Time-Reversal Symmetric Bloch bundles:

topology & differential geometry

Giuseppe De Nittis

(*FAU, Universität Erlangen-Nürnberg*)

Topological Phases of Quantum Matter

ESI, Wien, Austria • *Aug 04 - Sep 12, 2014*

Collaborator:

K. Gomi

Reference:

arXiv:1402.1284

arXiv:1404.5804

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

Alexander von Humboldt Stiftung/Foundation

K-theory $[P] \in K(\mathbb{B})$

$\|$

$\boxed{\text{Chern pairing [topology]}}$

$\Downarrow$

cohomology $ch(P) \in H^{\bullet}(\mathbb{B}, \mathbb{Z})$

$\|$

$\boxed{\text{Chern-Weil theory [differential geometry]}}$

$\Downarrow$

differential cohomology $\omega(P) \in H^{\bullet}_{\mathrm{dR}}(\mathbb{B})$

$\|$

$\boxed{\text{homology-cohomology (Kronecker) pairing [integration]}}$

$\Downarrow$

numerical evaluation $C(P) \in \mathbb{Z}^k$

Outline

Let $\mathbb{B}$ a topological space, ("Brillouin zone"). Assume that:

- $\mathbb{B}$ is compact, Hausdorff and path-connected;
- $\mathbb{B}$ admits a CW-complex structure.

Definition (Topological Quantum System (TQS))

Let $\mathscr{H}$ be a separable Hilbert space and $\mathcal{K}(\mathscr{H})$ the algebra of compact operators. A TQS (of class **A**) is a self-adjoint map

$$\mathbb{B} \ni k \longmapsto H(k) = H(k)^* \in \mathcal{K}(\mathscr{H})$$

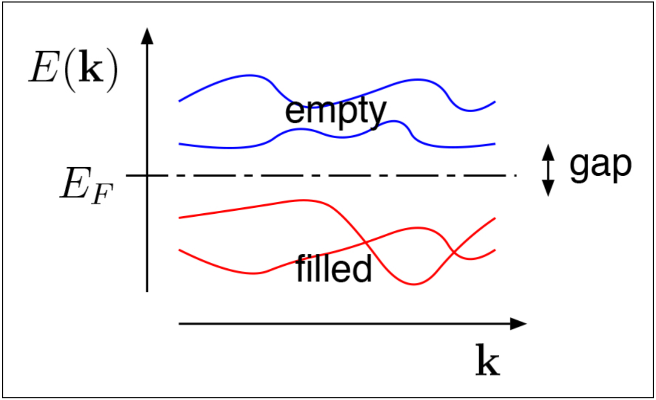
continuous with respect to the norm-topology.

☞ The spectrum $\sigma(H(k)) = \{E_j(k) \mid j \in \mathscr{I} \subseteq \mathbb{Z}\} \subset \mathbb{R}$, is a sequence of eigenvalues ordered according to

$$\ldots E_{-2}(k) \leqslant E_{-1}(k) < 0 \leqslant E_1(k) \leqslant E_2(k) \leqslant \ldots .$$

☞ The maps $k \mapsto E_j(k)$ are continuous by standard perturbative arguments and are called energy bands.

$$H(k)\,\psi_j(k) = E_j(k)\,\psi_j(k)\,, \qquad k \in \mathbb{B}$$



☞ Usually an energy gap separates the filled valence bands from the empty conduction bands. The Fermi level E_F characterizes the gap.

Gap condition and Fermi projection

- An isolated family of energy bands is any (finite) collection $\{E_{j_1}(\cdot),\ldots,E_{j_m}(\cdot)\}$ of energy bands such that

$$\min_{k\in\mathbb{B}} \operatorname{dist}\left(\bigcup_{s=1}^{m}\{E_{j_s}(k)\},\ \bigcup_{j\in\mathscr{I}\setminus\{j_1,\ldots,j_m\}}\{E_j(k)\}\right) = C_g > 0\,.$$

This is usually called “gap condition”.

- An isolated family is described by the “Fermi projection”

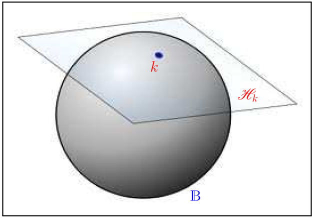
$$P_F(k) := \sum_{s=1}^{m} |\psi_{j_s}(k)\rangle\langle\psi_{j_s}(k)|\,.$$

This is a continuous projection-valued map

$$\mathbb{B} \ni k \longmapsto P_F(k) \in \mathcal{K}(\mathscr{H})\,.$$

- For each $k \in \mathbb{B}$
$$\mathscr{H}_k := \mathrm{Ran}\, P_F(k) \subset \mathscr{H}$$
is a subspace of $\mathscr{H}$ of fixed dimension m.

- The collection
$$\mathscr{E}_F := \bigsqcup_{k \in \mathbb{B}} \mathscr{H}_k$$
is a topological space (said total space) and the map
$$\pi : \mathscr{E}_F \longrightarrow \mathbb{B}$$
defined by $\pi(k, \mathrm{v}) = k$ is continuous (and open).



This is a complex vector bundle (of rank m) called "Bloch bundle".

- A morphism of rank m complex vector bundles over $\mathbb{B}$ is a map f

$$\begin{tikzcd} \mathscr{E}_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & \mathscr{E}_2 \arrow[dl, "\pi_2"] \\ & \mathbb{B} & \end{tikzcd}$$

such that $\pi_2 \circ f = \pi_1$.

- $\mathscr{E}_1 \simeq \mathscr{E}_2$ if there is an isomorphism $f : \mathscr{E}_1 \to \mathscr{E}_2$ (equivalence relation).

$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) :=$ set of equivalence classes of rank m vector bundles .

Definition (Topological phases)

Let $\mathbb{B} \ni k \longmapsto H(k)$ be a TQS with an isolated family of m energy bands and associated Bloch bundle $\mathscr{E}_F \longrightarrow \mathbb{B}$. The topological phase of the system is specified by

$$[\mathscr{E}_F] \in \mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) .$$

"Ordinary" quantum system:

TQS in a trivial phase

$\Updownarrow$

Trivial vector bundle, $\quad 0 \equiv [\mathbb{B} \times \mathbb{C}^m]$

$\Updownarrow$

$\boxed{\text{Exists a global frame of continuous Bloch functions}}$

Allowed (adiabatic) deformations:

Transformations which doesn't alter the nature of the system

$\Updownarrow$

Vector bundle isomorphism

$\Updownarrow$

$\boxed{\text{Stability of the topological phase}}$

For a TQS in class **A** the classification of topological phases agrees with the determination of $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B})$.

☞ Homotopy classification theorem:

$$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) \simeq [\mathbb{B}, \mathrm{Gr}_m(\mathbb{C}^N)] \qquad (N \gg 1)$$

where

$$\mathrm{Gr}_m(\mathbb{C}^N) := \mathbb{U}(N) \,/\, \big(\mathbb{U}(m) \times \mathbb{U}(N-m)\big)$$

is the Grasmannian of m-planes in $\mathbb{C}^N$.

☞ The computation of $[\mathbb{B}, \mathrm{Gr}_m(\mathbb{C}^N)]$ is a difficult task (non algorithmic **!!**).

☞ One introduces (Chern) characteristic classes which are the pullback of the cohomology ring of the classifying space $\mathrm{Gr}_m(\mathbb{C}^N)$ under the classifying homotopy (algorithmic **!!**).

☞ One uses the Chern-Weil theory to represent characteristic classes as de Rham forms builded in terms of connections and curvatures.

Theorem (Peterson, 1959)

If $\dim(\mathbb{B}) \leqslant 4$ *then*

$$\begin{aligned} \mathrm{Vec}^1_{\mathbb{C}}(\mathbb{B}) &\simeq H^2(\mathbb{B},\mathbb{Z}) \\ \mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) &\simeq H^2(\mathbb{B},\mathbb{Z}) \oplus H^4(\mathbb{B},\mathbb{Z}) \qquad (m \geqslant 2) \end{aligned}$$

and the isomorphism

$$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) \ni [\mathscr{E}] \longmapsto (c_1, c_2) \in H^2(\mathbb{B},\mathbb{Z}) \oplus H^4(\mathbb{B},\mathbb{Z})$$

is given by the first two Chern classes (notice $c_2 = 0$ *if* $m = 1$*).*

- The computation of the groups $H^k(\mathbb{B},\mathbb{Z})$ is algorithmic.
- In many cases the condition $\dim(\mathbb{B}) \leqslant 4$ covers all the physical interesting situations.
- When $\dim(\mathbb{B}) \geqslant 5$ (and the torsion is absent) Chern classes classify only vector bundles of stable rank $m \geqslant \frac{1}{2}\dim(\mathbb{B})$.

Classification: differential geometry

Let assume now that $\mathbb{B}$ has a manifold structure.

- As a consequence of the de Rham theorem

$$H^k(\mathbb{B},\mathbb{Z}) \hookrightarrow H^k(\mathbb{B},\mathbb{R}) \simeq H^k_{dR}(\mathbb{B})$$

one can interpret Chern classes as (classes of) differential forms.

- The Chern-Weil theory allows to construct the differential expression of Chern classes starting from the curvature 2-form Ω by means of the invariant polynomial

$$\det\left(\frac{\mathrm{i}\,s}{2\pi}\Omega + \mathbb{1}_m\right) = \sum_j s^j\, c_j\,.$$

- The curvature 2-form Ω can be computed from a connection 1-form $\omega := \{\mathscr{A}_\alpha\}$ by means of the structure equation

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega;\omega]\,.$$

The natural candidate is the Berry-(Bott-Chern) connection given by

$$\mathscr{A}^{i,j}_\alpha(k) := \mathrm{i}\,\langle\psi_i(k)|\nabla_k|\psi_j(k)\rangle\cdot\mathrm{d}k\,,\qquad k\in\mathfrak{U}_\alpha\,.$$

Definition (Involution on the Brillouin zone)

A homeomorphism $\tau : \mathbb{B} \to \mathbb{B}$ is called involution if $\tau^2 = \mathrm{Id}_{\mathbb{B}}$. The pair $(\mathbb{B}, \tau)$ is called an involutive space.

☞ Each space $\mathbb{B}$ admits the trivial involution $\tau_{\mathrm{triv}} := \mathrm{Id}_{\mathbb{B}}$.

Definition (TQS with time-reversal symmetry)

Let $(\mathbb{B}, \tau)$ be an involutive space, $\mathscr{H}$ a separable Hilbert space endowed with a complex conjugation C. A TQS $\mathbb{B} \ni k \mapsto H(k)$ has a time-reversal symmetry (TRS) of parity $\varepsilon \in \{\pm\}$ if there is a continuous unitary-valued map $k \mapsto U(k)$ such that

$$U(k)\,H(k)\,U(k)^* = C\,H(\tau(k))\,C\,, \qquad C\,U(\tau(k))\,C = \varepsilon U(k)^*\,.$$

A TQS of class **AI** (resp. **AII**) is endowed with an even (resp. odd) time-reversal symmetry.

The equivalence relation for the Fermi projection

$$U(k)\, P_F(k)\, U(k)^* \;=\; C\, P_F(\tau(k))\, C \qquad \forall\, k \in \mathbb{B}$$

induces an additional structure on the Bloch-bundle $\mathscr{E}_F \to \mathbb{B}$.

Definition (Atiyah, 1966 - Dupont, 1969)

Let $(\mathbb{B}, \tau)$ be an involutive space and $\mathscr{E} \to \mathbb{B}$ a complex vector bundle. Let $\Theta : \mathscr{E} \to \mathscr{E}$ an homeomorphism such that

$$\Theta \;:\; \mathscr{E}|_k \longrightarrow \mathscr{E}|_{\tau(k)} \qquad \text{is anti-linear} .$$

- The pair $(\mathscr{E}, \Theta)$ is a “Real”-bundle over $(\mathbb{B}, \tau)$ if

$$\Theta^2 \;:\; \mathscr{E}|_k \xrightarrow{+1} \mathscr{E}|_k \qquad \forall\, k \in \mathbb{B}\,;$$

- The pair $(\mathscr{E}, \Theta)$ is a “Quaternionic”-bundle over $(\mathbb{B}, \tau)$ if

$$\Theta^2 \;:\; \mathscr{E}|_k \xrightarrow{-1} \mathscr{E}|_k \qquad \forall\, k \in \mathbb{B}\,.$$

- A morphism of rank m “Real” or “Quaternionic” bundles over $(\mathbb{B},\tau)$ is a map f

$$\begin{tikzcd}
(\mathscr{E}_1,\Theta_1) \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & (\mathscr{E}_2,\Theta_2) \arrow[dl, "\pi_2"] \\
& (\mathbb{B},\tau) &
\end{tikzcd}$$

such that $\pi_2 \circ f = \pi_1, \quad \Theta_2 \circ f = f \circ \Theta_1.$

- $(\mathscr{E}_1,\Theta_1) \simeq (\mathscr{E}_2,\Theta_2)$ if there is an isomorphism f (equivalence relation)

$$\begin{aligned}
\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B},\tau) &:= \text{set of equivalence classes of rank } m \text{ “Real” bundles} \\
\mathrm{Vec}^m_{\mathscr{Q}}(\mathbb{B},\tau) &:= \text{set of equivalence classes of rank } m \text{ “Quaternionic” bundles .}
\end{aligned}$$

☞ These names are justified by the following isomorphisms:

$$\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B},\mathrm{Id}_{\mathbb{B}}) \simeq \mathrm{Vec}^m_{\mathbb{R}}(\mathbb{B})$$

$$\mathrm{Vec}^{2m}_{\mathscr{Q}}(\mathbb{B},\mathrm{Id}_{\mathbb{B}}) \simeq \mathrm{Vec}^m_{\mathbb{H}}(\mathbb{B}) .$$

Definition (Topological phases in presence of TRS)

Let $(\mathbb{B}, \tau)$ be an involutive space and $\mathbb{B} \ni k \longmapsto H(k)$ a TQS with a TRS of parity ε and an isolated family of m energy bands. Let $\mathscr{E}_F \longrightarrow \mathbb{B}$ be the associated Bloch bundle with equivariant structure Θ. The topological phase of the system is specified by

$$[(\mathscr{E}_F, \Theta)] \in \mathrm{Vec}^m_{\mathscr{T}_\varepsilon}(\mathbb{B}) , \qquad \mathscr{T}_+ \equiv \mathscr{R} , \ \mathscr{T}_- \equiv \mathscr{Q} .$$

CAZ	TRS	Category	VB
A	0	complex	$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B})$
AI	$+$	"Real"	$\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B}, \tau)$
AII	$-$	"Quaternionic"	$\mathrm{Vec}^m_{\mathscr{Q}}(\mathbb{B}, \tau)$

Problem :

1) Classification via topology;
2) Classification via differential geometry.

Relevant physical situations

- Periodic quantum systems (*e.g.* absence of disorder):
  - $\mathbb{R}^d$-translations $\Rightarrow$ free (Dirac) fermions;
  - $\mathbb{Z}^d$-translations $\Rightarrow$ crystal (Bloch) fermions.

- The Bloch-Floquet (or Fourier) theory exploits the invariance under translations of a periodic structure to describe the state of the system in terms of the quasi-momentum k on the Brillouin zone $\mathbb{B}$.

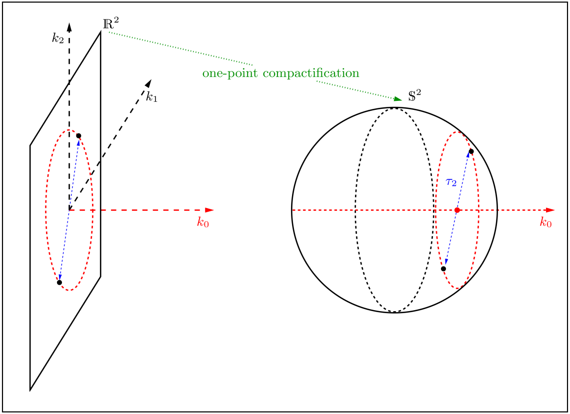
- Complex conjugation (TRS) endows $\mathbb{B}$ with an involution τ.

- Examples are:
  - Gapped electronic systems,
  - BdG superconductors,
  - Photonic crystals (M. Lein talk).

Continuous case $\mathbb{B} = \mathbb{S}^d$

$$\begin{array}{ccc} \mathbb{S}^d & \xrightarrow{\quad\quad\tau_d\quad\quad} & \mathbb{S}^d \\ (+k_0, +k_1, \ldots, +k_d) & \overset{\tau_d}{\longmapsto} & (+k_0, -k_1, \ldots, -k_d) \end{array}$$

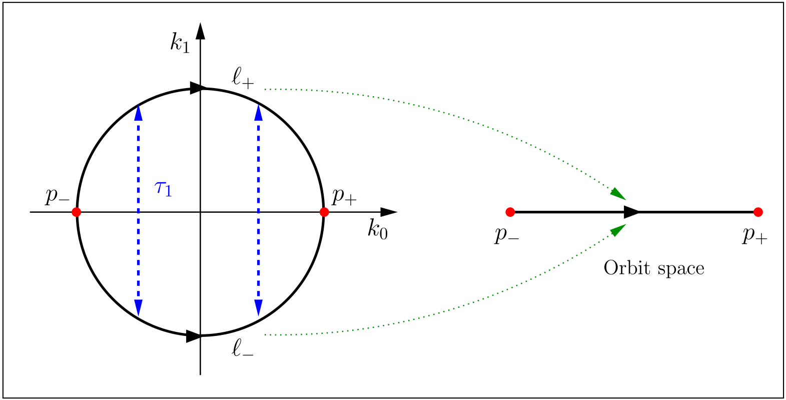
$$\tilde{\mathbb{S}}^d := (\mathbb{S}^d, \tau_d)$$



Periodic case $\mathbb{B} = \mathbb{T}^d$

$$\mathbb{T}^d = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1 \xrightarrow{\tau_d := \tau_1 \times \ldots \times \tau_1} \mathbb{T}^d = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$$

$$\tilde{\mathbb{T}}^d := (\mathbb{T}^d, \tau_d)$$



For a TQS in class **AI** the classification of topological phases agrees with the determination of $\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B},\tau)$.

- ☞ Homotopy classification theorem [Edelson, 1971]:

$$\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B},\tau) \simeq [(\mathbb{B},\tau),(\mathrm{Gr}_m(\mathbb{C}^N),\vartheta_R)]_{\mathbb{Z}_2} \qquad (N \gg 1)$$

  $\mathbb{Z}_2$-homotopy classes of equivariant maps $f : \mathbb{B} \to \mathrm{Gr}_m(\mathbb{C}^N)$ such that $f \circ \tau = \vartheta_R \circ f$. The involution ϑ_R acts by the complex conjugation, *i.e.*

$$\vartheta_R \; : \; \mathrm{span}(\mathrm{v}_1,\ldots,\mathrm{v}_m) \longmapsto \mathrm{span}(\overline{\mathrm{v}_1},\ldots,\overline{\mathrm{v}_m}) \, .$$

- ☞ The computation of $[(\mathbb{B},\tau),(\mathrm{Gr}_m(\mathbb{C}^N),\vartheta_R)]_{\mathbb{Z}_2}$ is a difficult task (non algorithmic **!!**).

- ☞ One introduces “Real” Chern classes [Kahn, 1987] which are the pullback of the equivariant Borel cohomology ring of the involutive classifying space $(\mathrm{Gr}_m(\mathbb{C}^N),\vartheta_R)$ under the equivariant classifying homotopy (algorithmic **!!**).

- ☞ One extends the Chern-Weil theory to the “Real” category in order to represent “Real” Chern classes as equivariant de Rham forms constructed in terms of equivariant connections and curvatures.

The Borel's construction

- $(\mathbb{B},\tau)$ any involutive space and $(\mathbb{S}^\infty,\theta)$ the infinite sphere (contractible space) with the antipodal (free) involution:

$$\mathbb{B}_{\sim\tau} := \frac{\mathbb{S}^\infty \times \mathbb{B}}{\theta \times \tau} \qquad \text{(homotopy quotient)}\,.$$

- $\mathscr{Z}$ any abelian ring (module, system of coefficients, ...)

$$H^j_{\mathbb{Z}_2}(\mathbb{B},\mathscr{Z}) := H^j(\mathbb{B}_{\sim\tau},\mathscr{Z}) \qquad \text{(eq. cohomology groups)}\,.$$

- $\mathbb{Z}(m)$ is the $\mathbb{Z}_2$-local system on $\mathbb{B}$ based on the module $\mathbb{Z}$

$$\mathbb{Z}(m) \simeq \mathbb{B}\times\mathbb{Z} \qquad \text{endowed with} \qquad (k,\ell) \mapsto (\tau(k),(-1)^m\ell)\,.$$

More precisely: $\mathbb{Z}(m)$ is the module with underlying group (isomorphic to) $\mathbb{Z}$ and scalars in the group ring $\mathbb{Z}[\pi_1(\mathbb{B}_{\sim\tau})]$ induced by the (two) possible representations $\rho_m : \pi_1(\mathbb{B}_{\sim\tau}) \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}) \simeq \mathbb{Z}_2$.

Outline

Theorem (D. & Gomi, 2014)

Let $(\mathbb{B},\tau)$ such that $d := \dim(\mathbb{B}) \leqslant 3$ and the fixed point set $\mathbb{B}^\tau$ has only isolated points. There is an isomorphism

$$\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B},\tau) \simeq H^2_{\mathbb{Z}_2}(\mathbb{B},\mathbb{Z}(1)) \qquad \textit{if} \quad \dim(\mathbb{B}) \leqslant 3$$

given by the "Real" (first) Chern class $[\mathscr{E}] \mapsto \tilde{c}(\mathscr{E})$.

Proof.

- "Real" stable-range condition

$$\mathrm{Vec}^m_{\mathscr{R}}(\mathbb{B},\tau) \simeq \mathrm{Vec}^{[d/2]}_{\mathscr{R}}(\mathbb{B},\tau)\,, \qquad [\cdot] = \text{integer part}\,.$$

- "Real" line bundles description [Kahn, 1987]

$$\mathrm{Vec}^1_{\mathscr{R}}(\mathbb{B},\tau) =: \mathrm{Pic}(\mathbb{B},\tau) \simeq H^2_{\mathbb{Z}_2}(\mathbb{B},\mathbb{Z}(1))\,. \qquad \blacksquare$$

☞ **Corollary:** $\tilde{c}(\mathscr{E}) = 0 \Leftrightarrow$ exists a global frame of continuous Bloch functions such that $\psi(\tau(k)) = (\Theta\psi)(k)$ ("Real" frame).

Theorem (D. & Gomi, 2014)

$$H^2_{\mathbb{Z}_2}(\tilde{\mathbb{S}}^d, \mathbb{Z}(1)) = H^2_{\mathbb{Z}_2}(\tilde{\mathbb{T}}^d, \mathbb{Z}(1)) = 0 \qquad \forall\, d \in \mathbb{N}\ .$$

The proof requires an equivariant generalization of the Gysin sequence and the suspension periodicity.

VB	CAZ	$d=1$	$d=2$	$d=3$	$d=4$	
$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{S}^d)$	**A**	0	$\mathbb{Z}$	0	$0 \quad (m=1)$ $\mathbb{Z} \quad (m \geqslant 2)$	Free
$\mathrm{Vec}^m_{\mathscr{R}}(\tilde{\mathbb{S}}^d)$	**AI**	0	0	0	$0 \quad (m=1)$ $2\mathbb{Z} \quad (m \geqslant 2)$	systems
$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^d)$	**A**	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6 \quad (m=1)$ $\mathbb{Z}^7 \quad (m \geqslant 2)$	Periodic
$\mathrm{Vec}^m_{\mathscr{R}}(\tilde{\mathbb{T}}^d)$	**AI**	0	0	0	$0 \quad (m=1)$ $2\mathbb{Z} \quad (m \geqslant 2)$	systems

The case $d = 4$ is interesting for the magneto-electric response (space-time variables)

Theorem (D. & Gomi, 2014)

Class AI topological insulators in $d = 4$ are completely classified by the 2-nd "Real" Chern class. These classes are representable as even integers and the isomorphisms

$$\mathrm{Vec}^m_{\mathscr{R}}(\tilde{\mathbb{T}}^4) \simeq 2\mathbb{Z}\,, \qquad \mathrm{Vec}^m_{\mathscr{R}}(\tilde{\mathbb{S}}^4) \simeq 2\mathbb{Z}$$

are given by the (usual) 2-nd Chern number.

☞ **Remark:** In $d = 4$ to have an even 2-nd Chern number is a necessary condition for a complex vector bundle to admit a "Real"-structure **!!**

Outline

Just a summary of the main results

- On can introduce consistently the notions of "Real" principal bundle and "Real" connection (*i.e.* $\tau^*\omega = \overline{\omega}$).

- Let $(\mathbb{B}, \tau)$ be a smooth manifold with a smooth involution. The de Rham cohomology groups $H^k_{\mathrm{dR}}(\mathbb{B})$ split as

$$H^k_{\mathrm{dR}}(\mathbb{B}) \;=\; H^k_{\mathrm{dR},+}(\mathbb{B}, \tau) \,\oplus\, H^k_{\mathrm{dR},-}(\mathbb{B}, \tau)$$

where $\phi \in H^k_{\mathrm{dR},\pm}(\mathbb{B}, \tau)$ if and only if $\tau^*\phi = \pm\phi$.

- One has the following isomorphisms

$$H^k_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{R}(0)) \;\simeq\; H^k_{\mathrm{dR},+}(\mathbb{B}, \tau)\,, \qquad H^k_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{R}(1)) \;\simeq\; H^k_{\mathrm{dR},-}(\mathbb{B}, \tau)\,.$$

- There is a map

$$H^k_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(j)) \;\longrightarrow\; H^k_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{R}(j)) \;\simeq\; H^k_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(j)) \otimes_{\mathbb{Z}} \mathbb{R}$$

induces by the homomorphism $\mathbb{Z}(j) \to \mathbb{R}(j)$. The torsion is lost !!

☞ Let ω a "Real" connection and Ω the related curvature. The differential forms c_j obtained from the invariant polynomial

$$\det\left(\frac{\mathrm{i}\,s}{2\pi}\Omega + \mathbb{1}_m\right) = \sum_j s^j\, c_j\ .$$

verify $\tau^* c_j = (-1)^j c_j$, hence

$$c_j \in H^{2j}_{\mathrm{dR},(-)^{\mathrm{j}}}(\mathbb{B},\tau)\ .$$

Theorem (D. & Gomi, 2014)

The de Rham cohomology class c_j agrees with the image of the "Real" Chern class $\tilde{c}_j \in H^{2j}_{\mathbb{Z}_2}(\mathbb{B},\mathbb{Z}(j))$ unde the homomorphism

$$H^k_{\mathbb{Z}_2}(\mathbb{B},\mathbb{Z}(j)) \longrightarrow H^k_{\mathbb{Z}_2}(\mathbb{B},\mathbb{R}(j)) \simeq H^{2j}_{\mathrm{dR},(-)^{\mathrm{j}}}(\mathbb{B},\tau)\ .$$

The torsion part can be reconstructed by the "Real" Holonomy.

☞ **Conclusion:** We can compute invariants for the "Real" category by using usual differential geometric formulas applied to "Real" connections.**!!**

For a TQS in class **AII** the classification of topological phases agrees with the determination of $\mathrm{Vec}^m_{\mathscr{Q}}(\mathbb{B},\tau)$.

- If the fixed point set $\mathbb{B}^\tau \neq \emptyset$ the rank of a $\mathscr{Q}$-bundle has to be even (Kramers degeneracy).

- Homotopy classification theorem (adaption of [Edelson, 1971]):
$$\mathrm{Vec}^{2m}_{\mathscr{Q}}(\mathbb{B},\tau) \simeq [(\mathbb{B},\tau),(\mathrm{Gr}_{2m}(\mathbb{C}^{2N}),\vartheta_Q)]_{\mathbb{Z}_2} \qquad (N \gg 1)$$
$\mathbb{Z}_2$-homotopy classes of equivariant maps $f : \mathbb{B} \to \mathrm{Gr}_m(\mathbb{C}^{2N})$ such that $f \circ \tau = \vartheta_Q \circ f$. The involution ϑ_Q acts as
$$\vartheta_Q : \mathrm{span}(\mathrm{v}_1,\ldots,\mathrm{v}_{2m}) \longmapsto \mathrm{span}(Q\cdot\overline{\mathrm{v}_1},\ldots,Q\cdot\overline{\mathrm{v}_{2m}})$$
where $Q := \mathbb{1}_N \otimes \mathrm{i}\sigma_2$ verifies $Q^2 := -\mathbb{1}_{2N}$.

- The computation of $[(\mathbb{B},\tau),(\mathrm{Gr}_{2m}(\mathbb{C}^{2N}),\vartheta_Q)]_{\mathbb{Z}_2}$ is a difficult task (non algorithmic **!!**).

- One introduces the FKMM-invariant [Furuta, Kametani, Matsue, Minami, 2000] which can be interpreted as the pullback of a universal class for $(\mathrm{Gr}_{2m}(\mathbb{C}^{2N}),\vartheta_Q)$ under the equivariant classifying homotopy (algorithmic **!!**).

- One interprets the FKMM-invariant in terms of integration of differential forms (Wess-Zumino in $d=2$, Chern-Simons in $d=3$, ...)

Construction and interpretation of the FKMM-invariant

- Let $(\mathbb{B}, \tau)$ be an involutive space and $\mathbb{Y} \subset \mathbb{B}$ τ-invariant. The relative equivariant cohomology groups

$$H^k_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{Y}, \mathbb{Z}(m))$$

can be consistently defined.

- One has the geometric interpretations

$$H^1_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)) \simeq [(\mathbb{B}, \tau), (\mathbb{U}(1), c)]_{\mathbb{Z}_2} \qquad \text{[Gomi, 2013]}$$
$$H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)) \simeq \mathrm{Vec}^1_{\mathscr{R}}(\mathbb{B}, \tau) \qquad \text{[Kahn, 1987]}$$

where c is the complex conjugation.

- We have also the geometric interpretation [D. & Gomi, 2013]

$$H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{Y}, \mathbb{Z}(1)) \simeq \mathrm{Vec}^1_{\mathscr{R}}(\mathbb{B}|\mathbb{Y}, \tau) =: \mathrm{Pic}(\mathbb{B}|\mathbb{Y}, \tau)$$

where $\mathrm{Vec}^1_{\mathscr{R}}(\mathbb{B}|\mathbb{Y}, \tau)$ is the (Picard) group of isomorphism classes of pairs (L, s) where $L \to \mathbb{B}$ is a “Real” line bundle and $s : \mathbb{Y} \to L|_{\mathbb{Y}}$ is a nowhere vanishing, equivariant section $s \circ \tau = \Theta \circ s$.

Theorem (D. & Gomi, 2014)

Each $[(\mathscr{E},\Theta)] \in \mathrm{Vec}^{2m}_{\mathscr{Q}}(\mathbb{B},\tau)$ *defines a unique element in* $\mathrm{Vec}^{1}_{\mathscr{R}}(\mathbb{B}|\mathbb{B}^{\tau},\tau)$ *and the map*

$$\mathrm{Vec}^{2m}_{\mathscr{R}}(\mathbb{B},\tau) \ni [(\mathscr{E},\Theta)] \xrightarrow{\kappa} \kappa[\mathscr{E}] \in H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^{\tau},\mathbb{Z}(1))$$

(defined by the above isomorphism) is the FKMM-invariant.

Proof.

- Determinant construction: $L := \det(\mathscr{E})$ is a line bundle over $\mathbb{B}$ and Θ induces a "Real"-structure $\det\Theta : L \to L$.
- Because $\mathbb{B}^{\tau}$ is made by fixed points $L|_{\mathbb{B}^{\tau}}$ has a unique canonical trivialization $h : L|_{\mathbb{B}^{\tau}} \to \mathbb{B}^{\tau} \times \mathbb{C}$ which define a canonical (nowhere vanishing, equivariant) section $s(k) := h^{-1}(k,1)$.
- $[(\mathscr{E},\Theta)]$ defines $[(L,s)] \in \mathrm{Vec}^{1}_{\mathscr{R}}(\mathbb{B}|\mathbb{B}^{\tau},\tau)$. ∎

☞ $\mathbb{B}$ is a FKMM-involutive-space if:

- $(\mathbb{B}, \tau)$ Hausdorff, path-connected, $\ldots$;
- $\mathbb{B}^{\tau} = \{k_1, \ldots, k_N\}$;
- $H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)) = 0$.

Theorem (D. & Gomi, 2014)

Let $(\mathbb{B}, \tau)$ be a FKMM-involutive-space:

$$\kappa \; : \; \mathrm{Vec}^{2m}_{\mathscr{Q}}(\mathbb{B}, \tau) \;\to\; H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^{\tau}, \mathbb{Z}(1)) \qquad \text{if} \quad \dim(\mathbb{B}) \leqslant 3$$

κ, called FKMM-invariant, is an injective map .

☞ **Corollary:** $\kappa(\mathscr{E}) = 0 \Leftrightarrow$ exists a global frame of continuous Bloch functions such that $\psi(\tau(k)) = (\Theta\psi)(k)$ ("Quaternionic" frame).

Theorem (D. & Gomi, 2014)

For $\mathbb{B} = \tilde{\mathbb{S}}^d$ or $\mathbb{B} = \tilde{\mathbb{T}}^d$ with $d \leqslant 3$ the FKMM-invariant κ is an isomorphism.

VB	CAZ	$d=1$	$d=2$	$d=3$	$d=4$	
$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{S}^d)$	**A**	0	$\mathbb{Z}$	0	0 $(m=1)$ $\mathbb{Z}$ $(m \geqslant 2)$	Free
$\mathrm{Vec}^{2m}_{\mathscr{Q}}(\tilde{\mathbb{S}}^d)$	**AII**	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	systems
$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{T}^d)$	**A**	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$ $(m=1)$ $\mathbb{Z}^7$ $(m \geqslant 2)$	Periodic
$\mathrm{Vec}^{2m}_{\mathscr{Q}}(\tilde{\mathbb{T}}^d)$	**AII**	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}$	systems

☞ **Remark:** In these two special cases the FKMM-invariant κ can be written (and so agrees with) as the Fu-Kane-Mele invariant.

In the case $\dim(\mathbb{B}) = 4$ we consider the pair of maps

$$(\kappa, c_2) \;:\; \mathrm{Vec}^{2m}_{\mathscr{Q}}(\mathbb{B}, \tau) \longrightarrow H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)) \oplus H^4(\mathbb{B}, \mathbb{Z})$$

where c_2 is the second Chern class.

Theorem (D. & Gomi, 2014)

Class AII topological insulators in $d = 4$ are classified by the 2-nd Chern class c_2 and the FKMM-invariant κ. The maps

$$(\kappa, c_2) \;:\; \mathrm{Vec}^{2m}_{\mathscr{Q}}(\tilde{\mathbb{S}}^4) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}$$
$$(\kappa, c_2) \;:\; \mathrm{Vec}^{2m}_{\mathscr{Q}}(\tilde{\mathbb{T}}^4) \longrightarrow \mathbb{Z}_2^{11} \oplus \mathbb{Z}$$

are injective. $\mathrm{Ran}(\kappa, c_2)$ *is fixed by the constraint*

$$(-1)^{c_2(\mathscr{E})} = \prod_{j=1}^{N} \kappa(\mathscr{E})_j \,, \qquad \mathscr{E} \text{ is a } \mathscr{Q}-\text{vector bundle}$$

and $N = 1$ for $\tilde{\mathbb{S}}^4$ or $N = 11$ for $\tilde{\mathbb{T}}^4$.

Just a summary of the main results

- On can introduce consistently the notions of "Quaternionic" principal bundle and "Quaternionic" connection, (*i.e.* $\tau^*\omega = -Q\cdot\overline{\omega}\cdot Q$).

- Let $(\mathbb{B},\tau)$ be a smooth manifold with a smooth involution and a finite number of fixed points. Let ω a "Quaternionic" connection associated with the "Quaternionic" vector bundle $(\mathscr{E},\Theta)$.

- Let $\dim(\mathbb{B}) = 3$ and consider the Chern-Simons form

$$CS(\mathscr{E}) := \frac{1}{8\pi^2}\int_{\mathbb{B}} \mathrm{Tr}\left(\omega\wedge \mathrm{d}\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\right)$$

then

$$\kappa_{\text{strong}}(\mathscr{E}) = \mathrm{e}^{\mathrm{i}2\pi\, CS(\mathscr{E})}$$

is a topological invariant of $(\mathscr{E},\Theta)$ which provides the strong part of the FKMM-invariant (already known for $\mathbb{B}=\mathbb{T}^3$!!).

If $\dim(\mathbb{B}) = 2$ then

$$\kappa_{\text{strong}}(\mathscr{E}) = \mathrm{e}^{\mathrm{i}2\pi\, WS(\mathscr{E})}$$

where $WS(\mathscr{E})$ is the Wess-Zumino term computed with respect to any $3d$-$\tilde{\mathbb{B}}$ such that $\partial\tilde{\mathbb{B}} = \mathbb{B}$ (already known for $\mathbb{B}=\mathbb{T}^2$!!).

Thank you for your attention