Incompressibility Estimates in the Laughlin Phase

Jakob Yngvason, University of Vienna

with Nicolas Rougerie, University of Grenoble

ESI, September 8, 2014

References

N. Rougerie, JY, *Incompressibility Estimates for the Laughlin Phase*, arXiv:1402.5799; to appear in CMP.

N. Rougerie, JY, *Incompressibility Estimates for the Laughlin Phase II*, in preparation

Also:

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall states of bosons in rotating anharmonic traps, Phys. Rev. A* **87**, 023618 (2013); arXiv:1212.1085

N. Rougerie, S. Serfaty, J.Y., *Quantum Hall Phases and the Plasma Analogy in Rotating Trapped Bose Gases, J. Stat. Phys*, **154**, 2–50 (2014), arXiv:1301.1043

IQHE and FQHE

The Integer Quantum Hall Effect can essentially be understood in terms of single particle physics and the Pauli principle for the electrons. In a magnetic field B there is a maximal number, $\frac{B}{2\pi}$, of fermions per unit area that can be accommodated in a given Landau level, corresponding to a filling factor 1.

In the Fractional Quantum Hall Effect, on the other hand, the correlations induced by strong repulsive interactions are essential and reduce the particle density in the ground state. The filling factor(s) in the state(s) introduced by Laughlin is $1/3$ (more generally $1/\ell$).

This talk concerns estimates for the 1-partice density, and also some higher marginals of the N-particle density, in correlated many-body states in the lowest Landau level (LLL), related to the Laughlin state. The results apply also to bosons, that may show features of the FQHE under rapid rotation.

Upper bounds on the density are a manifestation of the incompressibility of the quantum fluid.

Landau levels

Magnetic Hamiltonian:

$$H = -\tfrac{1}{2}\left(\nabla_\perp + \mathrm{i}\,\mathbf{A}(\mathbf{r})\right)^2$$

with $\mathbf{A}(\mathbf{r}) = \frac{B}{2}(-y, x)$. Choose units so that $B = 2$. (For rotating systems $B = 2\Omega$.)

Complex coordinates: $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, $\partial = \frac{d}{dz}$, $\bar{\partial} = \frac{d}{d\bar{z}}$.

We can write

$$H = a^\dagger a + \tfrac{1}{2}$$

with

$$a = \left(\bar{\partial} + \tfrac{1}{2}z\right), \qquad a^\dagger = \left(-\partial + \tfrac{1}{2}\bar{z}\right), \qquad [a, a^\dagger] = 1.$$

The Spectrum of H is $\varepsilon_n = 2(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$.

H commutes with another set of creation and annihilation operators

$$b = (\partial + \tfrac{1}{2}\bar{z}) \qquad \text{and} \qquad b^\dagger = \left(-\bar{\partial} + \tfrac{1}{2}z\right)$$

and hence also with the angular momentum operator

$$\mathcal{L} = b^\dagger b - a^\dagger a = z\partial - \bar{z}\bar{\partial}.$$

The joint eigenfunctions of H and $\mathcal{L}$ have the form

$$\psi_{n,l}(z,\bar{z}) = P_{n,l}(z,\bar{z})e^{-|z|^2/2}$$

with associated Laguerre polynomials $P_{n,l}(z,\bar{z})$.

The LLL, characterized by $a\,\psi = (\bar{\partial} + \frac{1}{2}z)\psi = 0$, is generated by

$$\psi_{0,l}(z,\bar{z}) = (\pi l!)^{-1/2}\, z^l e^{-|z|^2/2}.$$

Maximum density droplet

The density $|\psi_{0,l}(z,\bar{z})|^2$ is concentrated around the maximum at radius

$$r_l = \sqrt{l}.$$

Thus, if $l \gg 1$, then l is the number of orthonormal states whose wave functions can be accommodated within a disc of area $\pi r_l^2 = \pi l$, so the density of states per unit area in the LLL is π^{-1}. The same holds for the higher Landau levels.

The "Maximum density droplet" (MDD) in the LLL is the Slater determinant

$$\Psi_{\mathrm{MDD}} = (N!)^{-1/2}\psi_{0,0} \wedge \cdots \wedge \psi_{0,N-1}$$

The 1-particle density $\sum_l |\psi_{0,l}|^2$ is essentially π^{-1} up to radius $\sqrt{N}$, i.e., the filling factor is 1.

The Laughlin wave function(s)

The wave function of the MDD is, apart from the gaussian factor, a Vandermonde determinant and can be written as

$$\Psi_{\mathrm{MDD}} = (N!)^{-1/2} \prod_{i<j} (z_i - z_j) e^{-\sum_{i=1}^{N} |z_i|^2/2}.$$

The Laughlin wave function(s), on the other hand, have the form

$$\Psi_{\mathrm{Laugh}}^{(\ell)} = C_{N,\ell} \prod_{i<j} (z_i - z_j)^{\ell} e^{-\sum_{i=1}^{N} |z_i|^2/2}$$

with ℓ odd ≥ 3 and $C_{N,\ell}$ a normalization constant. For Bosons ℓ is even and ≥ 2.

In his 1983 paper Laughlin claims that the 1-particle density of $\Psi_{\mathrm{Laugh}}^{(\ell)}$ within its support is close to $(\ell\pi)^{-1}$.

Methaphoric picture; plasma analogy

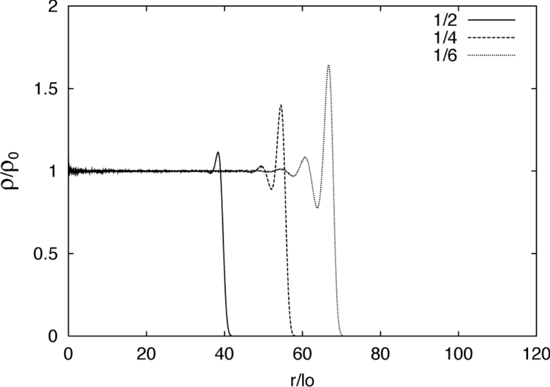
Methaphoric picture of the N-particle density (not due to Laughlin!):

The particles "move" in a correlated way, as tightly packed as the factors $(z_i - z_j)^\ell$ allow, like huddling emperor penguins during an Antarctic winter. Each "penguin" claims on the average an area of radius $\sqrt{\ell}$.

Laughlin's argument for the density $(\ell\pi)^{-1}$ is more mathematical. It is based on the "plasma analogy":

The N-particle density $|\Psi^{(\ell)}_{\rm Laugh}|^2$ can be interpreted as the Boltzmann-Gibbs factor at temperature $T = N^{-1}$ of classical 2D jellium, i.e., a 2D Coulomb gas in a uniform neutralizing background. A mean field approximation leads to the claimed density.

Numerical calculations (O. Ciftja) show, however, that the density may be considerably larger than $(\ell\pi)^{-1}$ close to the edge. The result can thus only hold in a suitable weak sense in the limit $N \to \infty$.



1/2
1/4
1/6
ρ/ρ₀
r/lo
0
0.5
1
1.5
2
0
20
40
60
80
100
120

A proof of Laughlin's claim amounts to a rigorous study of the mean field limit of the jellium model, including error estimates. More generally, we consider Laughlin's "quasi hole" states of the form

$$\Psi_{\mathrm{qh}}^{(\ell,m)} = C_m \prod_{i=1}^{N} z_i^m \, \Psi_{\mathrm{Laugh}}^{(\ell)}$$

where m may depend on N.

We denote $(z_1, \dots, z_N)$ by Z for short and consider the scaled N particle probability density (normalized to 1)

$$\mu^{(N)}(Z) = N^N \left| \Psi_{\mathrm{qh}}^{(\ell,m)}(\sqrt{N} Z) \right|^2 .$$

The density as a Boltzmann-Gibbs factor

We can write

$$\begin{aligned}\mu^{(N)}(Z) &= \mathcal{Z}_N^{-1}\exp\left(\sum_{j=1}^{N}(-N|z_j|^2+2m\log|z_j|)+2\ell\sum_{i<j}\log|z_i-z_j|\right)\\ &= \mathcal{Z}_N^{-1}\exp\left(-\frac{1}{T}\mathcal{H}_N(Z)\right),\end{aligned}$$

with $T = N^{-1}$ and

$$\mathcal{H}_N(Z) = \sum_{j=1}^{N}\left(|z_j|^2 - \frac{2m}{N}\log|z_j|\right) - \frac{2\ell}{N}\sum_{i<j}\log|z_i-z_j|.$$

Mean field limit

The Hamiltonian $\mathcal{H}_N(Z)$ defines a classical 2D Coulomb gas ('plasma') in a uniform background of opposite charge and a point charge $(2m/N)$ at the origin, corresponding respectively to the $|z_i|^2$ and the $-\frac{2m}{N}\log|z_j|$ terms.

The probability measure $\mu^{(N)}(Z)$ minimizes the free energy functional

$$\mathcal{F}(\mu) = \int \mathcal{H}_N(Z)\mu(Z) + T\int \mu(Z)\log\mu(Z)$$

for this Hamiltonian at $T = N^{-1}$.

The $N \to \infty$ limit is in this interpretation a mean field limit where at the same time $T \to 0$. It is thus not unreasonable to expect that for large N, in a suitable sense

$$\mu^{(N)} \approx \rho^{\otimes N}$$

with a one-particle density ρ minimizing a mean field free energy functional.

Mean field limit (cont.)

The mean field free energy functional is defined as

$$\mathcal{E}_N[\rho] = \int_{\mathbb{R}^2} W\,\rho - \ell \int\int \rho(z)\log|z-z'|\rho(z') + N^{-1}\int_{\mathbb{R}^2}\rho\log\rho$$

with

$$W(z) = |z|^2 - 2\frac{m}{N}\log|z|.$$

It has a minimizer ρ_N among probability measures on $\mathbb{R}^2$ and this minimizer should be a good approximation for the scaled 1-particle probability density of the trial wave function, i.e.,

$$\mu_N^{(1)}(z) = \int_{\mathbb{R}^{2(N-1)}} \mu^{(N)}(z, z_2, \ldots, z_N)\mathrm{d}^2 z_2 \ldots \mathrm{d}^2 z_N.$$

Properties of the Mean Field Density

The picture of the 1-particle density arises from asymptotic formulas for the mean-field density: If $m \lesssim N^2$, then ρ_N is well approximated by a density $\hat{\rho}_N$ that minimizes the mean field functional without the entropy term.

The variational equation satisfied by this density is:

$$|z|^2 - 2(m/N)\log|z| - 2\ell\rho * \log|z| - C \geq 0$$

with "$=$" where $\rho > 0$ and "$>$" where $\rho = 0$.

Applying the Laplacian gives

$$1 - \pi(m/N)\delta(z) - \ell\pi\,\rho(z) = 0$$

where $\rho > 0$. Hence ρ takes the constant value $(\ell\pi)^{-1}$ on its support.

Using the radial symmetry and the maximum principle for the electrostatic potential one deduces that the support of $\hat{\rho}_N$ is an annulus with inner and outer radia

$$R_- = (m/N)^{1/2} \quad \text{and} \quad R_+ = (2 + m/N)^{1/2}$$

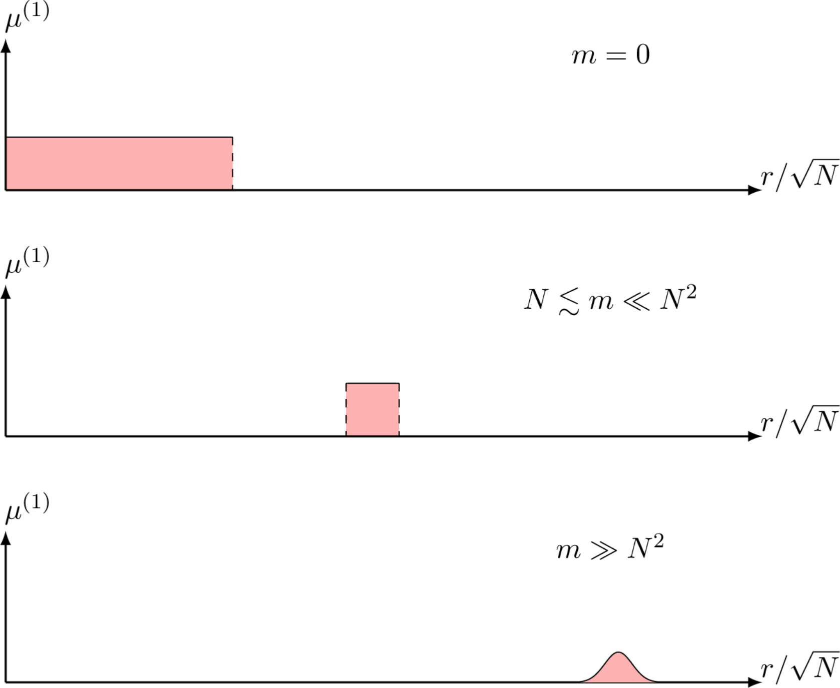
and is zero otherwise.

For $m \gtrsim N^2$ the entropy term dominates the interaction term $\int\int \rho(z) \log|z - z'|\rho(z')$.

The density is then well approximated by a gaussian

$$\rho^{\text{th}}(z) \sim |z|^{2m} \exp(-N|z|^2)$$

that in the radial variable is centered around $\sqrt{m/N}$.



$\mu^{(1)}$
$m = 0$
$r/\sqrt{N}$
$\mu^{(1)}$
$N \lesssim m \ll N^2$
$r/\sqrt{N}$
$\mu^{(1)}$
$m \gg N^2$
$r/\sqrt{N}$

General states in the Laughlin phase

Consider now general states in the Laughlin phase of the form

$$\Psi(z_1, \ldots, z_N) = \phi(z_1, \ldots, z_N) \prod_{i<j} (z_i - z_j)^\ell e^{-\sum_{j=1}^N |z_j|^2/2}$$

with ℓ even ≥ 2 for bosons, or ℓ odd ≥ 3 for fermions, and $\phi(z_1, \ldots, z_N)$ a symmetric holomorphic function.
These states constitute the kernel $\ker \mathcal{I}_N$ of

$$\mathcal{I}_N = \sum_{i<j} \delta(z_i - z_j)$$

that is well defined (even a bounded operator) in the LLL. In fact, "$\delta(z_i - z_j)$" is equivalent to

$$(\delta_{ij}\varphi)(z_i, z_j) = \frac{1}{2\pi} \varphi\left(\tfrac{1}{2}(z_i + z_j), \tfrac{1}{2}(z_i + z_j)\right).$$

For other repulsive potentials peaked at the origin, e.g. $|z_i - z_j|^{-1}$, the interaction is not zero, but suppressed by the $(z_i - z_j)^\ell$ factors.

Define as before

$$\mu^{(N)}(Z) = N^N |\Psi(\sqrt{N} Z)|^2,$$

and the n-marginal

$$\mu^{(n)}(z_1, \dots, z_n) = \int_{\mathbb{C}^{N-n}} \mu^{(N)}(z_1, \dots, z_n; z_{n+1}, \dots, z_N) dz_{n+1} \cdots dz_N.$$

We would like to prove that in a suitable sense, for $N \to \infty$ but $n \ll N$,

$$\mu^{(n)} \approx (\mu^{(1)})^{\otimes n}$$

and that the density $\mu^{(1)}$ satisfies the "incompressibility bound"

$$\mu^{(1)}(z) \leq \frac{1}{\ell\pi}.$$

Comparison with 'bathtub energy'

An "incompressibility bound" leads to a lower bound to the potential energy (and hence to the ground state energy in the Laughlin phase!) in an external potential,

$$\int V(z)\mu^{(1)}(z)dz,$$

in terms of the 'bathtub energy'

$$E^{\mathrm{bt}}(V) = \inf\left\{\int V(z)\rho(z)dz \,:\, 0 \leq \rho \leq (\ell\pi)^{-1},\ \int \rho = 1\right\}.$$

The 'bathtub minimizer' is

$$\rho_{\mathrm{bt}}(z) = \begin{cases} (\ell\pi)^{-1} & \text{if } V(z) \leq \lambda \\ 0 & \text{if } V(z) > \lambda \end{cases}$$

We have proved that the potential energy is, indeed, bounded below by this bathtub energy, with $(\ell\pi)^{-1}$ as bound on the dentisty, in the limit $N \to \infty$ if ϕ has the special form

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\ldots,N\}} f_2(z_i, z_j) \ldots \prod_{(i_1,\ldots,i_n)\in\{1,\ldots,N\}} f_n(z_{i_1}, \ldots, z_{i_n}).$$

with fixed n (or n not growing too fast with N).

Consider the special case

$$\prod_{j=1}^{N} f_1(z_j) \prod_{(i,j)\in\{1,\dots,N\}} f_2(z_i, z_j).$$

The proof has two parts:

- Comparison of the free energy with the energy defined by the functional (MF energy functional without entropy term)

$$\hat{\mathcal{E}}[\rho] = \int_{\mathbb{R}^2} \left(|z|^2 - \frac{2}{N} \log |g_1(z)| \right) \rho(z) dz + \int_{\mathbb{R}^4} \rho(z) \big(-\ell \log |z - z'| - \log |g_2(z, z')| \big) \rho(z') dz dz'$$

where $g_1(z) = f_1(\sqrt{N} z)$, $g_2(z, z') = f_2(\sqrt{N} z, \sqrt{N} z')$.

- Bound on the density of the minimizer of the MF functional.

Bound on the MF density

The variational equation for the modified MF functional is

$$|z|^2 - \frac{2}{N}\log|g_1(z)| - 2\int \log|g_2(z,z')|\rho(z')dz' - 2\ell\rho * \log|z| - C = 0$$

on the support of ρ.

$$1 - (1/2N)\Delta\log|g_1(z)| - \tfrac{1}{2}\int \Delta_z \log|g_2(z,z'|)\rho(z')dz' - \ell\pi\,\rho(z) = 0.$$

But $\Delta\log|g_1(z)| \geq 0$ and $\Delta_z\log|g_2(z,z'| \geq 0$, so

$$\rho(z) \leq \frac{1}{\ell\pi}.$$

Justification of the mean field approximation

The first step, however, i.e., the justification of the mean field approximation, is less simple.

Note that $-\log|g_2(z,z')|$ can be much more intricate than $-\log|z-z'|$ and is not of positive type in general so previous arguments for the mean field limit do not apply.

Instead we use a theorem of Diaconis and Freedman. This is a quantitative version of the Hewitt-Savage theorem.

The latter says essentially that the n-th marginals of a symmetric probability measure on a S^N can, for $N\to\infty$, be approximated by a convex combination of pure tensor products, $\rho^{\otimes n}$.

The application to the problem at hand requires some regularizations and is not entirely simple, but the result is the expected lower bound in terms of the bathtub energy with density $(\ell\pi)^{-1}$.

General correlation factors

The approach via the Diaconis Freedman Theorem and mean field limits does not work for functions of the general form

$$\Psi(z_1, \dots, z_N) = \phi(z_1, \dots, z_N) \prod_{i<j} (z_i - z_j)^\ell e^{-\sum_{j=1}^N |z_j|^2/2}$$

Using a different method we can, however, prove:

Theorem

For V smooth and tending to ∞ for $|z| \to \infty$

$$\int V(z) \mu^{(1)}(z) dz \geq E^{\mathrm{bt}}(V)(1 - o(1))$$

where

$$E^{\mathrm{bt}}(V) = \inf \left\{ \int V(z)\rho(z) dz \; : 0 \leq \rho \leq 4(\ell\pi)^{-1}, \; \int \rho = 1 \right\}.$$

Lieb's Lemma

A crucial ingredient is a lemma, that is a slight generalization of an unpublished remark of Elliott Lieb:

Lemma

For every minimizing configuration $(z_1^0, \dots, z_N^0)$ *of the Hamiltonian function*

$$\mathcal{H}_N(Z) = \sum_{j=1}^N |z_j|^2 - \frac{2\ell}{N} \sum_{i<j} \log|z_i - z_j| - \frac{2}{N} \log|\phi(\sqrt{N}Z)|$$

we have

$$\inf_{i \neq j} |z_i^0 - z_j^0| \geq (\ell/N)^{1/2}\,.$$

Corollary

Let $(z_1^0, \ldots, z_N^0)$ *be a minimizing configuration and*

$$\rho_0(z) = \frac{1}{N} \sum_{j=1}^{N} \delta(z - z_j^0)$$

the corresponding 1-particle density. There exists a $\tilde{\rho} \in L^\infty(\mathbb{R}^2)$ *with*

$$\int_{\mathbb{R}^2} \tilde{\rho} = 1, \qquad 0 \le \tilde{\rho}_0 \le \frac{4}{\ell\pi}(1 - N^{-1})$$

such that for any smooth V

$$\left| \int_{\mathbb{R}^2} (\rho_0 - \tilde{\rho}_0) V \right| \le C N^{-1/2}.$$

The proof of the lemma relies on the subharmonicity in each variable of the function

$$F(z_1, \ldots, z_N) = \frac{2\ell}{N} \sum_{i<j} \log |z_i - z_j| + \frac{2}{N} \log |\phi(\sqrt{N} Z)|,$$

i.e.,

$$\Delta_j F(z_1, \ldots, z_N) \geq 0$$

for all j.

The corollary is obtained by smearing the measure ρ_0 over a ball of radius $\frac{1}{2}(\ell/N)^{1/2}$.

The proof of the theorem is based on upper and lower bounds to the free energy corresponding to a modified Hamiltonian function

$$\mathcal{H}_N^{\varepsilon}(Z) = \sum_{j=1}^{N} \left(|z_j|^2 + \varepsilon V(z_i)\right) - F(z_1, \ldots z_N).$$

with $N^{-1} \ll \varepsilon \ll N^{-1/2}$. The subharmonicity of F is again essential.

For the estimate of the entropy term, positivity of the relative entropy of the minimizer of the free energy w.r.t. a trial density derived by smearing the minimizing configuration of the modified Hamiltonian is used.

Conclusions

Using a notion of incompressibility based on a lower bound to the potential energy in terms of a 'bath tub' energy, we have shown that this property holds for a large class of states derived from the Laughlin state(s).

Open problem:

Show that for general external potentials V the variational problem of minimizing the energy within the constrained class of functions $\mathrm{Ker}(\mathcal{I}_N)$ is, in the limit $N \to \infty$, solved by wave functions of the form

$$\prod_{j=1}^{N} f_1(z_j)\Psi^{(\ell)}_{\mathrm{Laugh}}(z_1, \ldots, z_N).$$

with $\ell = 2$ for bosons and $\ell = 1$ for fermions. For sufficiently strong Coulomb repulsion $\ell = 3$ should be favored for fermions.