the same state
- $g > 0.8$ large-loop phase
  $|\Psi_\alpha\rangle$ are four diff. states
- For the second topo. order:



g

g=0.802

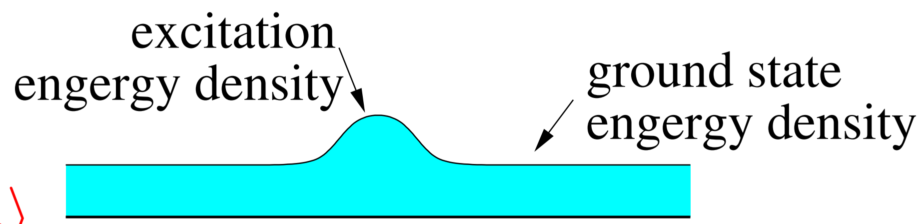
$$S = \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix} \quad T = \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix} \qquad S = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix} \quad T = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}$$

(c)

$$S = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&-1 \end{pmatrix}, \quad T = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&-1&0 \end{pmatrix}.$$

Local and topological quasiparticle excitations

In a system: $H = \sum_{\vec{x}} H_{\vec{x}}$



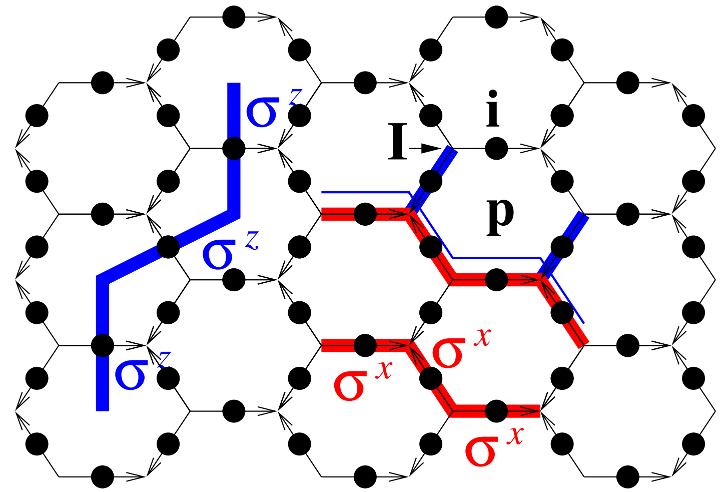
- a particle-like excitation:
  energy density $= \langle \Psi_{exc} | H_{\vec{x}} | \Psi_{exc} \rangle$

  $|\Psi_{exc}\rangle$ is the gapped ground state of $H + \delta H^{\text{trap}}(\vec{x})$.
- Local quasiparticle excitation: $|\Psi_{exc}\rangle = \hat{O}(\vec{x})|\Psi_{grnd}\rangle$
- Topological quasiparticle excitations $|\Psi_{exc}\rangle \neq \hat{O}(\vec{x})|\Psi_{grnd}\rangle$ for any local operators $\hat{O}(\vec{x})$
- Topological quasiparticle types:
  if $|\Psi'_{exc}\rangle = \hat{O}(\vec{x})|\Psi_{exc}\rangle$, then $|\Psi'_{exc}\rangle$ and $|\Psi_{exc}\rangle$ belong to the same type.

Number of topological quasiparticle types is an important topological invariant that characterizes the topological order.

Only topological quasiparticles can carry fractional statistics and fractional quantum numbers.

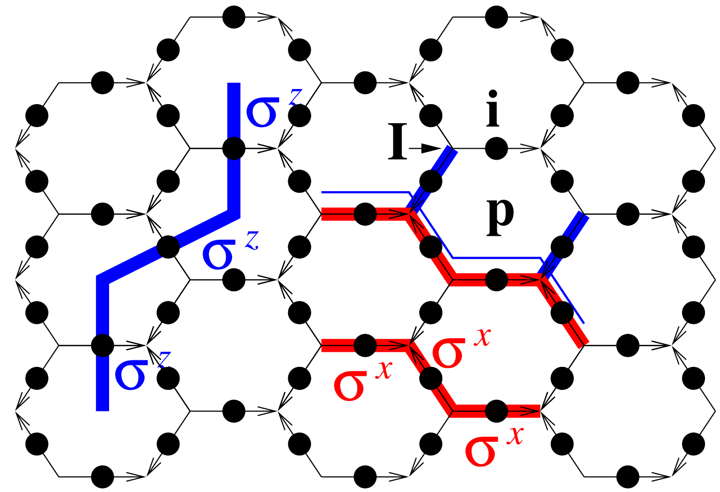
The string operators in the Z_2 topologically ordered state: the creation operator of topological quasiparticle



- Toric code model:
  $H = -U \sum_{\mathbf{I}} Q_{\mathbf{I}} - g \sum_{\mathbf{p}} F_{\mathbf{p}}$
  $Q_{\mathbf{I}} = \prod_{\text{legs of } \mathbf{I}} \sigma^z_{\mathbf{i}}$,
  $F_{\mathbf{p}} = \prod_{\text{edges of } \mathbf{p}} \sigma^x_{\mathbf{i}}$
- Topological excitations:
  $Q_{\mathbf{I}} = 1 \to Q_{\mathbf{I}} = -1$
  $F_{\mathbf{p}} = 1 \to F_{\mathbf{p}} = -1$
- Type-I string operator $W_{\mathrm{I}} = \prod_{\text{string}} \sigma^x_i$
- Type-II string operator $W_{\mathrm{II}} = \prod_{\text{string}*} \sigma^z_i$
- Type-III string op. $W_{\mathrm{III}} = \prod_{\text{string}} \sigma^x_i \prod_{\text{legs}} \sigma^z_i$
- $[H, W_{\mathrm{I}}^{\text{closed}}] = [H, W_{\mathrm{II}}^{\text{closed}}] = [H, W_{\mathrm{III}}^{\text{closed}}] = 0$.
  $W_{\mathrm{I}}^{\text{closed}}|\Psi_{\text{grnd}}\rangle = W_{\mathrm{II}}^{\text{closed}}|\Psi_{\text{grnd}}\rangle = W_{\mathrm{III}}^{\text{closed}}|\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$
- $F_{\mathbf{p}}$: closed type-I string opertors. $Q_{\mathbf{I}}$: closed type-II string opertors.

The string operators in the Z_2 topologically ordered state: the creation operator of topological quasiparticle

- Toric code model:
  $H = -U \sum_{\mathbf{I}} Q_{\mathbf{I}} - g \sum_{\mathbf{p}} F_{\mathbf{p}}$
  $Q_{\mathbf{I}} = \prod_{\text{legs of } \mathbf{I}} \sigma_{\mathbf{i}}^z$,
  $F_{\mathbf{p}} = \prod_{\text{edges of } \mathbf{p}} \sigma_{\mathbf{i}}^x$



- Topological excitations:
  $Q_{\mathbf{I}} = 1 \to Q_{\mathbf{I}} = -1$
  $F_{\mathbf{p}} = 1 \to F_{\mathbf{p}} = -1$
- Type-I string operator $W_{\mathrm{I}} = \prod_{\text{string}} \sigma_i^x$ $\to$ e-type. $e \times e = 1$
- Type-II string operator $W_{\mathrm{II}} = \prod_{\text{string}*} \sigma_i^z$ $\to$ m-type. $m \times m = 1$
- Type-III string op. $W_{\mathrm{III}} = \prod_{\text{string}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$ $\to$ ϵ-type $= e \times m$
- $[H, W_{\mathrm{I}}^{\text{closed}}] = [H, W_{\mathrm{II}}^{\text{closed}}] = [H, W_{\mathrm{III}}^{\text{closed}}] = 0$.
  $W_{\mathrm{I}}^{\text{closed}}|\Psi_{\text{grnd}}\rangle = W_{\mathrm{II}}^{\text{closed}}|\Psi_{\text{grnd}}\rangle = W_{\mathrm{III}}^{\text{closed}}|\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$
- $F_{\mathbf{p}}$: closed type-I string opertors. $Q_{\mathbf{I}}$: closed type-II string opertors.
- Open string operators create topological excitations.
  Open string operators are hopping operators of topo. excitations

Emergence of fractional spin/statistics

- Why electron carry spin-1/2 and Fermi statistics?
- Ends of strings (type-I) are point-like excitations, which can carry spin-1/2 and Fermi statistics?

Fidkowski-Freedman-Nayak-Walker-Wang 06

- $\Phi_{\text{str}}\left(\;\right) = 1$ string liquid $\Phi_{\text{str}}\left(\;\right) = \Phi_{\text{str}}\left(\;\right)$

  360° rotation: $\to$ and $=$ $\to$: $R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

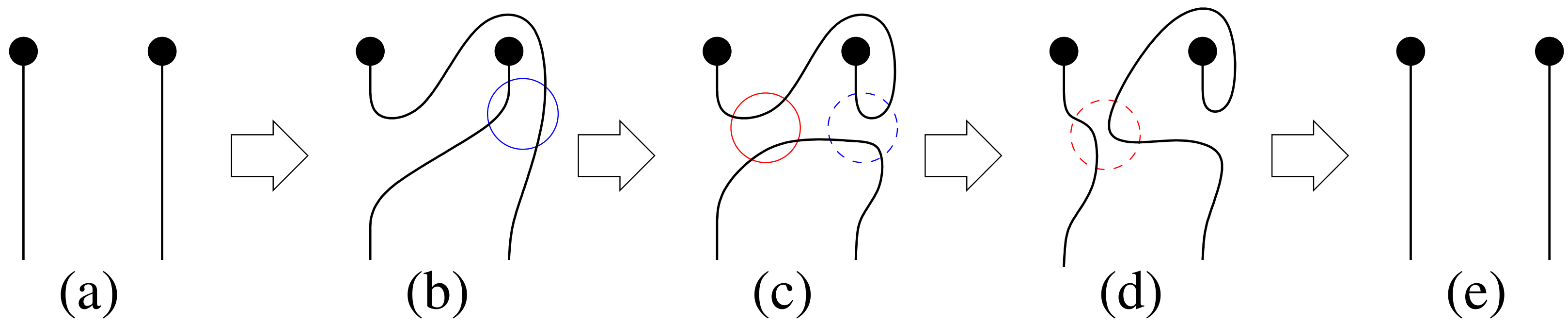
  $+$ has a spin 0 mod 1. $-$ has a spin $1/2$ mod 1.

- $\Phi_{\text{str}}\left(\;\right) = (-)^{\#\text{ of loops}}$ string liquid $\Phi_{\text{str}}\left(\;\right) = -\Phi_{\text{str}}\left(\;\right)$

  360° rotation: $\to$ and $= -$ $\to -$: $R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  $+\,\mathrm{i}$ has a spin $-1/4$ mod 1. $-\,\mathrm{i}$ has a spin $1/4$ mod 1.

Spin-statistics theorem

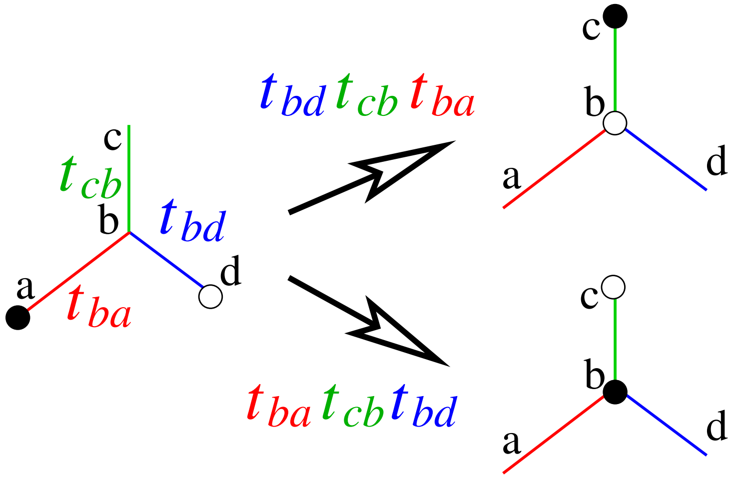


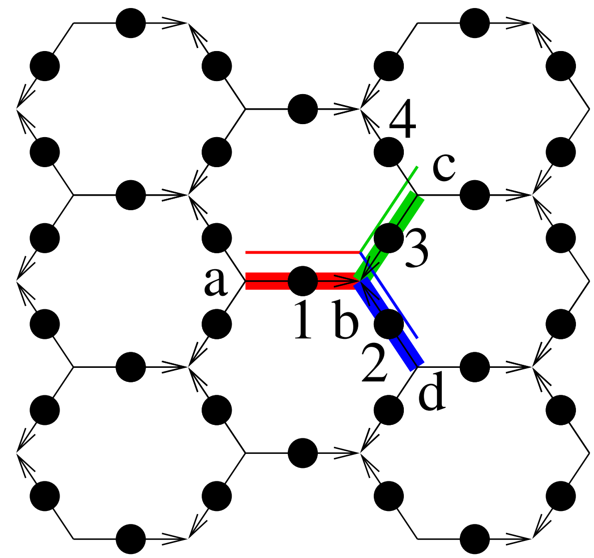
- (a) $\to$ (b) $=$ exchange two string-ends.
- (d) $\to$ (e) $= 360^\circ$ rotation of a string-end.
- Amplitude (a) $=$ Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

$\to$ **Spin-statistics theorem**

Statistics of ends of strings

- The statistics is determined by particle hopping operators Levin-Wen 03:





- An open string operator is a hopping operator of the 'ends'. The algebra of the open string operator determine the statistics.
- For type-I string: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x$
  We find $t_{bd}t_{cb}t_{ba} = t_{ba}t_{cb}t_{bd}$
  The ends of type-I string are bosons
- For type-III strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \underline{\sigma_3^x}\sigma_4^z$, $t_{bd} = \sigma_2^x\underline{\sigma_3^z}$
  We find $t_{bd}t_{cb}t_{ba} = -t_{ba}t_{cb}t_{bd}$
  The ends of type-III strings are fermions