Quantum Simulation and Many-Body Physics
with Light
June 4-11, 2016
Chania, Crete

Keldysh Field Theory
for
Driven Open Quantum Systems,
and some applications

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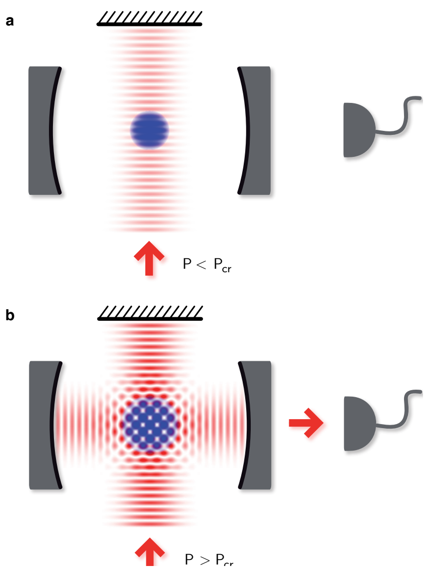
based on review:

L. Sieberer, M. Buchhold, SD,
Keldysh Field Theory for Driven Open Quantum Systems,
arxiv (2015), to appear in Reports on Progress in Physics

€ € € :

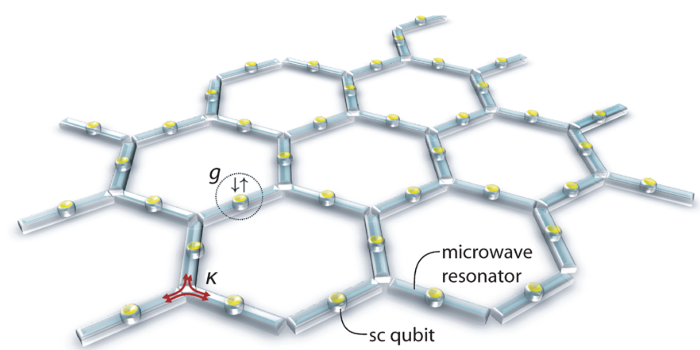
Motivation: Driven open many-body dynamics

- experimental systems on the interface of quantum optics and many-body physics
  - driven-open Dicke models



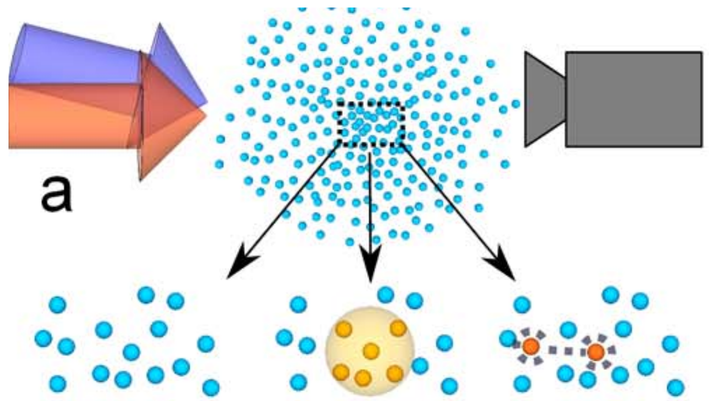
Baumann et al., Nature 2010
Ritsch et al., RMP 2013

  - coupled microcavity arrays



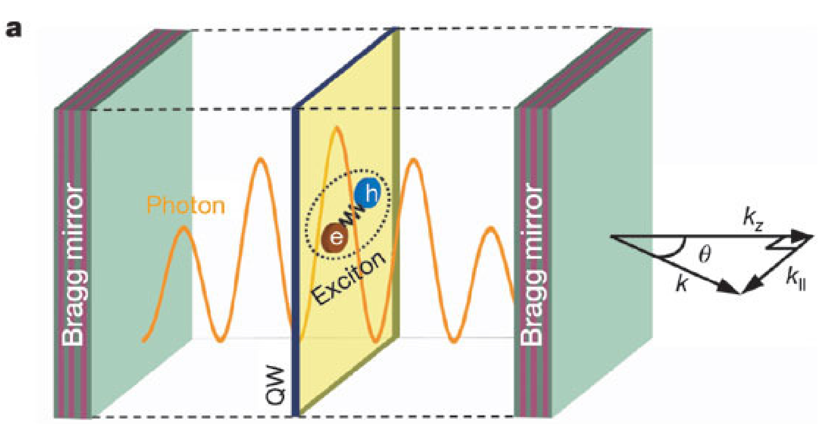
Koch et al., PRA 2010
Houck, Türeci, Koch, Nat. Phys. 2012

  - driven-dissipative Rydberg systems



Carr et al. PRL 2013
Malossi et al. PRL 2014

  - exciton-polariton systems in semiconductor quantum wells



Kasprzak et al., Nature 2006
Carusotto, Ciuti RMP 2013

- other platforms (light-matter):
  - ➡ polar molecules — Zhu et al. PRL 2013
  - ➡ photon BECs — Klaers et al. Nature 2010
  - ➡ trapped ions — Kim et al., Nature 2010; Islam et al., Nature 2011; Barreiro et al. Nature 2011; Britton et al. Nature 2012

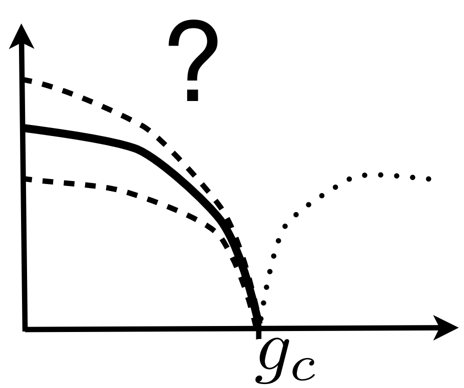
Non-Equilibrium Physics with Driven Open Quantum Systems (DOQS)

- Interdisciplinary research area: physics at various length scales

Quantum Optics	Many-body physics	Statistical mechanics
coherent and driven-dissipative dynamics on equal footing	continuum of spatial degrees of freedom	
Microscopic	"Thermodynamic"	Long wavelength

- Questions and Challenges:

Novel universal phenomena ?

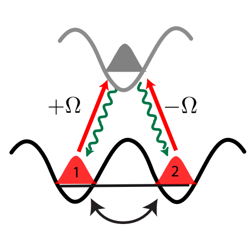


Efficient theoretical tools ?

$$Z[J] = \int \mathcal{D}\varphi \, e^{i(S[\varphi] + \int J\varphi)}$$

perform the transition form micro-to macrophysics:
quantum field theory out of equilibrium

Experimental platforms ?



cold atoms, light-driven semiconductors, microcavity arrays, trapped ions ...

Outline

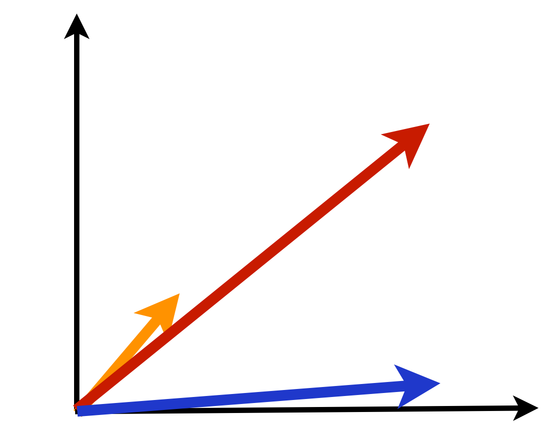
L. Sieberer, M. Buchhold, SD,
Keldysh Field Theory for Driven Open Quantum Systems,
arxiv (2015), to appear in Reports on Progress in Physics

Part I: Theoretical background

- From the quantum master equation to the Keldysh functional integral
    - construction
    - semiclassical limit, connection to exciton-polariton systems
    - “what is non-equilibrium about it?”

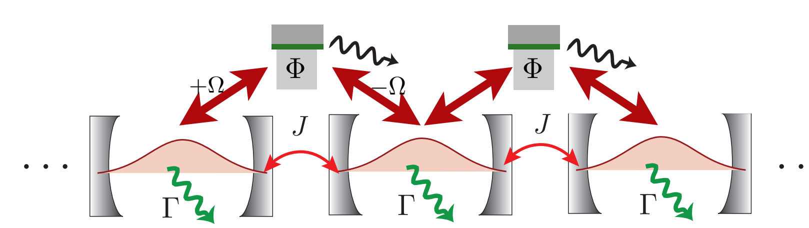
$$\partial_t \rho = -i[H,\rho] + \mathcal{L}[\rho]$$

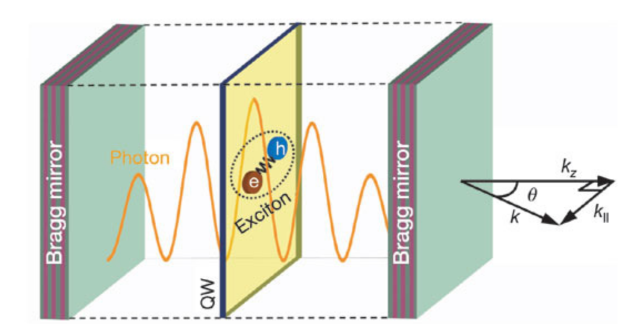
$$e^{i\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi e^{iS_M[\Phi+\delta\Phi]}$$



Part II: Applications

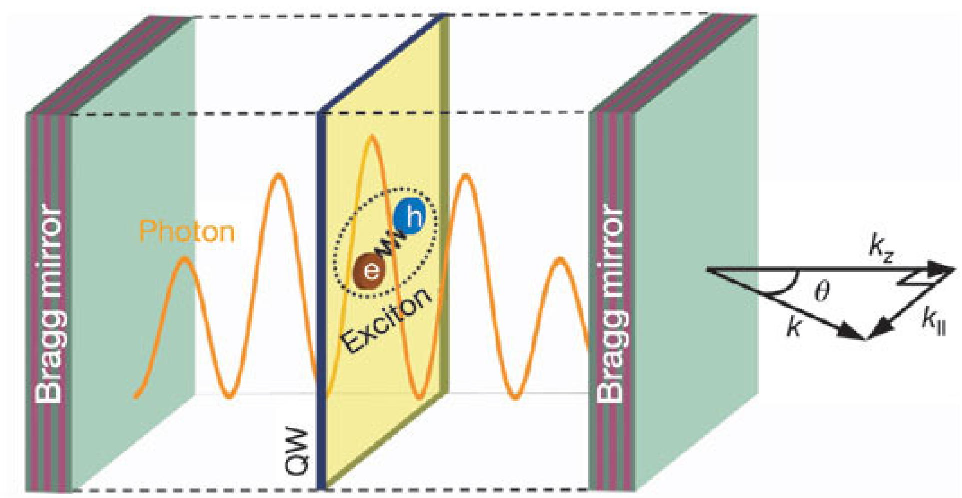
- Critical behavior in driven open quantum systems
    - classical
    - quantum
- Universal long wavelength behavior in low dimension
    - physics of and mapping to KPZ equation
    - non-linear Goldstone mode vs. vortex unbinding (2D & 1D)



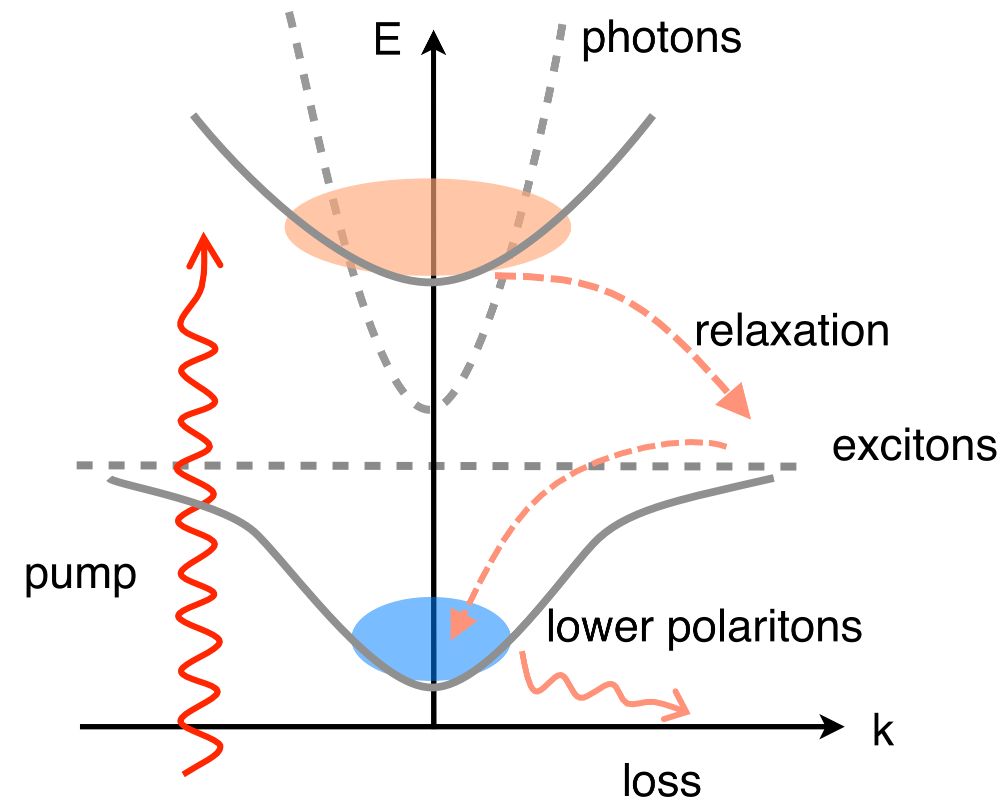


An Example: Exciton-Polariton Systems

Kasprzak et al., Nature 2006



Imamoglu et al., PRA 1996



- phenomenological description: stochastic driven-dissipative Gross-Pitaevskii-Eq

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2 \right] \phi + \zeta$$

propagation

pump & loss rates

elastic collisions

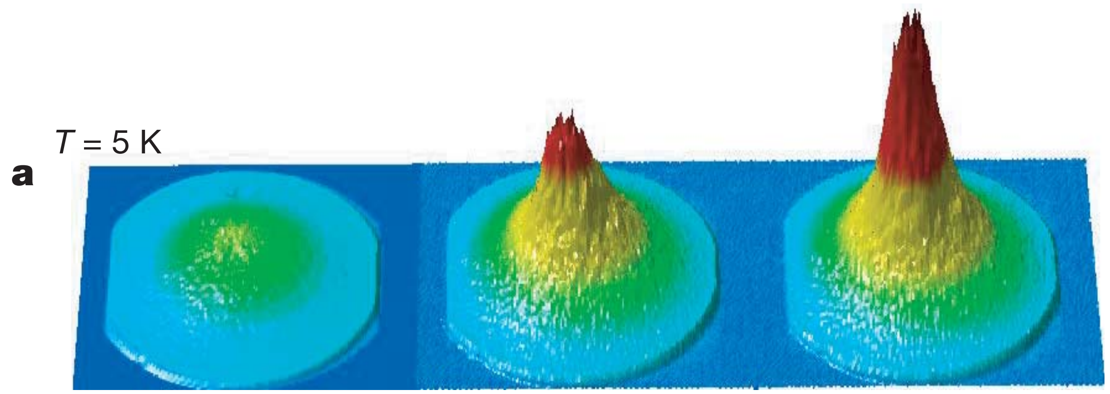
two-body loss

$$\langle \zeta^*(t,\mathbf{x})\zeta(t',\mathbf{x}') \rangle = \gamma\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$$

microscopic derivation and linear fluctuation analysis:
Szymanska, Keeling, Littlewood PRL (04, 06); PRB (07));
Wouters, Carusotto PRL (07,10)

An Example: Exciton-Polariton Systems

- Bose condensation seen despite non-equilibrium conditions



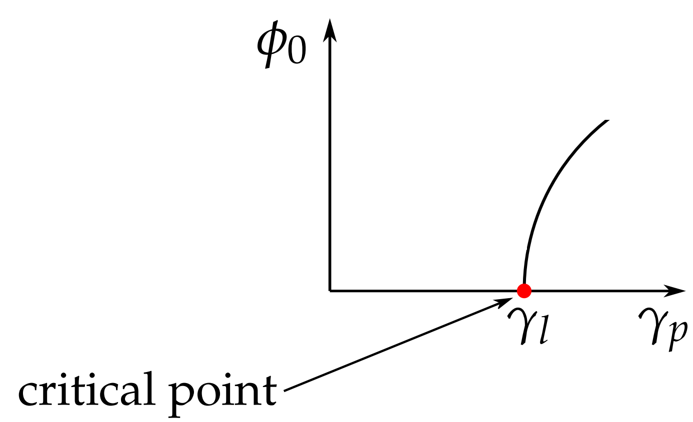
Kasprzak et al., Nature 2006

- stochastic driven-dissipative Gross-Pitaevskii-Eq

$$\cancel{i\partial_t\phi} = \left[-\cancel{\frac{\nabla^2}{2m}} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2\right]\phi + \cancel{\zeta}$$

Szymanska, Keeling, Littlewood PRL (04, 06); PRB (07)); Wouters, Carusotto PRL (07,10)

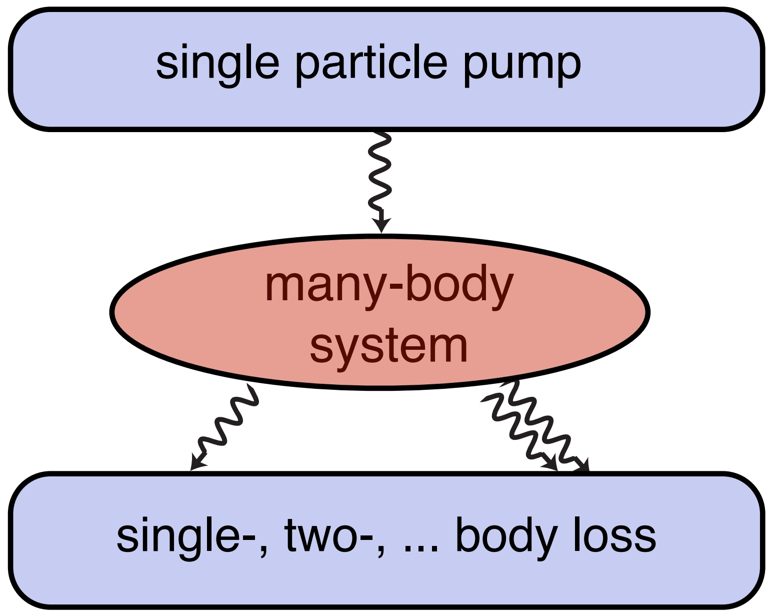
- mean field
  - neglect noise
  - homogeneous solution $\phi(\mathbf{x},t) = \phi_0$



- naively, just as Bose condensation in equilibrium!
- Q: What is “non-equilibrium” about it?

Microscopic Description: Quantum Master Equation

- generic microscopic many-body model:



$$\partial_t \rho = -i[H, \rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

$$H = \int_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right) \hat{\phi}_{\mathbf{x}} + \frac{\lambda}{2} (\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}})^2$$

$$\mathcal{D}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat{\phi}^\dagger_{\mathbf{x}}\, \rho\, \hat{\phi}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}_{\mathbf{x}} \hat{\phi}^\dagger_{\mathbf{x}}, \rho\}]}_{\text{single particle pump}} + \underbrace{\gamma_l \int_{\mathbf{x}} [\hat{\phi}_{\mathbf{x}}\, \rho\, \hat{\phi}^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^\dagger_{\mathbf{x}} \hat{\phi}_{\mathbf{x}}, \rho\}]}_{\text{single particle loss}} +$$

$$\underbrace{\kappa \int_{\mathbf{x}} [\hat{\phi}^2_{\mathbf{x}}\, \rho\, \hat{\phi}^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat{\phi}^{\dagger\,2}_{\mathbf{x}} \hat{\phi}^2_{\mathbf{x}}, \rho\}]}_{\text{two particle loss}}$$

➡ how to detect non-equilibrium conditions?

➡ how does this model relate to the exciton-polariton systems?

➡ how to do efficient (semi-analytical) calculations for such systems?

Part I:
Theoretical Background

$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, \rho\}$$

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

Keldysh Functional Integral for stationary states of driven open quantum systems

- Construction from quantum master equation
- Semiclassical limit
- “What is non-equilibrium about it?”

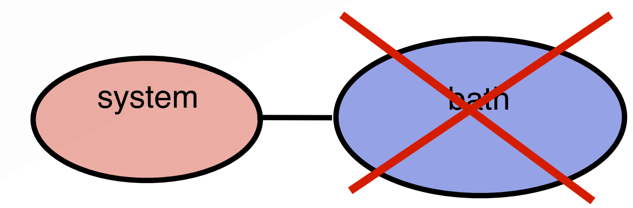
$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

Quantum master equation

- eliminate bath in second order perturbation theory: Master equation

Lindblad operators

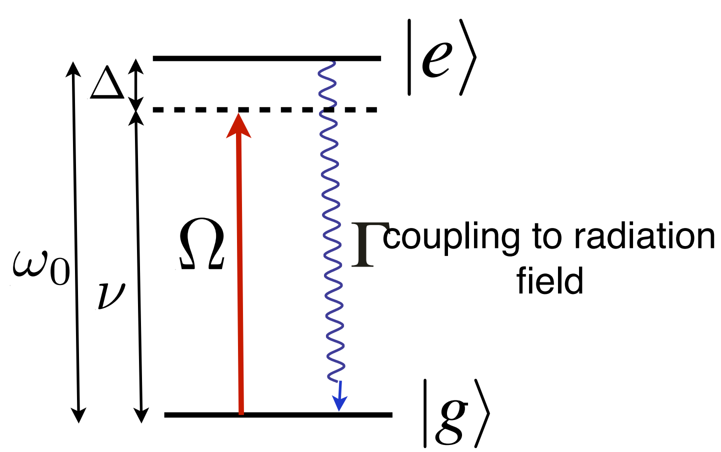
$$\partial_t \rho = \underbrace{-i[H,\rho]}_{\text{coherent evolution}} + \kappa \underbrace{\sum_i \left(L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}\right)}_{\text{dissipative evolution}}$$



$$\equiv \mathcal{L}[\rho]$$ -- Liouvillian operator

- Lindblad form: most general time-local meaningful (trace preserving & completely positive) time evol. of density matrix

- driven nature:



$$H = (|e\rangle, |g\rangle) \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} \langle e| \\ \langle g| \end{pmatrix}$$

$$L_i = |g\rangle\langle e| = \sigma^-$$

- simple facts:
  - system energy not conserved: $[H, L_i] \neq 0$
  - drive essential to access upper level

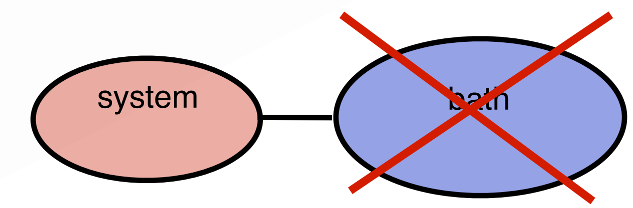
- Implications:
  - no guarantee for detailed balance
  - no obedience of the second law of thermodynamics (state purification)

Many-body quantum master equation

- eliminate bath in second order perturbation theory: Master equation

Lindblad operators

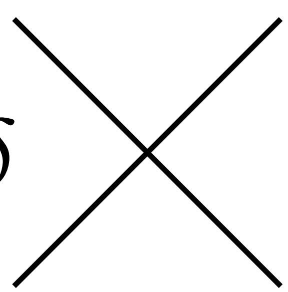
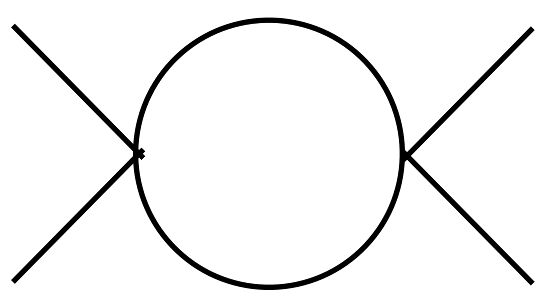
$$\partial_t \rho = \underbrace{-i[H,\rho]}_{\text{coherent evolution}} + \underbrace{\kappa \sum_i \left(L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}\right)}_{\text{dissipative evolution}}$$



$\equiv \mathcal{L}[\rho]$ -- Liouvillian operator

- Lindblad form: most general time-local meaningful (trace preserving & completely positive) time evol. of density matrix

- The many-body problem: given continuum of degrees of freedom, smallness of coupling does not guarantee convergence of perturbation theory
    - e.g. second order correction to local interaction:

$\delta\lambda$ = δ  =  $\sim \lambda \cdot \int \frac{d^d q}{(2\pi)^2} \frac{T}{(q^2+\Delta)^2} \cdot \lambda$

sum over intermediate states

divergence if $\Delta \to 0, d < 4$

where: phase transitions, ordered phases..

➡ harness many-body techniques in quantum optics context! -> Keldysh functional integral

Keldysh Functional Integrals: Why?

- Feynman's formulation of quantum mechanics

REVIEWS OF

MODERN PHYSICS

VOLUME 20, NUMBER 2 APRIL, 1948

Space-Time Approach to Non-Relativistic Quantum Mechanics

R. P. FEYNMAN

Cornell University, Ithaca, New York

Non-relativistic quantum mechanics is formulated here in a different way. It is, however, mathematically equivalent to the familiar formulation. In quantum mechanics the probability of an event which can happen in several different ways is the absolute square of a sum of complex contributions, one from each alternative way. The probability that a particle will be found to have a path $x(t)$ lying somewhere within a region of space time is the square of a sum of contributions, one from each path in the region. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of $\hbar$) for the path in question. The total contribution from all paths reaching x, t from the past is the wave function $\psi(x, t)$. This is shown to satisfy Schroedinger's equation. The relation to matrix and operator algebra is discussed. Applications are indicated, in particular to eliminate the coordinates of the field oscillators from the equations of quantum electrodynamics.

1. INTRODUCTION

IT is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the differential equation of Schroedinger, and the matrix algebra of Heisenberg. The two, apparently dissimilar approaches, were proved to be mathematically equivalent. These two points of view were destined to complement one another and to be ultimately synthesized in Dirac's transformation theory.

This paper will describe what is essentially a classical action[3] to quantum mechanics. A probability amplitude is associated with an entire motion of a particle as a function of time, rather than simply with a position of the particle at a particular time.

The formulation is mathematically equivalent to the more usual formulations. There are, therefore, no fundamentally new results. However, there is a pleasure in recognizing old things from a new point of view. Also, there are problems for which the new point of view offers a distinct advantage. For example, if two systems

- Useful language for systems with many degrees of freedom
  - general: powerful techniques
    - diagrammatic perturbation theory;
    - collective variables;
    - renormalization group
  - non-equilibrium Keldysh
    - closer to the real-time formulations of quantum mechanics
    - yields directly observable quantities (responses and correlations)
    - indispensable for non-Hamiltonian systems:
      - disorder
      - dissipation

      infinite harmonic baths!
    - open the powerful toolbox of quantum field theory for many-body non-equilibrium situations

Keldysh functional integral

$$\hbar = 1$$

- The basic idea in three steps:

$$U(t, t_0) = e^{-iH(t-t_0)}$$

1. Schroedinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

2. Heisenberg-von Neumann equation: evolving a state (density) matrix

$$\partial_t \rho(t) = -i[H, \rho(t)] \quad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

   - identical for pure (factorizable) states $\quad \rho = |\psi\rangle\langle\psi|$

3. The same is true for the Master Equation:

$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\} \;\equiv \mathcal{L}[\rho]$$

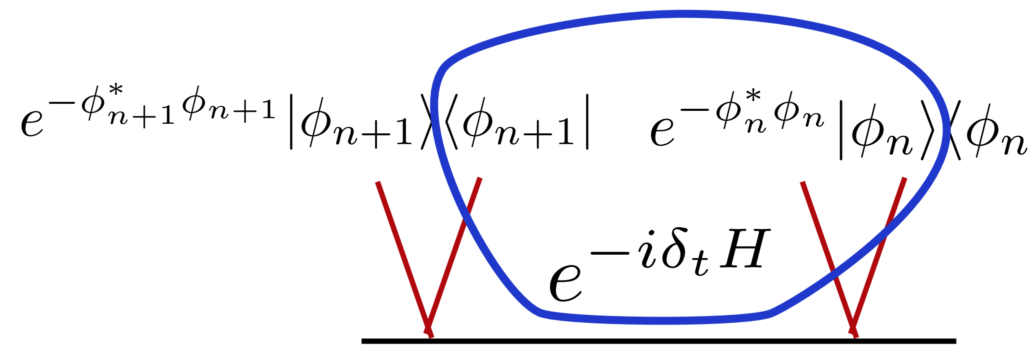
$$\Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

Keldysh functional integral

1. Functional integral idea:

➡ "Trotterization" of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N$

- one time step



$$e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|e^{-i\delta_t H[a^\dagger,a]}|\phi_n\rangle$$

$$\approx e^{-\phi^*_n\phi_n}\langle\phi_{n+1}|1-i\delta_t H[a^\dagger,a]|\phi_n\rangle$$

H normally ordered

$$= e^{-\phi^*_n\phi_n}e^{+\phi^*_{n+1}\phi_n}(1-i\delta_t H[\phi^*_{n+1},\phi_n])$$

$$\approx e^{i\delta_t[-i\frac{(\phi^*_{n+1}-\phi^*_n)}{\delta_t}\phi_n - H[\phi^*_{n+1},\phi_n]]}$$

continuum limit

$$dt \quad -i\partial_t\phi^*(t)\cdot\phi(t) \qquad H[\phi^*(t),\phi(t)]$$

coherent states:

$$a|\phi\rangle = \phi|\phi\rangle$$

$$\langle\phi'|\phi\rangle = e^{\phi'^*\phi}$$

$$\mathbf{1} = \int\frac{d\phi^* d\phi}{\pi}e^{-\phi^*\phi}|\phi\rangle\langle\phi|$$

Keldysh functional integral

1. Functional integral idea:

➡ “Trotterization” of time interval and insertion of coherent states: $e^{iH(t-t_0)} = \lim_{N\to\infty} (1 + i\delta_t H)^N$

- many time steps

$$\underbrace{\int \prod_t \frac{d\phi^*(t)d\phi(t)}{\pi}}_{=: \int \mathcal{D}(\phi^*,\phi)} e^{i\int_{t_0}^{t_f} dt[-i\partial_t\phi^*(t)\cdot\phi(t)-H[\phi^*(t),\phi(t)]]}$$

functional integral measure

- Discussion
  - ➡ operator H -> complex, time dependent functional H
  - ➡ time evolution from overlap of neighbouring states
  - ➡ no reference to single particle or many-body Hamiltonian, lattice or continuum!
  - ➡ analogous for fermions (spins: more involved, but see M. Maghrebi, A. V. Gorshkov, PRB (2016))
  - ➡ single set of degrees of freedom for vector evolution

Keldysh functional integral

2. Schroedinger vs. Heisenberg-von Neumann

$$U(t,t_0) = e^{-iH(t-t_0)}$$

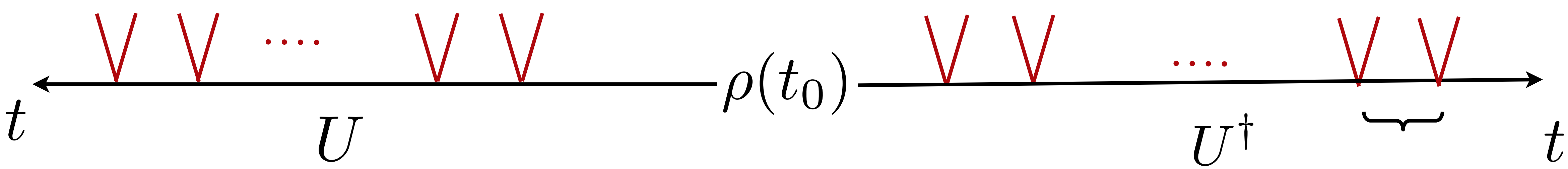
- Schroedinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \qquad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

- Heisenberg-von Neumann equation: evolving a state (density) matrix

$$\partial_t\rho(t) = -i[H,\rho(t)] \qquad \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

- Second case: “Trotterization” on both sides:



$$e^{iH(t-t_0)} = \lim_{N\to\infty}(1+i\delta_t H)^N \qquad \delta_t = \frac{t-t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

Keldysh functional integral

3. Schroedinger vs. Quantum Master

$$U(t,t_0) = e^{-iH(t-t_0)}$$

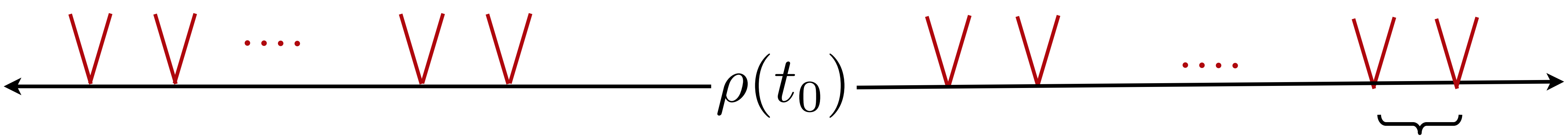
- Schroedinger equation: evolving a state vector

$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \qquad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

- Quantum Master equation: evolving a state (density) matrix

$$\partial_t\rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- Identical program for Liouville generator of dynamics (left and right action on density matrix)



$$\rho(t) = e^{(t-t_0)\mathcal{L}}\,\rho_0 = \lim_{N\to\infty}(1+\delta_t\mathcal{L})^N\,\rho_0 \qquad \delta_t = \frac{t-t_0}{N}$$

➡ two sets of degrees of freedom for matrix evolution

Keldysh functional integral

3. Schroedinger vs. Quantum Master

$$U(t,t_0) = e^{-iH(t-t_0)}$$

- Schroedinger equation: evolving a state vector

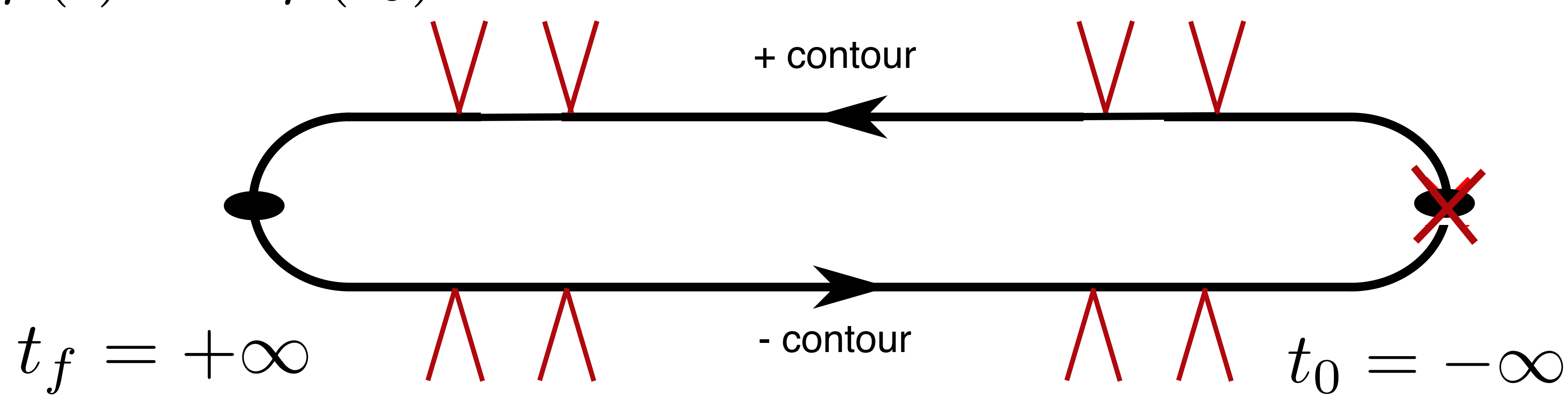
$$i\partial_t|\psi\rangle(t) = H|\psi\rangle(t) \quad \Rightarrow |\psi\rangle(t) = U(t,t_0)|\psi\rangle(t_0)$$

- Quantum Master equation: evolving a state (density) matrix

$$\partial_t\rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho] \qquad \Rightarrow \rho(t) = e^{\mathcal{L}(t-t_0)}\rho(t_0)$$

- final step: Keldysh "partition function"

$$Z = \mathrm{tr}\rho(t) = \mathrm{tr}\rho(t_0) = 1$$



$t_0 \to -\infty, t_f \to +\infty$

information on all stages; stationarity reached (boundary conditions irrelevant)

Keldysh functional integral: Final result

- quantum master equation: $\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$

$$= -i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2}\rho L_i^\dagger L_i)$$

- equivalent Keldysh functional integral:

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+,\Phi_-]} \qquad \Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi_\pm^* \end{pmatrix}$$

$$S_M[\Phi_+, \Phi_-] = \int dt (\phi_+^* i\partial_t \phi_+ - \phi_-^* i\partial_t \phi_- - i\mathcal{L}[\Phi_+, \Phi_-])$$

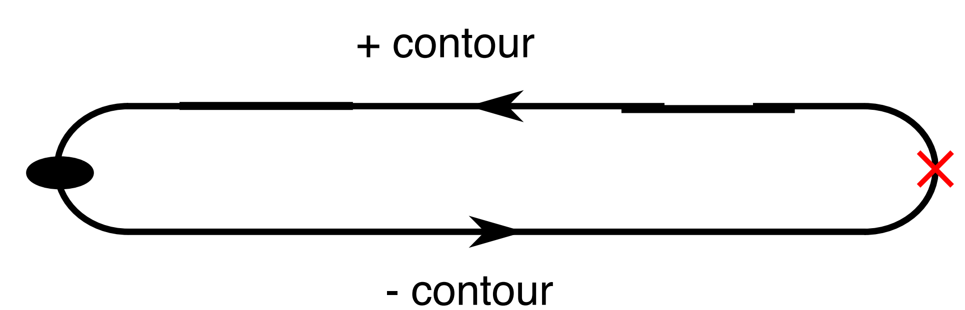
$$\boxed{\mathcal{L}[\Phi_+, \Phi_-] = -i\,(H_+ - H_-) - \kappa \sum_i \left(L_{i,+} L_{i,-}^\dagger - \tfrac{1}{2} L_{i,+}^\dagger L_{i,+} - \tfrac{1}{2} L_{i,-}^\dagger L_{i,-} \right)}$$

$H_\pm = H(\Phi_\pm)$ etc.

➡ recognize Lindblad structure

➡ simple translation table (for normal ordered Liouvillian)

- operator right of density matrix -> - contour
- operator left of density matrix -> + contour



Keldysh functional integral: Probability conservation / "Causality"

- quantum master equation: $\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$

$$= -i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2} \rho L_i^\dagger L_i)$$

- equivalent Keldysh functional integral:

$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) e^{\mathrm{i}(S_M[\Phi_+,\Phi_-]}$$

$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi_\pm^* \end{pmatrix}$$

$$S_M[\Phi_+, \Phi_-] = \int dt(\phi_+^* i\partial_t \phi_+ - \phi_-^* i \partial_t \phi_- - i\mathcal{L}[\Phi_+, \Phi_-])$$

$$\boxed{\mathcal{L}[\Phi_+, \Phi_-] = -i\,(H_+ - H_-) - \kappa \sum_i \left(L_{i,+} L_{i,-}^\dagger - \tfrac{1}{2} L_{i,+}^\dagger L_{i,+} - \tfrac{1}{2} L_{i,-}^\dagger L_{i,-} \right)}$$

$H_\pm = H(\Phi_\pm)$ etc.

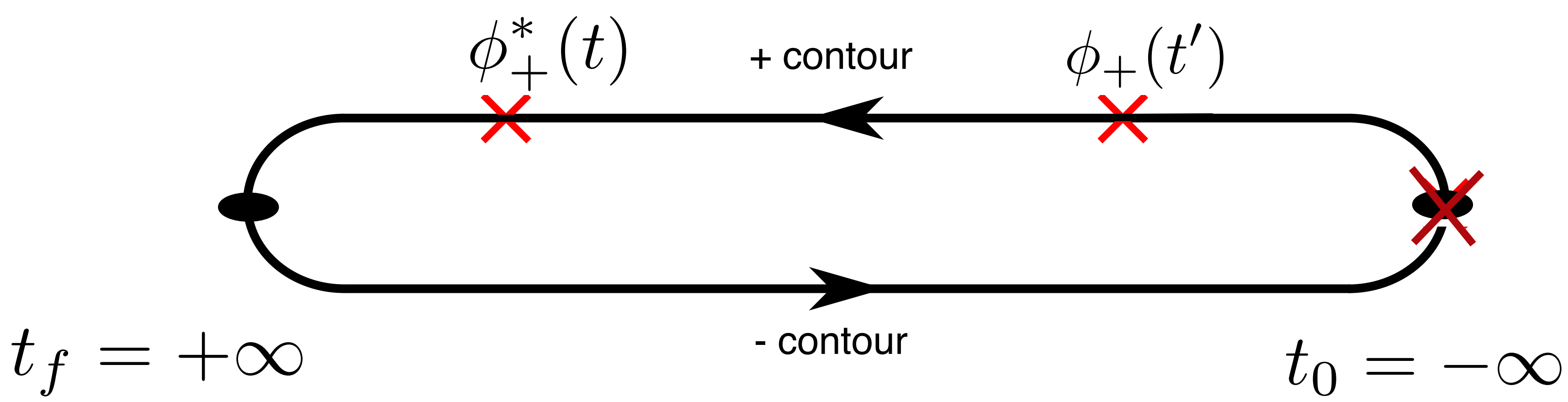
- trace preservation:
- QME: $\partial_t \mathrm{tr}\rho = \mathrm{tr}\big(-i(H\rho - \rho H) + \kappa \sum_i (L_i \rho L_i^\dagger - \tfrac{1}{2} L_i^\dagger L_i \rho - \tfrac{1}{2} \rho L_i^\dagger L_i)\big) = 0$
- Keldysh: $Z = \mathrm{tr}\rho(t) = 1$

cyclicity

- mnemonic: taking trace = ignoring contour order: $\Phi_+ = \Phi_- \Rightarrow S_M[\Phi_+, \Phi_-] = 0$

Physical Observables

- correlation functions: field insertions on the contour



- compute them: introduce sources (cf. Stat Mech)

$$Z = \mathrm{Tr}(1 \cdot \rho) = \langle 1 \rangle$$

$$Z[j_+, j_-] = \langle e^{\mathrm{i} \int (j_+ \phi_+^* - j_- \phi_-^* + c.c.)} \rangle$$

$$Z[0,0] = \langle 1 \rangle = 1$$ normalization

- example

$$\langle \mathcal{T}_C [\hat{\phi}^\dagger(t) \hat{\phi}(t')] \rangle = \frac{\delta^2 Z[j_+, j_-]}{\delta j_+(t) \delta j_+^*(t')} \Big|_{j=0}$$

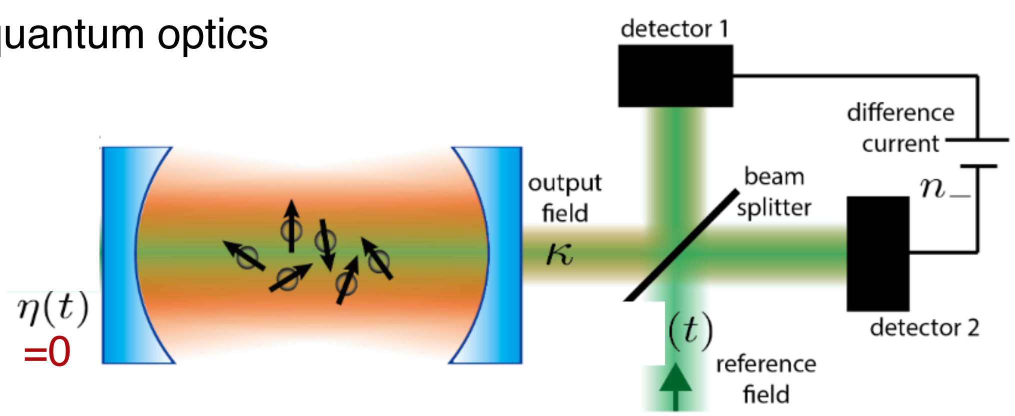
NB: Functional integrals always compute time-ordered correlation functions

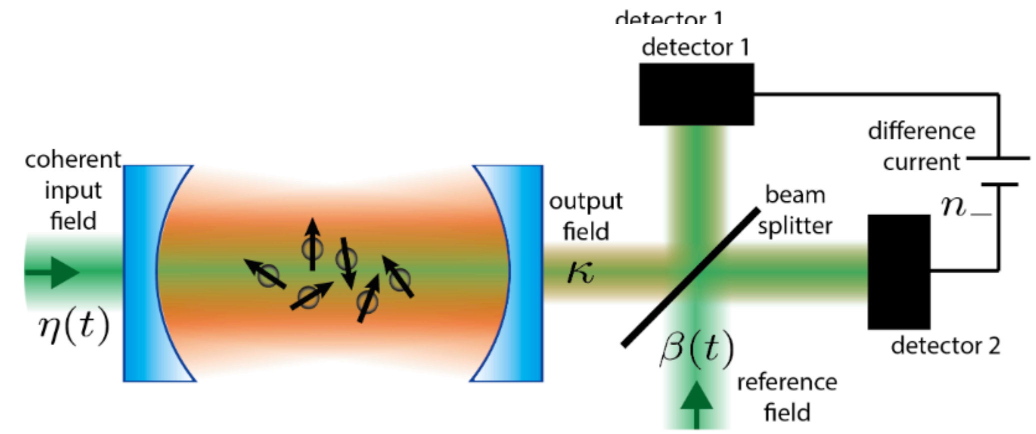
➡ there is a more intuitive basis to do computations

Correlation vs. response functions

- two basic types of experiments:
  - correlation measurements: study without disturbing
  - (linear) response measurements: probe system with (weak) external fields

eg. quantum optics





e.g. photon quadrature component at vacuum input field
(or: $g^{(1)}(\tau)$)

e.g. coherent input field

in homodyne detection: retarded response of quadrature components

- directly delivered in the functional framework via basis transformation: “Keldysh rotation”

$$\begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_+ + \phi_- \\ \phi_+ - \phi_- \end{pmatrix}$$

“classical field”: center-of-mass coordinate

“quantum field”: relative coordinate

  - classical field can acquire finite expectation value (e.g. lasing, Bose condensation)
  - quantum / noise field cannot
  - probability preservation: $\boxed{S_M[\Phi_c, \Phi_q = 0] = 0 \quad \forall \Phi_c}$

Correlation vs. response functions

- Partition function in new basis

$$Z[j] = \langle e^{\mathrm{i}\int (j_+ \phi_+^* - j_- \phi_-^* + c.c.)} \rangle = \langle e^{\mathrm{i}\int (j_c \phi_q^* + j_q \phi_c^* + c.c.)} \rangle$$

- order parameter:

$$\langle \phi_c(t,\mathbf{x}) \rangle = -i \frac{\delta Z[j]}{\delta j_q^*(t,\mathbf{x})} \Big|_{j=0}$$

q,c appear as conjugate pairs for the source

homodyne detection: vacuum input

- Single particle response: how does the field react to external perturbations?

relation to operator formalism (once and for all)

response to coherent field

t = t'

$$G^R(t-t',\mathbf{x}-\mathbf{x}') = i \frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x}) \delta j_c(t',\mathbf{x}')} \Big|_{j=0} = -i \langle \phi_c(t,\mathbf{x}) \phi_q^*(t',\mathbf{x}') \rangle = -i\theta(t-t') \langle [\hat\phi(t,\mathbf{x}), \hat\phi^\dagger(t',\mathbf{x}')] \rangle = 1$$

- Single particle correlations: how are states occupied?

t = t', x=x'

$$G^K(t-t',\mathbf{x}-\mathbf{x}') = i \frac{\delta^2 Z}{\delta j_q^*(t,\mathbf{x}) \delta j_q(t',\mathbf{x}')} \Big|_{j=0} = -i \langle \phi_c(t,\mathbf{x}) \phi_c^*(t',\mathbf{x}') \rangle = -i \langle \{ \hat\phi(t,\mathbf{x}), \hat\phi^\dagger(t',\mathbf{x}') \} \rangle = 2\langle \hat n(\mathbf{x}) \rangle + 1$$

time and space translation invariance assumed

$g^{(1)}(\tau = 0)$

- total Green's function

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^A = (G^R)^\dagger, \quad (G^K)^\dagger = -G^K$$

Correlation vs. response functions

- action in this basis:

$$S = \int_{\omega,\mathbf{q}} \left(\phi_c^*, \phi_q^*\right) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + \text{interactions.}$$

➡ redundancy of the +/- basis eliminated (zero entry)

- the matrix is the inverse single particle Green's function:

  - equation of motion (action principle):

$$\begin{pmatrix} \frac{\delta S}{\delta \phi_c^*} \\ \frac{\delta S}{\delta \phi_q^*} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix}}_{G^{-1}} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} \stackrel{!}{=} 0$$

(exact for free theory only)

  - Green's function $G^{-1} \circ G = \mathbf{1}\delta(\omega - \omega')\delta(\mathbf{q} - \mathbf{q})$ (diagonal in frequency/ momentum space)

- single particle Green's function:

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix} \qquad G^K = -G^R P^K G^A$$

Correlation vs. response: single degree of freedom

- master equation for decaying cavity:

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger\hat{a}, \rho\})$$

- action:

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

time domain $a_\nu(t)$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \underbrace{\omega - \omega_0 + i\kappa}_{G^R(\omega)^{-1}} & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain $a_\nu(\omega)$

- observables from the Green's functions:

$$G = \begin{pmatrix} G^K & G^R \\ G^A & 0 \end{pmatrix}$$

$$G^K = -G^R P^K G^A$$

response

- Lorentzian spectral density $\quad A(\omega) = \mathrm{Im} G^R(\omega) = \dfrac{2\kappa}{(\omega - \omega_0)^2 + \kappa^2}$

- decay of single-particle response: $\quad G^R(t-t') = \int_\omega e^{i\omega(t-t')} G^R(\omega) = \theta(t-t') e^{i\omega(t-t')} e^{-\kappa(t-t')}$

correlation

- cavity mode occupation in stationary state : $\quad 2\langle \hat{n}(t)\rangle + 1 = \langle \hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t)\rangle = iG^K(t-t) = i\int_\omega e^{i\omega(t-t)} G^K(\omega) = 1 \quad (t \to \infty)$

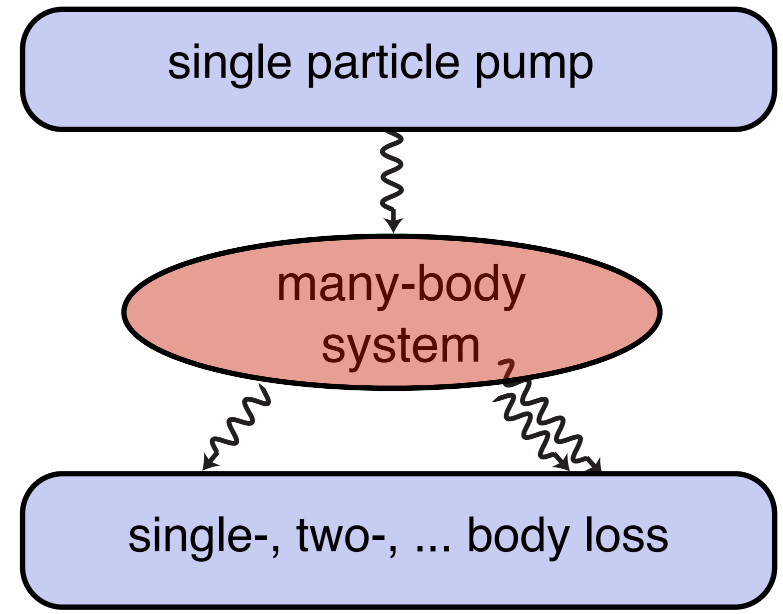
$$\langle \hat{n}(t \to \infty)\rangle = 0$$

➡ correlation / statistical properties: G^K

➡ response / spectral properties: G^R

Keldysh Action for Many-Body Model

- generic microscopic many-body model:



$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho] \equiv \mathcal{L}[\rho]$$

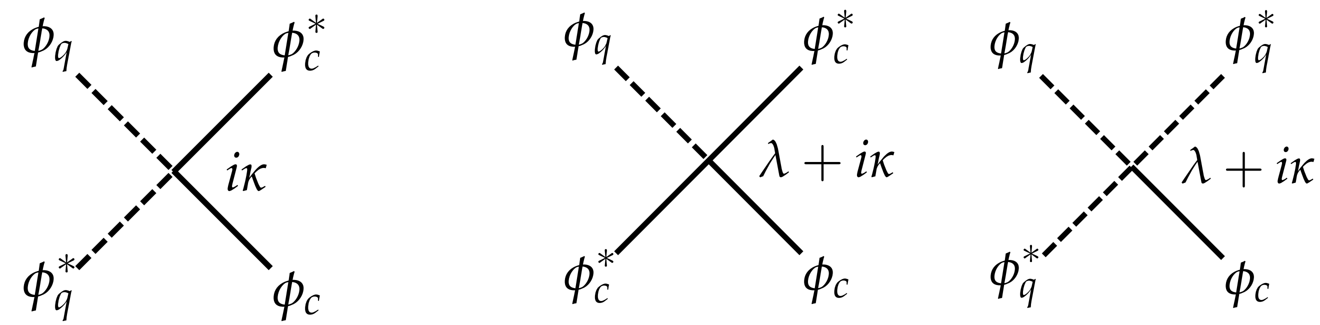
$$H = \int_{\mathbf{x}} \hat\phi^\dagger_{\mathbf{x}} \left(\frac{\triangle}{2M} - \mu\right)\hat\phi_{\mathbf{x}} + \frac{\lambda}{2}(\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}})^2$$

$$\mathcal{D}[\rho] = \underbrace{\gamma_p \int_{\mathbf{x}} [\hat\phi^\dagger_{\mathbf{x}}\,\rho\,\hat\phi_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi_{\mathbf{x}}\hat\phi^\dagger_{\mathbf{x}},\rho\}]}_{\text{single particle pump}} + \underbrace{\gamma_l \int_{\mathbf{x}} [\hat\phi_{\mathbf{x}}\,\rho\,\hat\phi^\dagger_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^\dagger_{\mathbf{x}}\hat\phi_{\mathbf{x}},\rho\}]}_{\text{single particle loss}} +$$

$$\underbrace{\kappa \int_{\mathbf{x}} [\hat\phi^2_{\mathbf{x}}\,\rho\,\hat\phi^{\dagger\,2}_{\mathbf{x}} - \tfrac{1}{2}\{\hat\phi^{\dagger\,2}_{\mathbf{x}}\hat\phi^2_{\mathbf{x}},\rho\}]}_{\text{two particle loss}}$$

Microscopic markovian dissipative action

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$



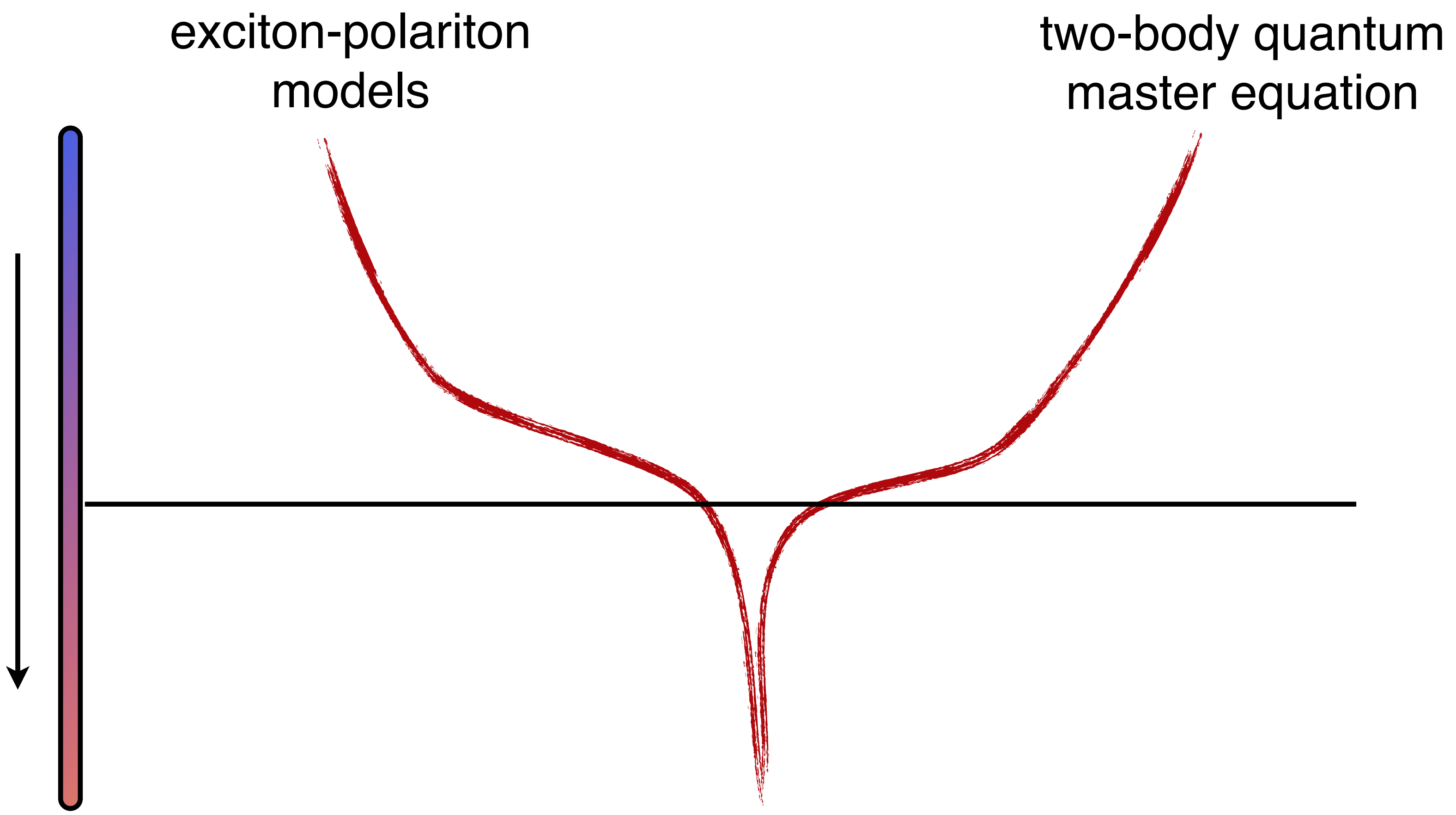
- Gaussian sector: inverse Green's function
  - retarded/advanced $\quad P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 - \mu + i\left(\gamma_l - \gamma_p\right)/2$
  - Keldysh component $\quad P^K = i\left(\gamma_l + \gamma_p\right)$

difference: distance from a phase transition

sum: noise of loss and pumping add up

- now: simplifications in the semiclassical limit:
  - sharp argument close to a critical point
  - provides intuition for a frequency regime $\omega \ll \gamma = \gamma_l + \gamma_p$

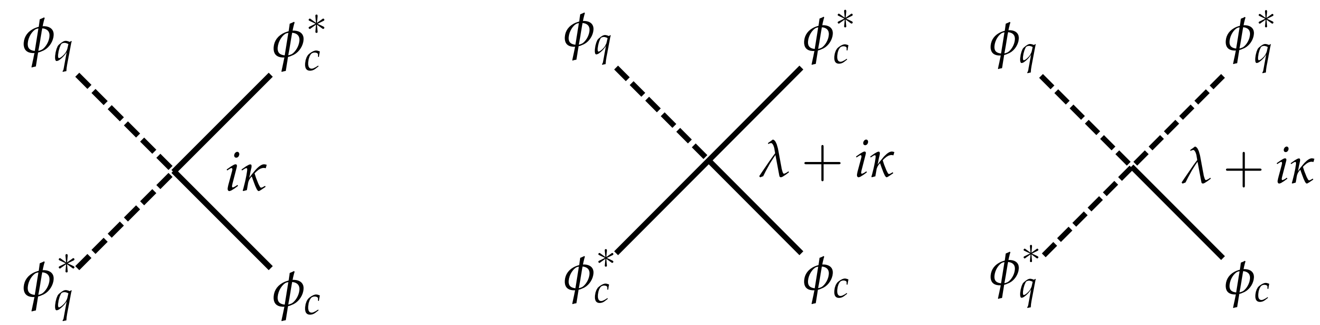
Semi-classical limit and Langevin equations



Semiclassical limit: power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ \left(\phi_c^*, \phi_q^*\right) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \phi_c^* \phi_c \phi_q^* \phi_q - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \phi_q^{*2} \phi_c \phi_q \right) + c.c. \right] \right\}$$

- Gaussian sector close to a critical point:



  - retarded/advanced $\quad P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i\left(\gamma_l - \gamma_p\right)/2}^{\to 0} \sim q^2$

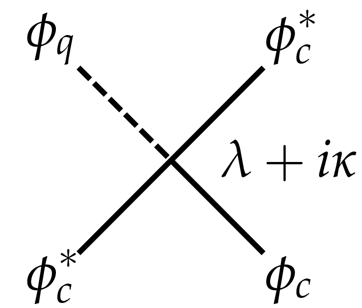
  - Keldysh component $\quad P^K = i\left(\gamma_l + \gamma_p\right) \sim q^0$

- Canonical field dimensions: $\boxed{[\phi_c] = \dfrac{d-2}{2} < [\phi_q] = \dfrac{d+2}{2}}$

  - action is dimensionless: phase e^{iS} in the functional integral
  - quadratic/Gaussian sector: scaling dimensions of inverse Green’s function known
  - intuitive: high order local couplings not relevant at large distances

Semiclassical limit: power counting

$$\mathcal{S} = \int_{t,\mathbf{x}} \left\{ (\phi_c^*, \phi_q^*) \begin{pmatrix} 0 & P^A \\ P^R & P^K \end{pmatrix} \begin{pmatrix} \phi_c \\ \phi_q \end{pmatrix} + 2i\kappa \cancel{\phi_c^* \phi_c \phi_q^*} \phi_q - \tfrac{1}{2} \left[(\lambda + i\kappa) \left(\phi_c^{*2} \phi_c \phi_q + \cancel{\phi_q^{*2} \phi_c \phi_q} \right) + c.c. \right] \right\}$$



- Gaussian sector at criticality:

  - retarded/advanced $$P^R(\omega, \mathbf{q}) = \omega - \mathbf{q}^2 \overbrace{- \mu + i\left(\gamma_l - \gamma_p\right)/2}^{\to 0} \sim q^2$$

  - Keldysh component $$P^K = i\left(\gamma_l + \gamma_p\right) \sim q^0$$

- Canonical field dimensions: $$[\phi_c] = \frac{d-2}{2} < [\phi_q] = \frac{d+2}{2}$$

➡ Local vertices with more than two quantum fields are irrelevant in the RG sense in d > 2

➡ Note preservance of probability in semiclassical limit $S_M[\Phi_c, \Phi_q = 0] = 0 \quad \forall \Phi_c$

➡ massive diagrammatic simplification

➡ identical to phenomenological models of exciton-polariton condensates (Wouters and Carusotto PRL 06; Szymanska, Keeling, Littlewood PRL 04)

Semiclassical limit: Equivalence to Langevin equation

- Keldysh integral after power counting $$Z = \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*]\, e^{iS[\phi_c,\phi_c^*,\phi_q,\phi_q^*]}$$
- with

$$S = \int_{t,\mathbf{x}} \left\{ \phi_q^* \frac{\delta \bar{S}[\phi_c]}{\delta \phi_c^*} + c.c. + i2\gamma \phi_q^* \phi_q \right\}$$

➡ phi_q only up to quadratic order

$$\bar{S} = \int_{t,\mathbf{x}} \{\phi_c^* i\partial_t \phi_c - \mathcal{H}_c + i\mathcal{H}_d\} \qquad \mathcal{H}_\alpha = r_\alpha |\phi_c|^2 + K_\alpha |\nabla \phi_c|^2 + \lambda_\alpha |\phi_c^* \phi_c|^4, \quad \alpha = c, d$$

- Hubbard-Stratonovich decoupling $$e^{-2\gamma \int_{t,\mathbf{x}} \phi_q^* \phi_q} = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi - i \int_{t,\mathbf{x}} (\phi_q^* \xi - \xi^* \phi_q)}$$

$$Z = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*, \phi_q, \phi_q^*] e^{i\left[\phi_q^* \left(i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c{}^*} + i \frac{\delta \mathcal{H}_d}{\delta \phi_c{}^*} - \xi \right) + c.c.\right]}$$

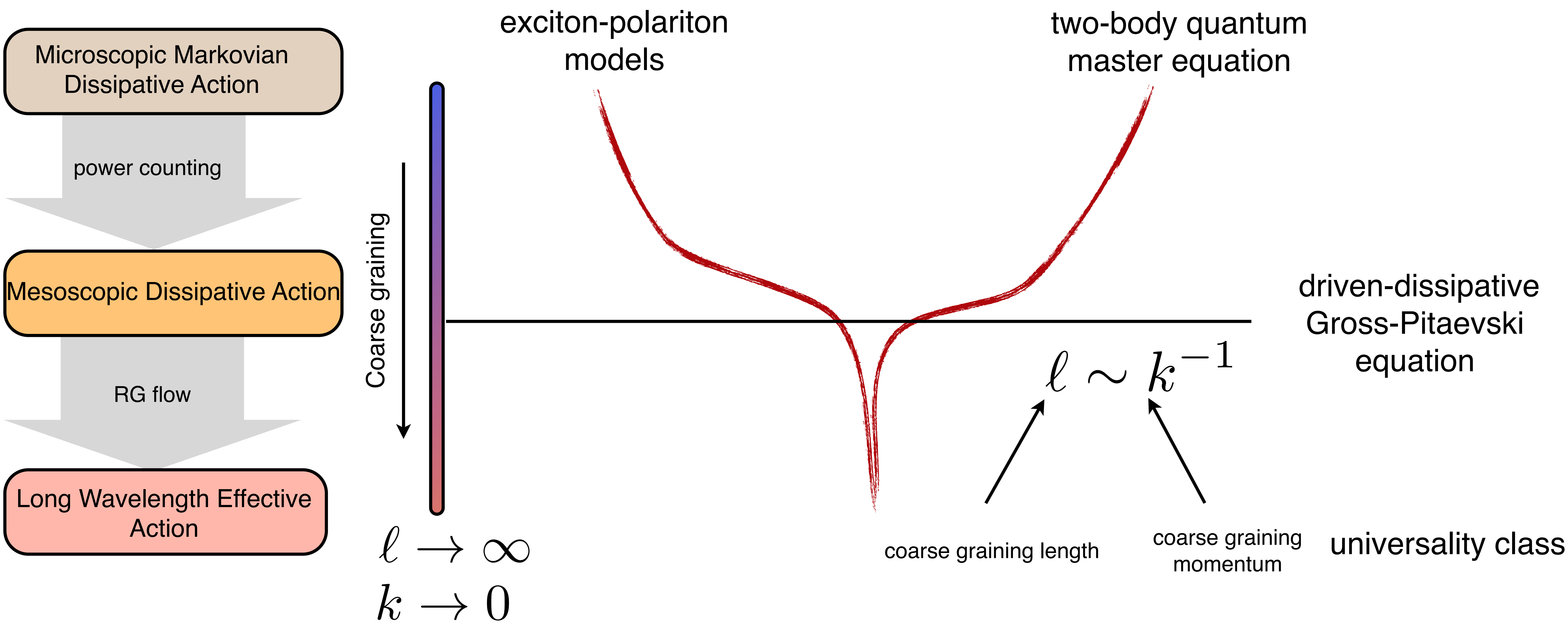
- linear in phi_q: Fourier representation of delta-functional

$$Z = \int \mathcal{D}[\xi, \xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c, \phi_c^*] \delta \left(\boxed{ i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} + i \frac{\delta \mathcal{H}_d}{\delta \phi_c^*} - \xi } \right) \delta\,(c.c.)$$

➡ noise averaging ➡ at each instant of time: ➡ driven-dissipative Gross-Pitaevski equation

Semiclassical limit and exciton-polariton model

- example of "weak" universality

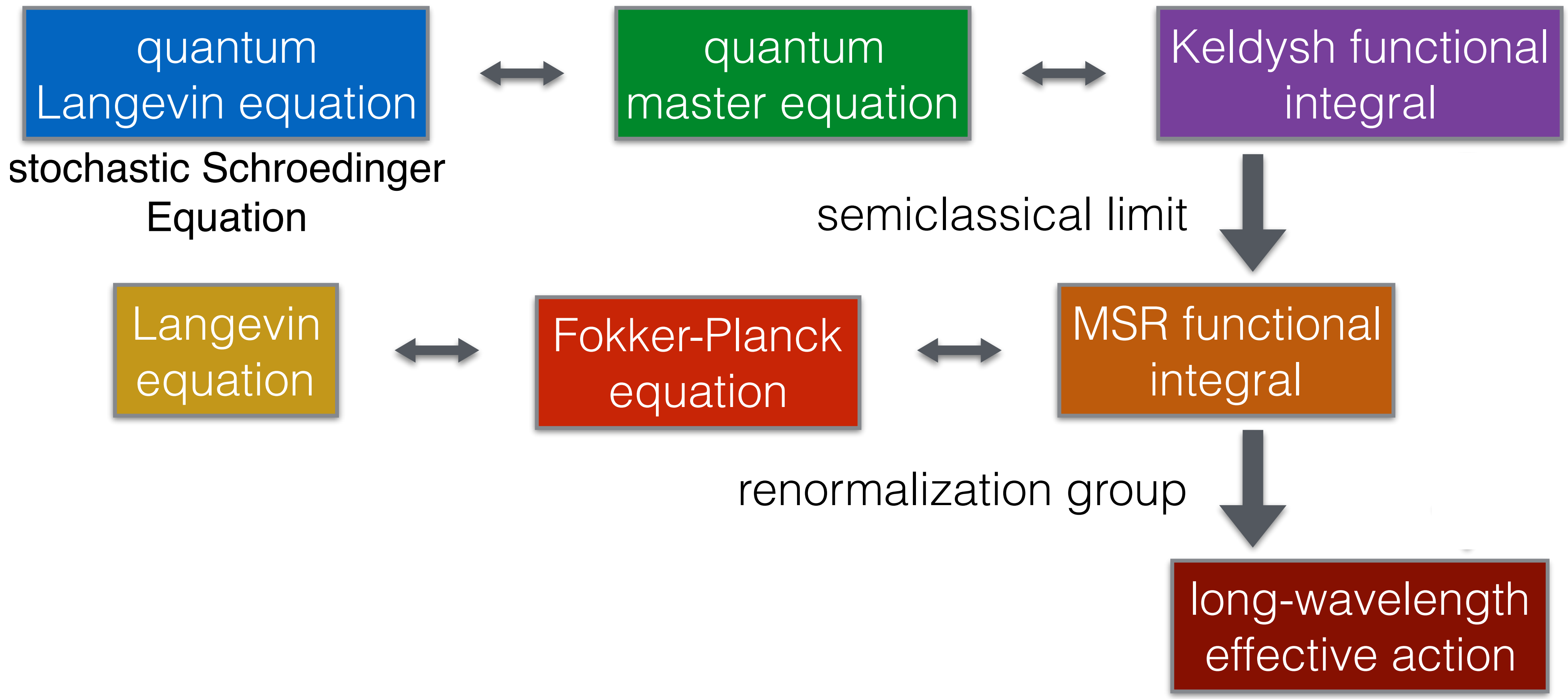


➡ many microscopic models collapse to an effective low energy model

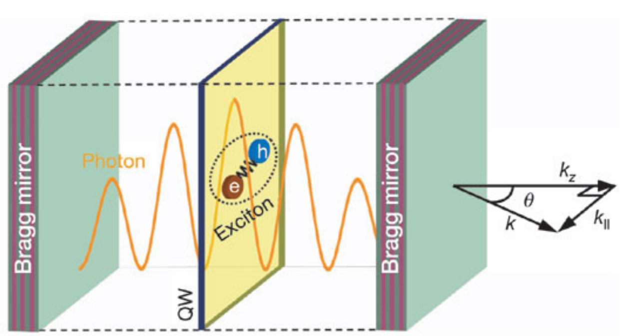
➡ form dictated by microscopic symmetries

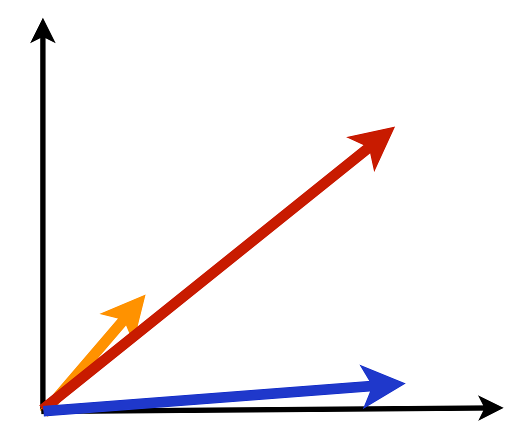
➡ longer wavelength behavior to be determined by calculation

Discussion: Langevin equations, Master equation, Keldysh integral



“What is Non-Equilibrium About It?”





$$Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$$

"What is non-equilibrium about it?"

even in stationary state!

- how to detect non-equilibrium conditions?

  - not straightforward: static observables

$$\rho = e^{-\beta H}/\mathrm{tr} e^{-\beta H}$$

    ➡ any positive semidefinite hermitean operator can be written like this

  - dynamical observables, e.g.:

$$\langle \psi^\dagger(t)\psi(0)\rangle \qquad \psi(t) = e^{iHt}\psi e^{-iHt}$$

    ➡ thermal equilibrium if generator of dynamics coincides with statistical weight

    ➡ otherwise must expect non-equilibrium conditions

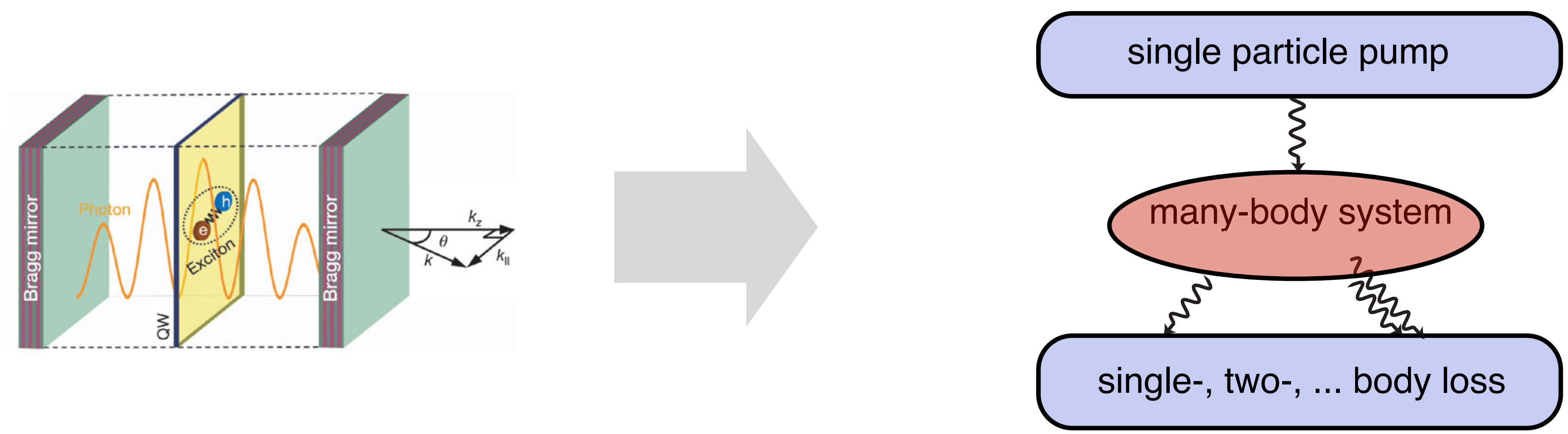
“What is non-equilibrium about it?”

even in stationary state!

- typical differences to closed equilibrium systems:

- absence of number conservation

  ➡ compatible with thermal equilibrium (Caldeira-Leggett Models)



- absence of energy conservation

  ➡ driven system, incompatible with thermal equilibrium

“What is non-equilibrium about it?”: Absence of energy conservation

- Energy conservation: equilibrium dynamics generated by a time-independent Hamiltonian

- more precisely: Given a time dependent Hamiltonian, characteristic scale of time dependence ω_0 There exists no rotating frame in which reference to this scale is gone./ No freedom of choice of the zero of energy.

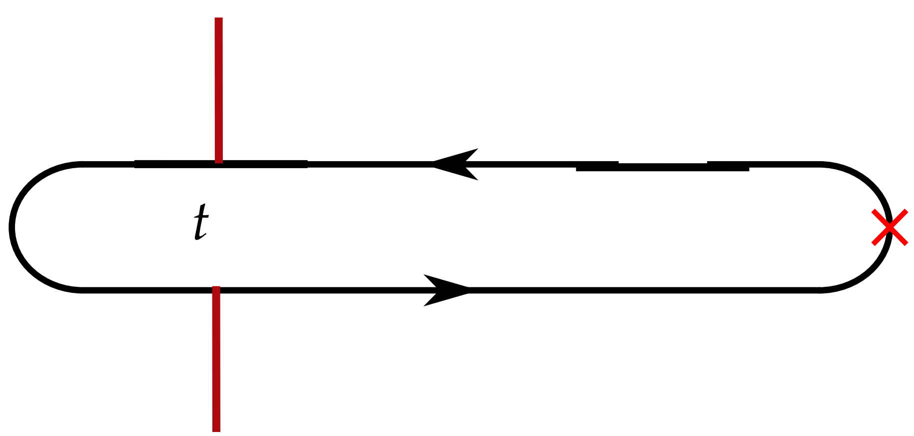
➡ formally: symmetry of Keldysh action under

L. Sieberer, A. Chiochetta, U. Tauber, A. Gambassi, SD, PRB (2015)

$$\mathcal{T}_\beta \Phi_\pm(t, \mathbf{x}) = \Phi^*_\pm(-t \pm \mathrm{i}\beta/2, \mathbf{x})$$

$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi^*_\pm \end{pmatrix}$$

$$\beta = 1/T$$



- discrete: $\mathcal{T}_\beta^2 = 1$

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- symmetry: invariance of $Z = \int \mathcal{D}(\Phi_+, \Phi_-) \mathrm{e}^{\mathrm{i}(S_M[\Phi_+, \Phi_-]}$

$$\mathcal{T}_\beta Z = Z \qquad \mathcal{T}_\beta S_M[\Phi] := S_M[\mathcal{T}_\beta \Phi] = S_M[\Phi], \quad \mathcal{T}_\beta \mathcal{D}(\Phi_+, \Phi_-) = \mathcal{D}(\Phi_+, \Phi_-)$$

- implies for correlation functions

$$\langle \mathcal{O}[\Psi] \rangle = \langle \mathcal{O}[\mathcal{T}_\beta \Psi] \rangle \qquad \langle \mathcal{O}[\Psi] \rangle = \int \mathcal{D}[\Psi] \mathcal{O}[\Psi] e^{iS[\Psi]} \qquad \Psi_\pm = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$$

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$$\mathcal{T}_\beta Z = Z \qquad \mathcal{T}_\beta S_M[\Phi] := S_M[\mathcal{T}_\beta \Phi] = S_M[\Phi], \quad \mathcal{T}_\beta \mathcal{D}(\Phi_+,\Phi_-) = \mathcal{D}(\Phi_+,\Phi_-)$$

- physical consequence: Fluctuation-dissipation relations, of any order, e.g. single particle sector:

$$\boxed{G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]}$$

correlations Bose distribution responses

any order <=> detailed balance
<=> global thermal equilibrium

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$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi^*_\pm \end{pmatrix}$$

- connection to operator formalism: compact functional formulation of Kubo-Martin-Schwinger boundary condition: for any two operators A,B,

$$\langle A(t)B(t')\rangle = \langle B(t' - i\beta)A(t)\rangle.$$

$$\langle \mathcal{O} \rangle = \mathrm{tr}(\mathcal{O}\rho)$$

- reason:

$$A(t) = e^{iHt} A e^{-iHt}, \rho = e^{-\beta H}/\mathrm{tr} e^{-\beta H}$$

$$\Rightarrow A(t)\rho = \rho A(t - i\beta)$$

& cyclic invariance

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$$\mathcal{T}_\beta \Phi_\pm(t, \mathbf{x}) = \Phi_\pm^*(-t \pm \mathrm{i}\beta/2, \mathbf{x})$$
$$= e^{\pm i \frac{\beta}{2} \partial_t} \Phi_\pm^*(-t, \mathbf{x})$$

$$\Phi_\pm = \begin{pmatrix} \phi_\pm \\ \phi_\pm^* \end{pmatrix}$$

$$\beta = 1/T$$

- semiclassical limit: T large => $e^{\pm i \frac{\beta}{2} \partial_t} \approx 1 \pm i \frac{\beta}{2} \partial_t$

irrelevant by power counting

$$\mathcal{T}_\beta \phi_c(t, \mathbf{x}) = \phi_c^*(-t, \mathbf{x}) + \frac{i}{2T} \partial_t \cancel{\phi_q^*(-t, \mathbf{x})},$$
$$\mathcal{T}_\beta \phi_q(t, \mathbf{x}) = \phi_q^*(-t, \mathbf{x}) + \frac{i}{2T} \partial_t \phi_c^*(-t, \mathbf{x})$$

reproduces classical result

H. K. Janssen (1976); C. Aron et al, J Stat. Mech (2011)

Geometric Interpretation

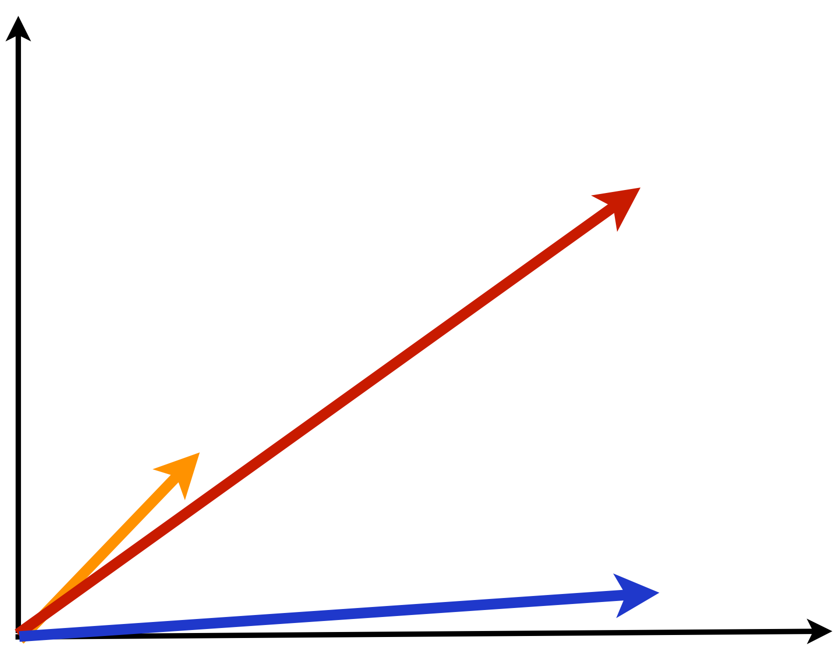
- couplings spanning the Keldysh action lie in the complex plane

$$\underbrace{\partial_t \rho = -i[H,\rho]}_{\Leftrightarrow\, S_H} + \underbrace{\mathcal{D}[\rho]}_{\Leftrightarrow\, S_{\mathcal{D}}} \qquad \Leftrightarrow \qquad Z = \int \mathcal{D}(\Phi_+,\Phi_-) e^{\mathrm{i}(S_H[\Phi_+,\Phi_-]+S_{\mathcal{D}}[\Phi_+,\Phi_-])}$$

Im

incoherent/ irrev. dynamics

$\Leftrightarrow S_{\mathcal{D}}$



Re

coherent/ reversible dynamics

$\Leftrightarrow S_H$

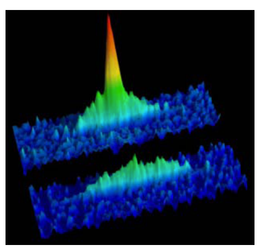
example: two-body processes λ

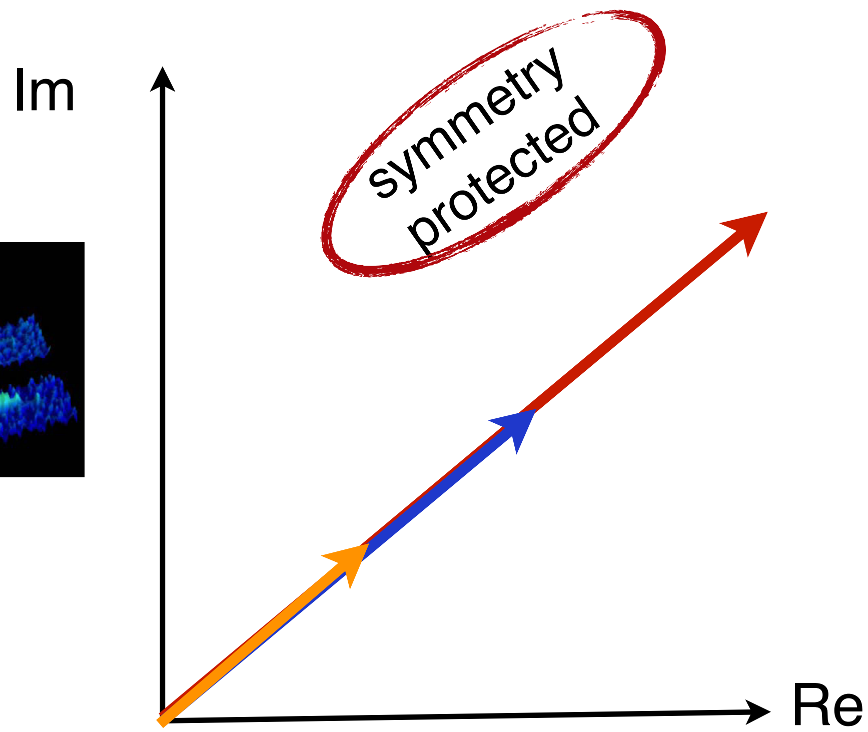
$\mathrm{Re}\lambda$ elastic two-body collisions

$\mathrm{Im}\lambda$ inelastic two-body losses

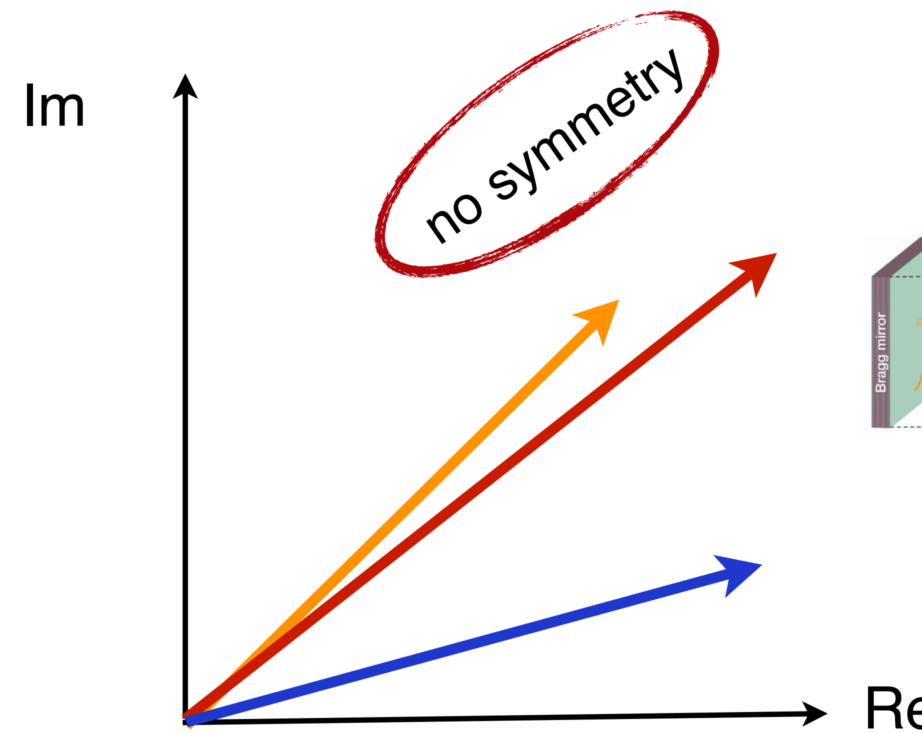
Equilibrium vs. Non-Equilibrium Dynamics

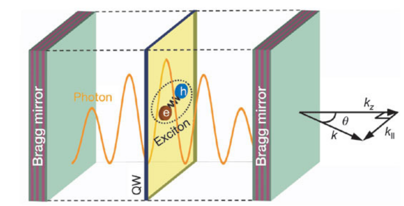
equilibrium dynamics





non-equilibrium dynamics





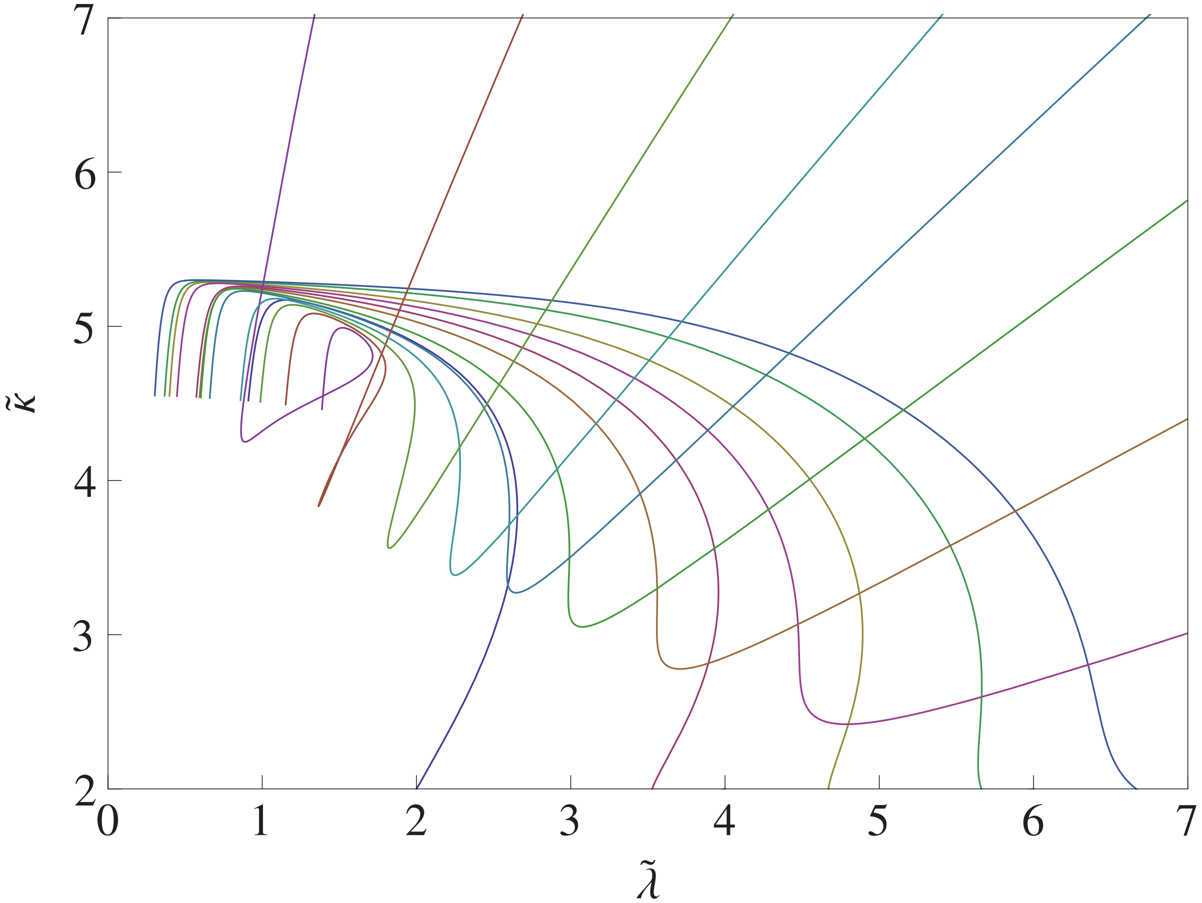
- coherent and dissipative dynamics may occur simultaneously
- but they are not independent

- coherent and dissipative dynamics do occur simultaneously
- they result from different dynamical resources

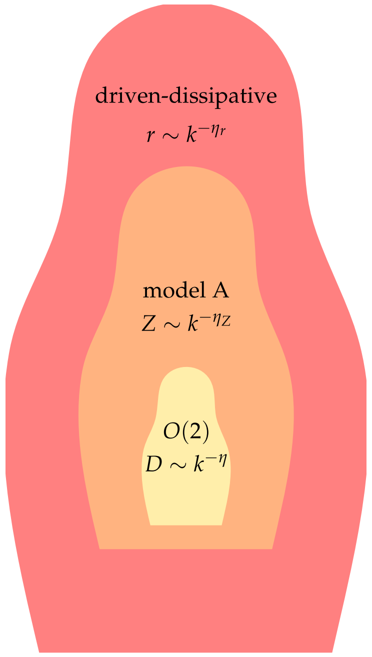
➡ measuring coupling ratios gives access to non-equilibrium conditions via static observables

➡ what are the physical consequences of the spread in the complex plane?

Part II:
Applications



Application I:
Driven Classical and Quantum Criticality



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PRL 110, 195301 (2013) and PRB 89, 134310 (2014);
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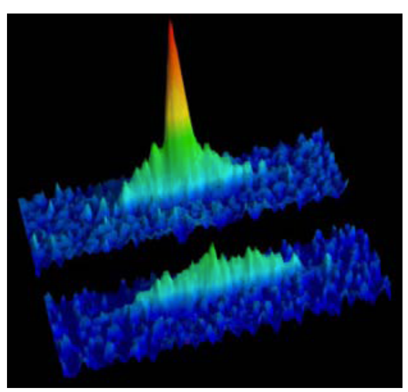
J. Marino, SD, PRL (2016) and arxiv:2016

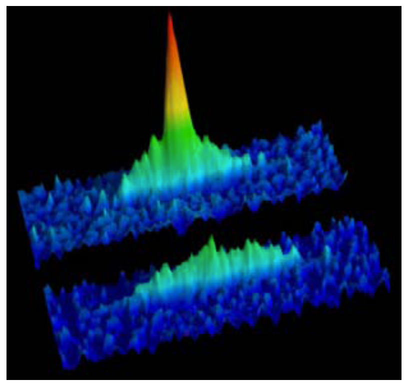
Critical Phenomena and Universality
(Equilibrium)



$\simeq$

Critical Phenomena and Universality (Equilibrium)

- Universality: The art of systematically forgetting about details



Bose-Einstein Condensate

$$\simeq$$

planar magnets

at the critical point

$$\tau = \frac{T-T_c}{T} \to 0$$

- The experimental witnesses: Critical exponents, e.g.

$$\langle \phi^*(r)\phi(0)\rangle \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}$$

correlation length

$$\xi \sim |\tau|^{-\nu} \to \infty$$

- The exponents:

ν “mass/gap exponent”

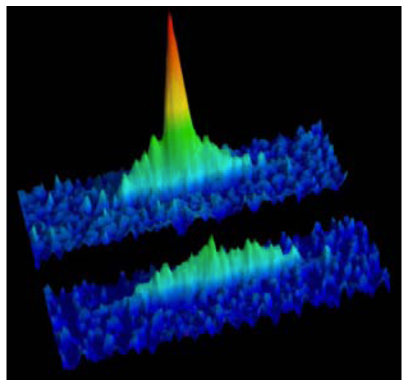
η “anomalous dimension”

nontrivial statement:
no more independent exponents *
than these!

* finite T equilibrium

Critical Phenomena and Universality (Equilibrium)

- Universality: The art of systematically forgetting about details

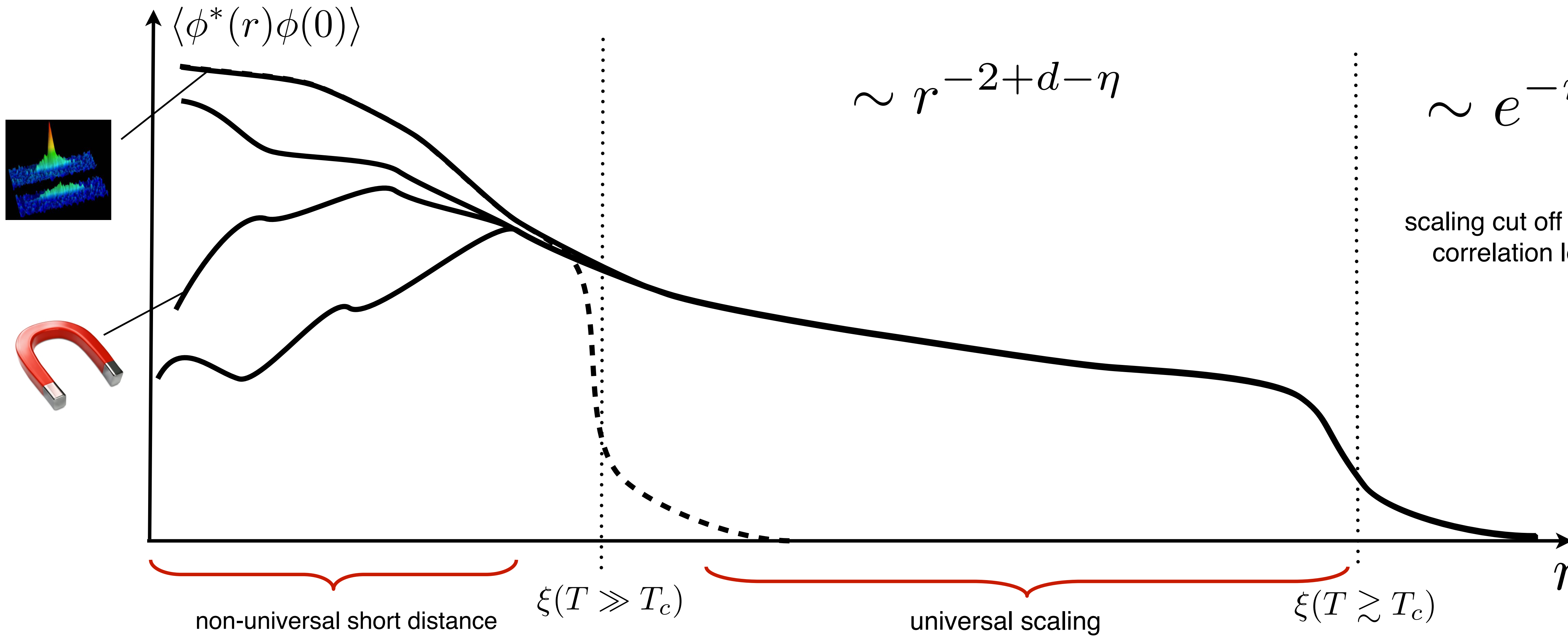


Bose-Einstein Condensate

$\simeq$

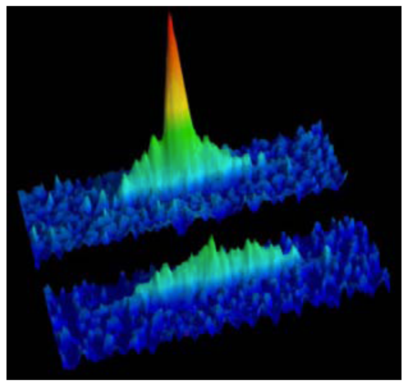
planar magnets

- The physical picture: universality induced by divergent correlation length



Critical Phenomena and Universality (Equilibrium)

- Universality: The art of systematically forgetting about details



Bose-Einstein Condensate

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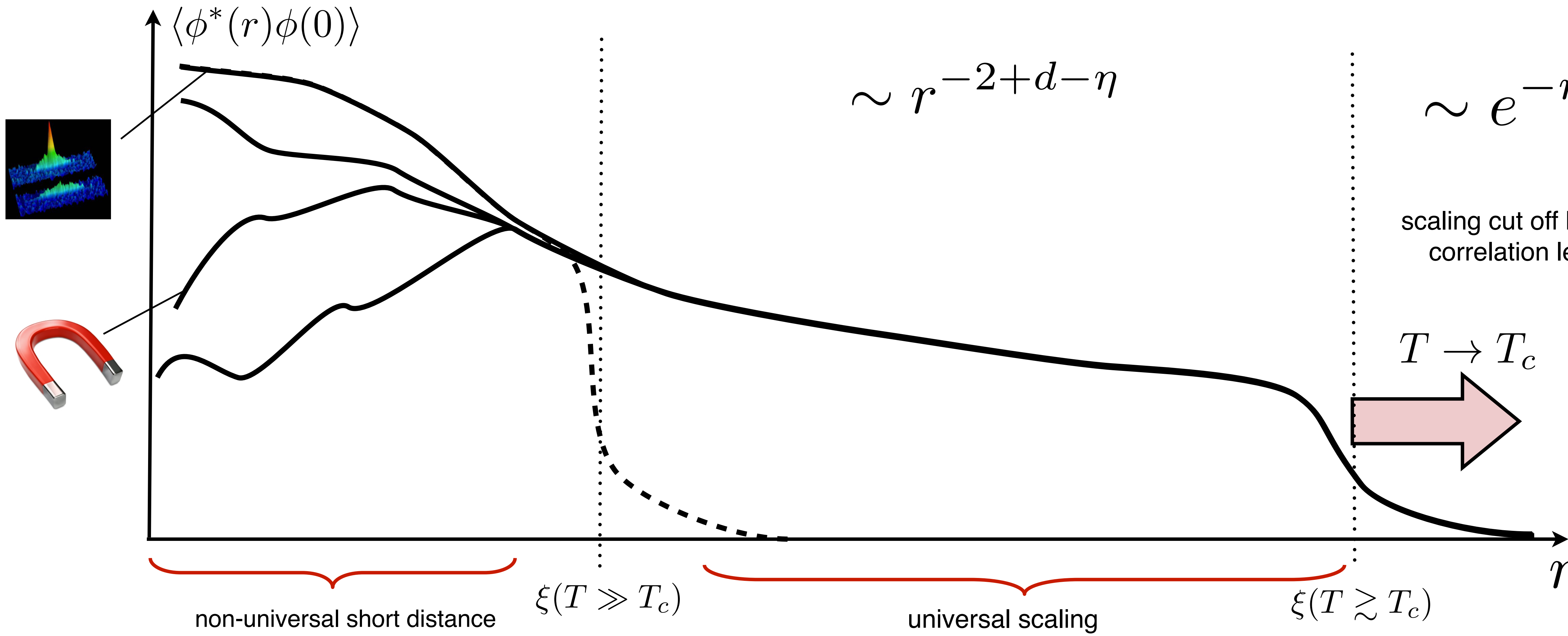
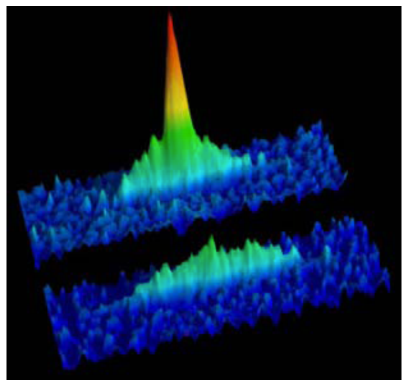


Critical Phenomena and Universality (Equilibrium)

- Universality: The art of systematically forgetting about details



Bose-Einstein Condensate

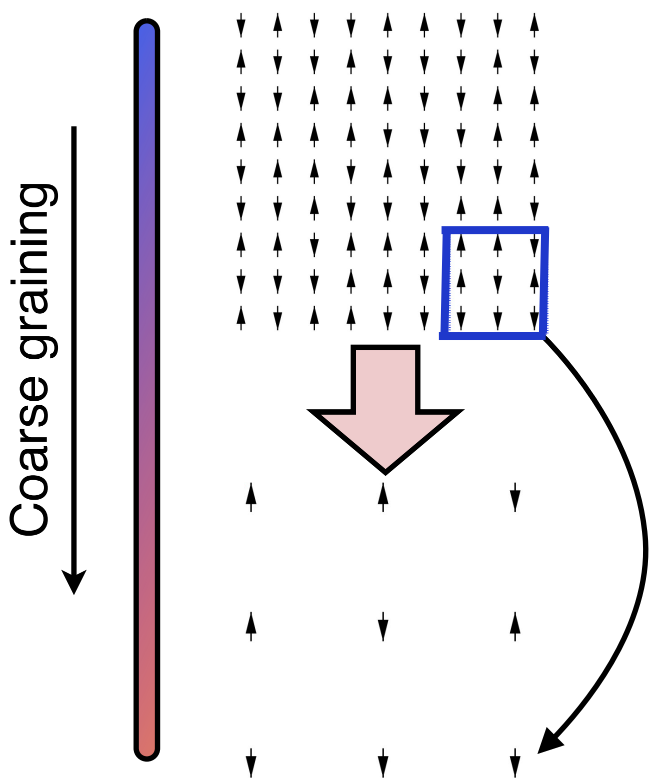
$\simeq$

planar magnets

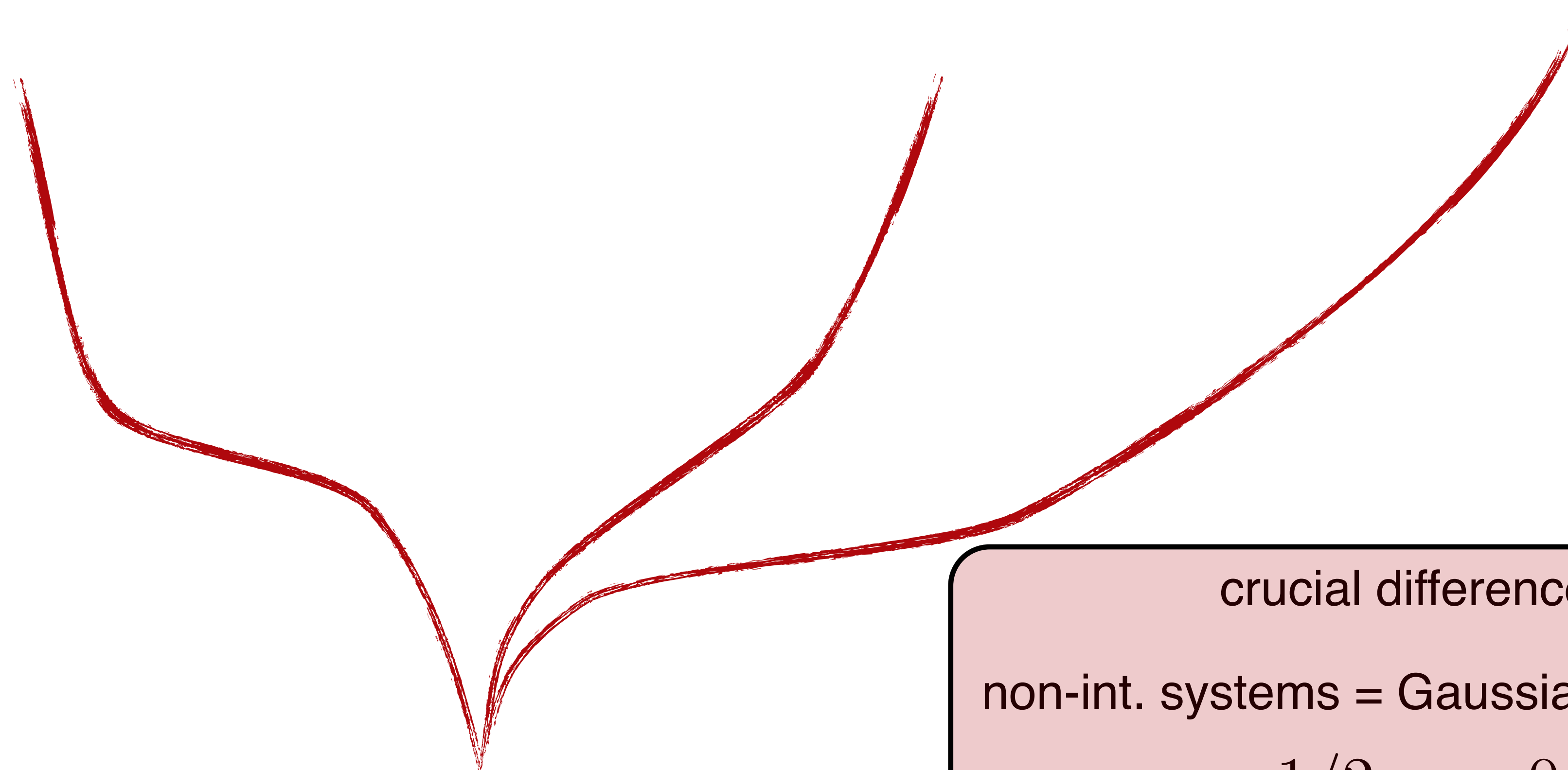
other systems...

- The description: Renormalization group

UV: microscopic physics



IR: long-wavelength physics



Wilson-Fisher fixed point

crucial difference:

non-int. systems = Gaussian fixed point

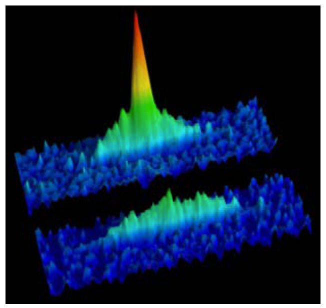
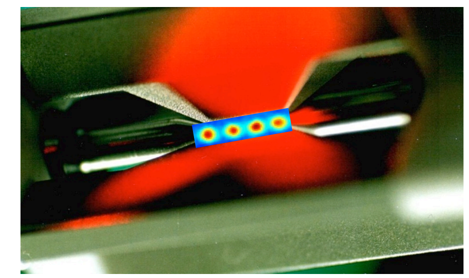
$$\nu = 1/2, \eta = 0$$

interacting systems = WF fixed point

ν, η non-rational

Universality Classes (Equilibrium)

- Universality classes: Memory of symmetries is kept

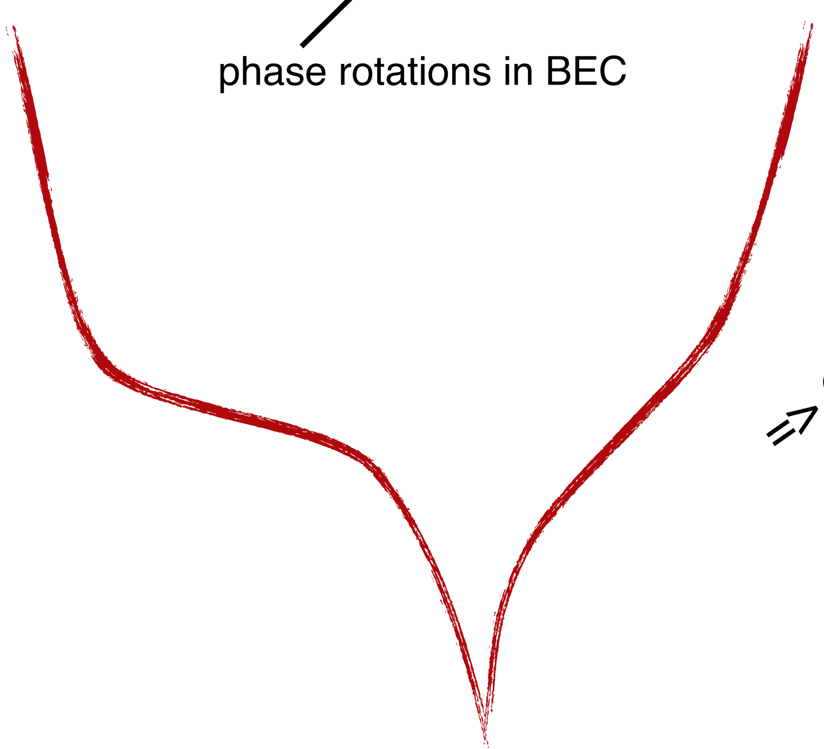
 $\simeq$  $\neq$  $\simeq$ 

Bose-Einstein Condensate | planar magnets | trapped ions | liquid-gas transition in carbon-dioxide

- Symmetries: $U(1) \simeq O(2)$ | Z_2

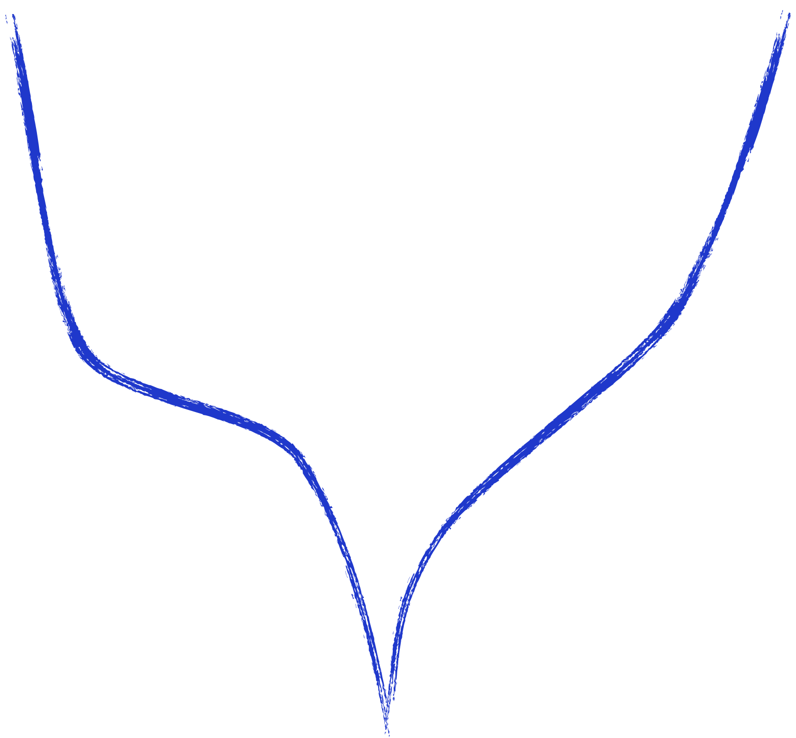
phase rotations in BEC

Coarse graining



~ 80 stable elements
=> O(10^10) possible compounds
~ 10^23 particles

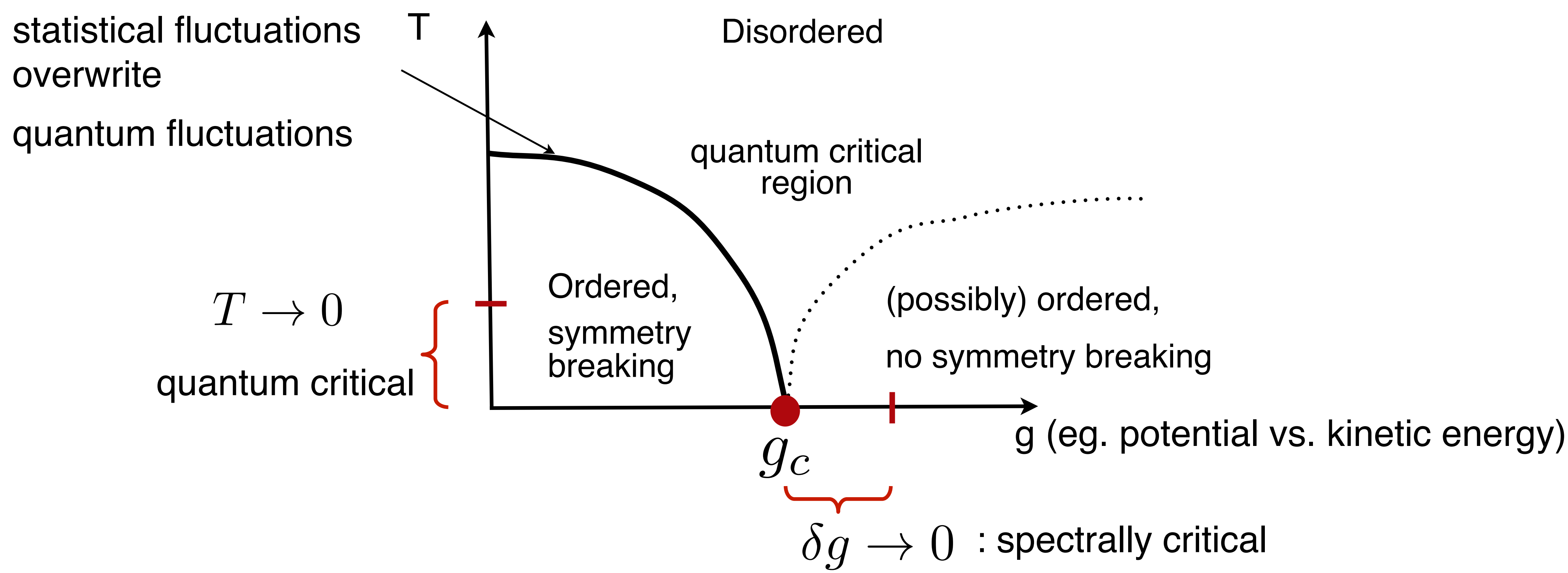
but only a handful universality classes



"O(2) universality class" | "Ising universality class"

Classical vs. Quantum Criticality

- generic quantum phase diagram



- double fine tuning, temperature is relevant perturbation to the quantum critical point

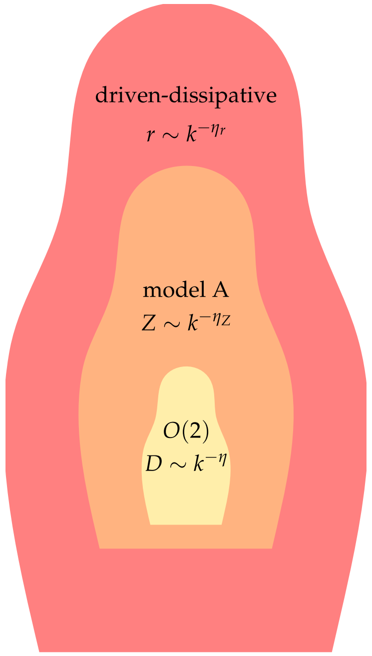
- quantum critical scaling for

$$T \ll \omega \ll \omega_G$$

quantum

non-gaussian

Driven Classical and Quantum Criticality



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J. Marino, SD, PRL (2016) and arxiv:2016

From Micro- to Macrophysics: Functional RG

microphysics → macrophysics

Many-Body Master Equation $\xrightarrow{\text{1-1 mapping}}$ Keldysh functional integral $\xrightarrow{\text{1-1 mapping}}$ Keldysh Functional Renormalization Group

$$\partial_t \rho = -i[H,\rho] + \mathcal{D}[\rho]$$

operator representation

$$e^{i\Gamma[\Phi]} = \int \mathcal{D}\delta\Phi e^{iS_M[\Phi+\delta\Phi]}$$

functional integral representation

$$\partial_k \Gamma_k = \frac{i}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_k R_k\right.$$

functional differential equation rep.

Wetterich, 93

closed system Keldysh:
Gasenzer, Pawlowski,PLB 08;
Berges, Hoffmeister, Nucl. Phys. B, 09

open system Keldysh review
Sieberer, Buchhold, SD, arxiv (2015)

From Micro- to Macrophysics: Functional RG

microphysics

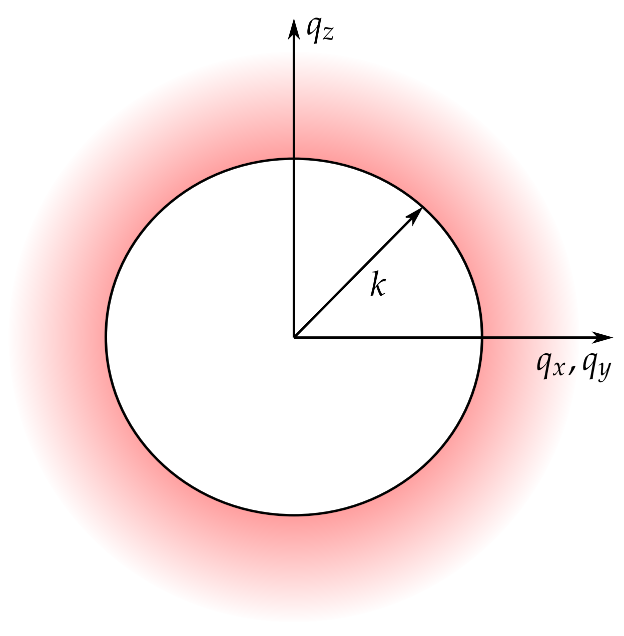
macrophysics

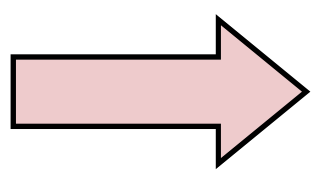
- how does it work?
  Smooth interpolation

$$\partial_k \Gamma_k = \frac{\mathrm{i}}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_k R_k \right]$$

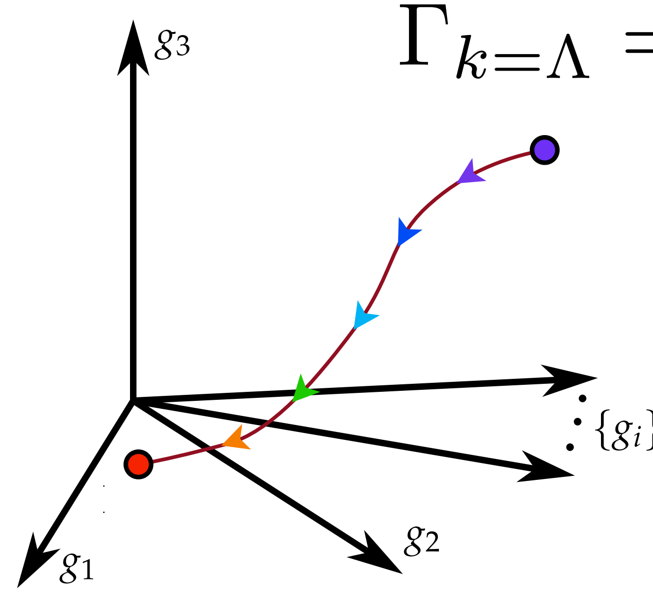
second field variation

infrared regulator





$\Gamma_{k=\Lambda} = S$ Markovian dissipative action



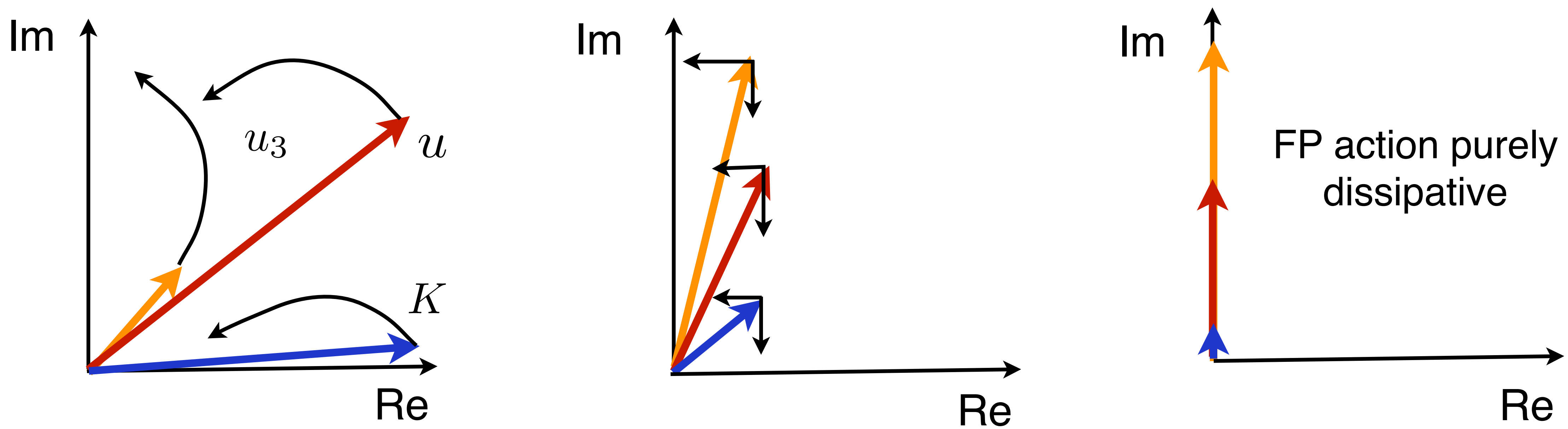
$\Gamma_{k=0} = \Gamma$ full effective action

coarse graining in real space = integrating out high modes in momentum space

mode elimination induces RG flow of coupling of effective action

Classical driven criticality: Schematic RG flow

- Flow in the complex plane of couplings



non-linear initial flow | linearized IR flow | fixed point

- initial values: $\Gamma_{k\approx\Lambda_0} \approx S$

- universal domain encoding universality class
- scaling of running couplings

$$g = ak^{\eta_a} + ibk^{\eta_b}$$

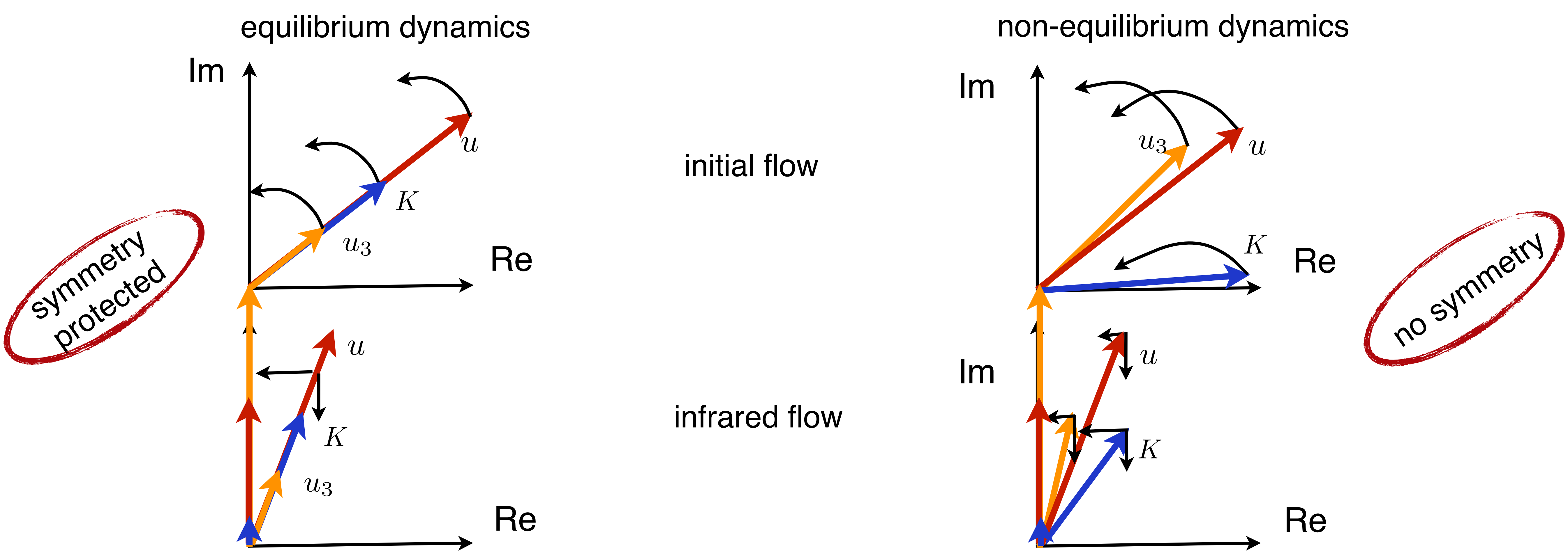
crit. exponent

- key results (classical):

➡ universal decoherence (new independent critical exponent)
➡ asymptotic thermalization
➡ reveals equilibrium vs. non-equilibrium fine structure

Universal decoherence, fine structure, and thermalization

- decoherence <=> purely imaginary fixed point action
- global thermal equilibrium is ensured by symmetry:



- eigenvalue of flow speed

$$\eta_R \approx -0.143$$

- lowest eigenvalue

$$\eta_r \approx -0.101$$

➡ equilibrium and driven systems are in different universality classes

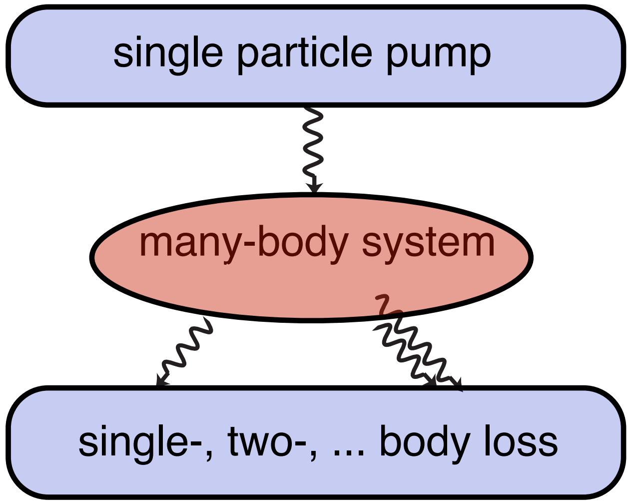
➡ physical reason: independence of coherent and dissipative dynamics

➡ asymptotic thermalization: all couplings aligned on Im axis

Non-equilibrium analogue of quantum criticality (1D)

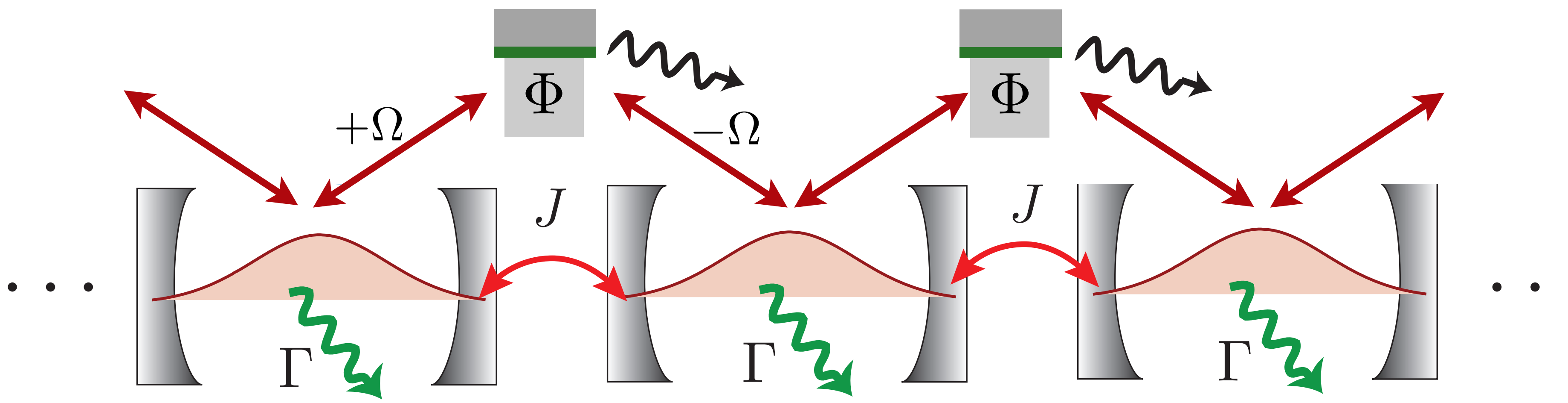
- Lindblad Master equation with additional strong quantum diffusion (1D)

$$\gamma_d \int_{\mathbf{x}} [\nabla a(x)\, \rho\, \nabla a^\dagger(x) - \tfrac{1}{2}\{\nabla a^\dagger(x) \nabla a(x), \rho\}]$$



- possible realization: microcavity arrays

cf. D. Marcos et al., NJP (2012)



$$H_c = \Omega \sum_i \sigma_i^+ (a_i - a_{i+1}) + h.c.$$

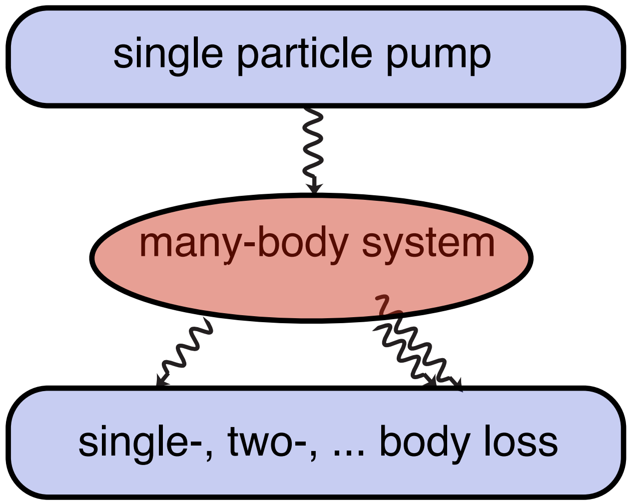
$$\Omega \ll \gamma_q$$

$$\mathcal{D}[\rho] = \gamma_q \sum_i [\sigma_i^- \,\rho\, \sigma_i^+ - \tfrac{1}{2}\{\sigma_i^+ \sigma_i^-, \rho\}]$$

Non-equilibrium analogue of quantum criticality (1D)

- Lindblad Master equation with additional strong quantum diffusion (1D)

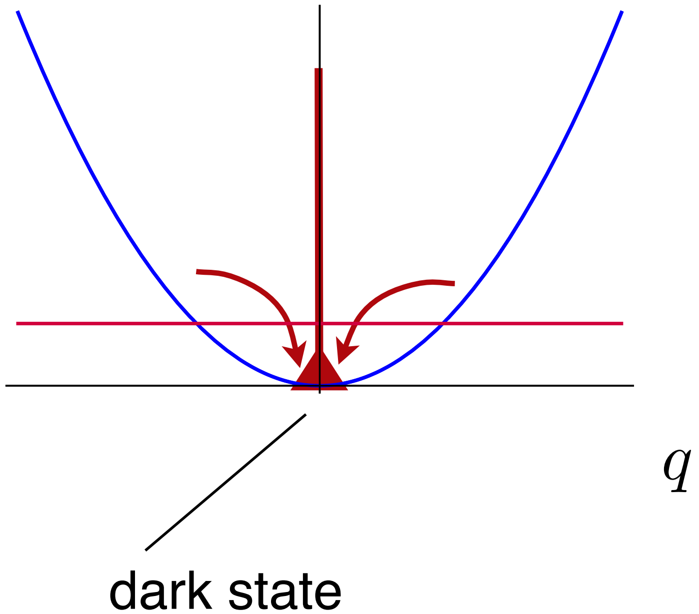
$$\gamma_d \int_{\mathbf{x}} [\nabla a(x)\, \rho\, \nabla a^\dagger(x) - \tfrac{1}{2}\{\nabla a^\dagger(x) \nabla a(x), \rho\}]$$



- physical interpretation: Dark state number conserving variant: SD et al., Nature Phys. (2008)

  - in Fourier space

$$\int_q \gamma_q [a_q\, \rho\, a_q^\dagger - \tfrac{1}{2}\{a_q^\dagger a_q, \rho\}]$$

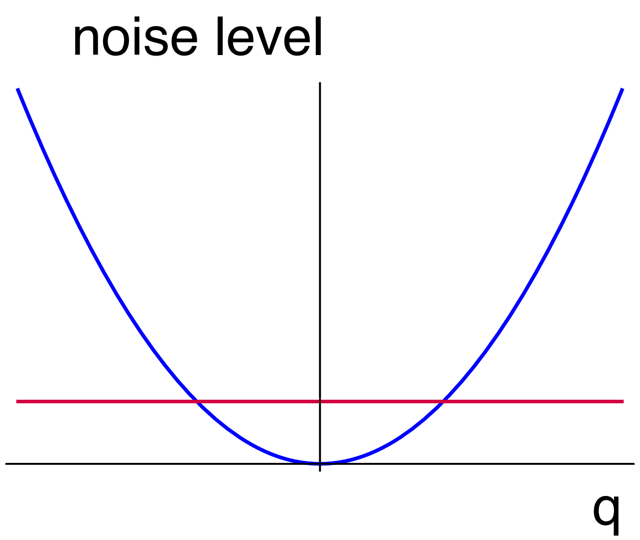


➡ noiseless "dark" state at q=0

➡ favors accumulation of bosons at q=0 ("BEC")

➡ competition w/ interactions yields phase transition

“What is quantum about it?”



- key point: scaling of noise level changes the field scaling dimensions

classical

$$P^R(\omega, \mathbf{q}) \sim q^2 \qquad [\phi_c] = \frac{d-2}{2}$$
$$P^K(\omega, \mathbf{q}) \sim q^0 \qquad [\phi_q] = \frac{d+2}{2}$$

quantum

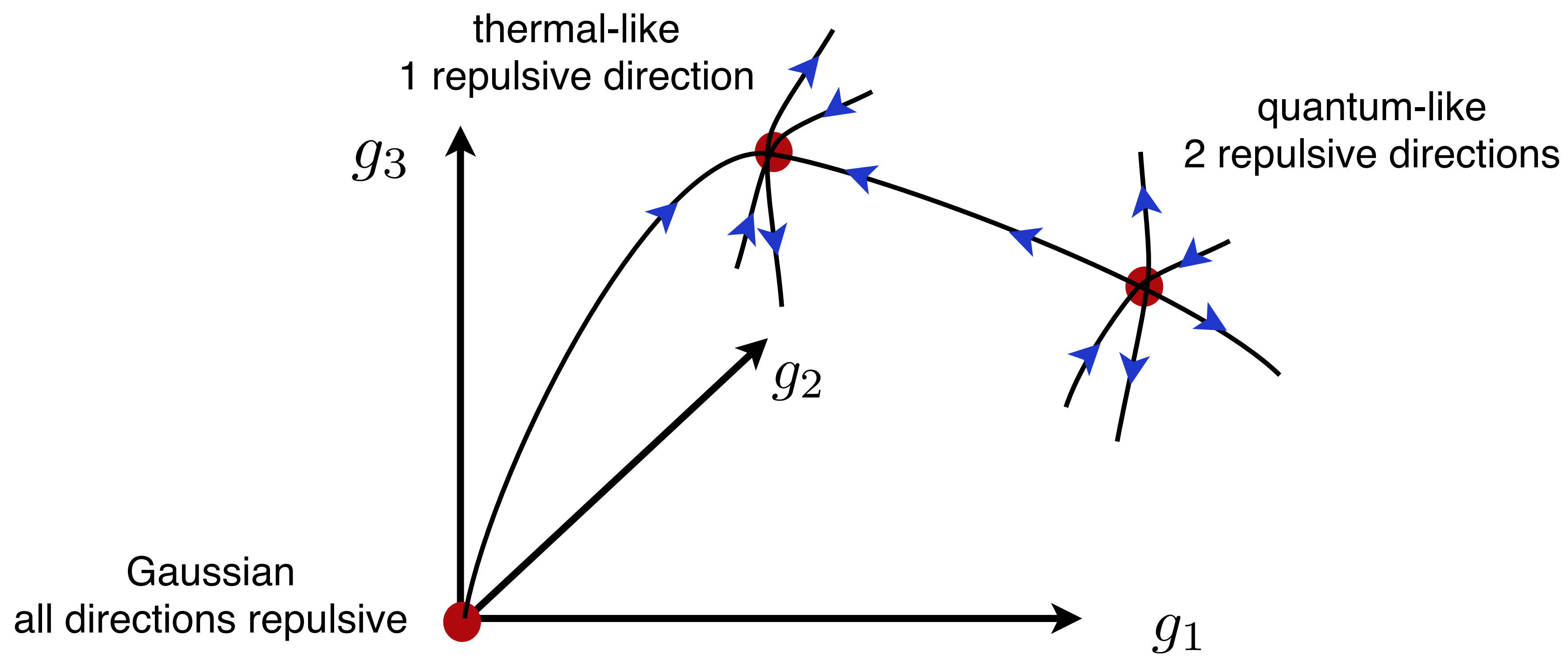
$$P^R(\omega, \mathbf{q}) \sim q^2 \qquad [\phi_c] = \frac{d}{2}$$
$$P^K(\omega, \mathbf{q}) \sim q^2 \qquad [\phi_q] = \frac{d}{2}$$

➡ identical scaling at a zero T quantum critical point

➡ needs full quantum dynamical field theory!

(1) No quantum-classical correspondence

- new fixed point with more repulsive directions (fine tuning of loss rate)



- results for critical exponents

	Crit. Exps.	ν	η_{K_R}	η_{K_I}	η_{ZR}	η_{ZI}	η_{γ_d}	η_γ
		static			dynamic		noise	
1+2 dimensions	DD Quantum	0.405	-0.025	-0.025	0.08	0.04	-0.26	$\times$
3 dimensions	DD SC	0.72	-0.22	-0.12	0.16	0	$\times$	-0.16

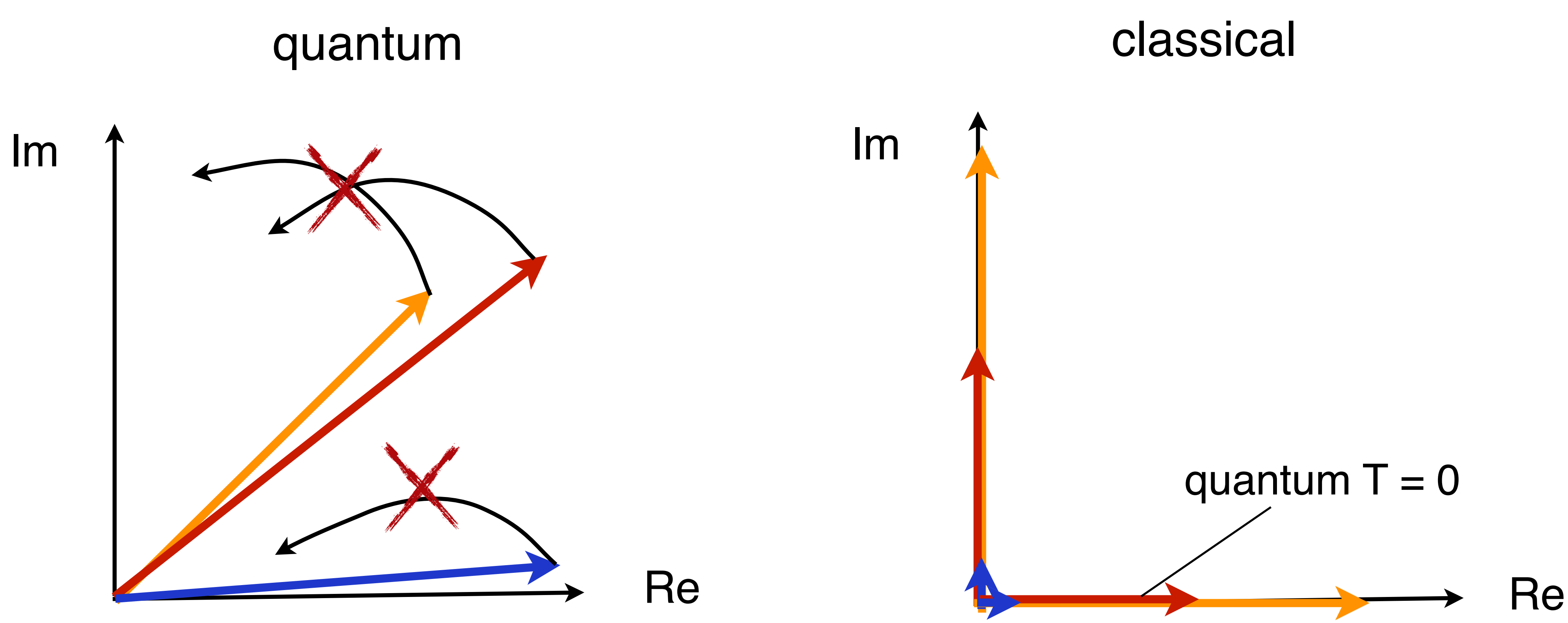
different degree of divergence of correlations length

$$\xi \sim (t - t_c)^{-\nu}$$

➡ new non-equilibrium universality class

(2) Absence of Asymptotic Decoherence

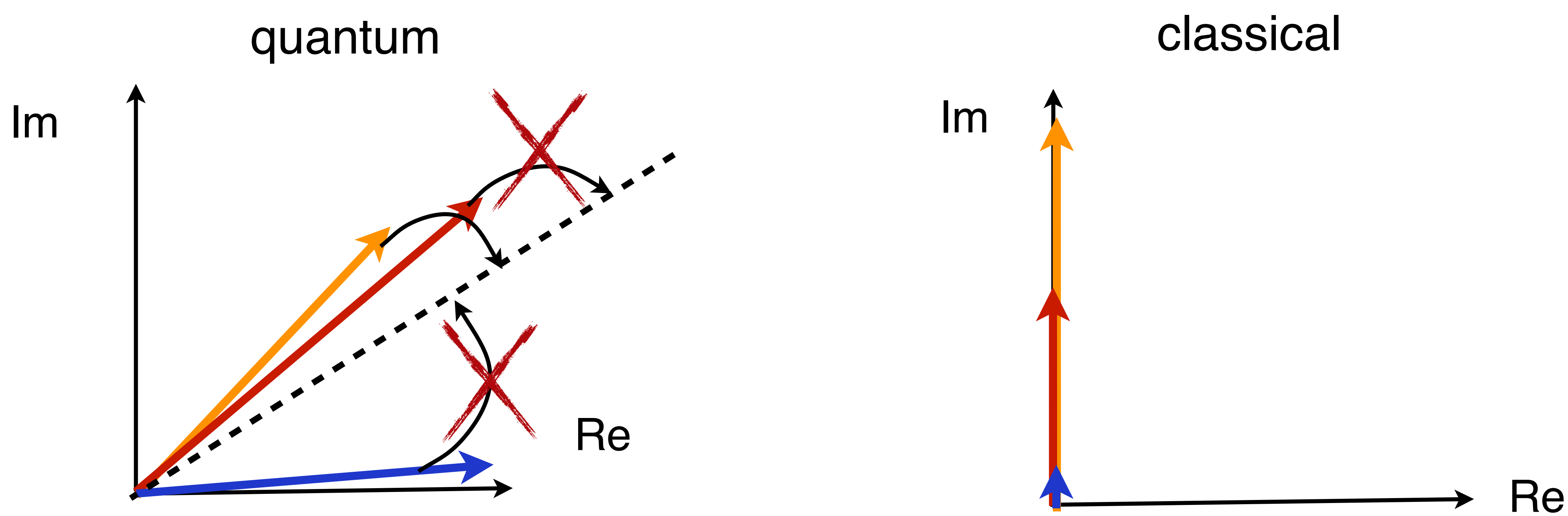
- coherent dynamics does not fade out:
- exponent degeneracy: $\eta_A = \eta_D = -0.03$ $\quad A \sim k^{\eta_A}$, "effective mass" $\quad D \sim k^{\eta_D}$ diffusion



➡ mixed fixed point with finite dissipative and coherent couplings

(3) Absence of Asymptotic Thermalization

- symmetry as straightforward diagnostic tool for Schwinger-Keldysh actions
- symmetry explicitly violated microscopically by markovian quantum dynamics
- not emergent:



- formally:

$$\boxed{Z \sim k^{\eta_Z} \qquad , \qquad \gamma_d \sim k^{\eta_{\gamma_d}}} \qquad\qquad Z \sim k^{\eta_Z}, \quad \gamma \sim k^{\eta_\gamma}$$

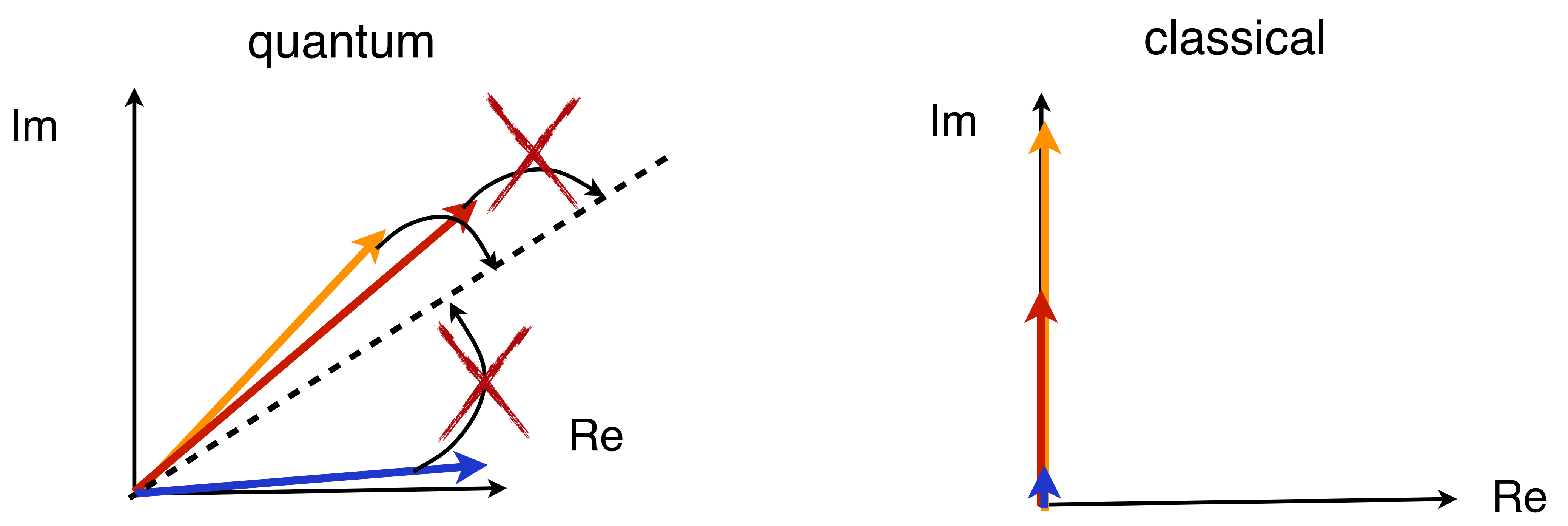
quasiparticle residue — noise level

$$\eta_Z = 0.08, \qquad , \quad \eta_{\gamma_d} = -0.26 \qquad\qquad \eta_Z = \eta_\gamma = 0.16$$

➡ microscopic and universal asymptotic violation of quantum FDR

(3) Absence of Asymptotic Thermalization

- symmetry as straightforward diagnostic tool for Schwinger-Keldysh actions
- symmetry explicitly violated microscopically by markovian quantum dynamics
- not emergent:



- formally:

$$\boxed{Z \sim k^{\eta_Z} e^{i\eta'_Z \log k/\Lambda}, \quad \gamma_d \sim k^{\eta_{\gamma_d}}} \qquad Z \sim k^{\eta_Z}, \quad \gamma \sim k^{\eta_\gamma}$$

quasiparticle residue   noise level

$$\eta_Z = 0.08, \quad \eta'_Z = 0.03, \quad \eta_{\gamma_d} = -0.26 \qquad \eta_Z = \eta_\gamma = 0.16$$

➡ limit-cycle like oscillations with (huge!) period
(observable: spectral density)

$$\frac{k_{n+1}}{k_n} = e^{\frac{2\pi}{\eta'_Z}}$$

Observable consequences of driven criticality

- static exponents: first order spatial coherence function

distance from phase transition

$$\langle \phi^*(r)\phi(0)\rangle \sim \frac{e^{-r/\xi}}{r^{1+\eta_D}} \qquad \xi \sim |\Delta|^{-\nu}$$

- dynamical exponents: experiments probing the dynamical single-particle renormalized response (RF spectroscopy for ultracold atoms, homodyne detection)

$$\chi(\omega,\mathbf{q}) \equiv G^R(\omega,\mathbf{q}) = \frac{Z^{-1}}{\omega - \omega_{\mathbf{q}}} \qquad \omega_{\mathbf{q}} \approx A\mathbf{q}^2 - iD\mathbf{q}^2$$

complex dispersion at criticality

- with anomalous behavior

$$Z \sim |\mathbf{q}|^{\eta_Z} e^{i\eta'_Z \log|\mathbf{q}|/\Lambda}$$

$$A \sim |\mathbf{q}|^{\eta_A}, \quad D \sim |\mathbf{q}|^{\eta_D} \qquad \eta_A = \eta_D$$

(absence of decoherence)

Recap

- construction: Many-Body Master Equation $\xrightarrow{\text{1-1 mapping}}$ Keldysh functional integral

$$\partial_t \rho = -i[H,\rho] + \mathcal{L}[\rho] \qquad Z = \int \mathcal{D}(\Phi_+,\Phi_-) \mathrm{e}^{\mathrm{i}(S_H[\Phi_+,\Phi_-]+S_{\mathcal{D}}[\Phi_+,\Phi_-])}$$

stochastic GPE

- semiclassical limit: $$Z = \int \mathcal{D}[\xi,\xi^*]\, e^{-\frac{1}{2\gamma}\int_{t,\mathbf{x}} \xi^* \xi} \int \mathcal{D}[\phi_c,\phi_c^*] \delta\left(i\partial_t \phi_c - \frac{\delta \mathcal{H}_c}{\delta \phi_c^*} + i\frac{\delta \mathcal{H}_d}{\delta \phi_c^*} - \xi \right) \delta\,(c.c.)$$

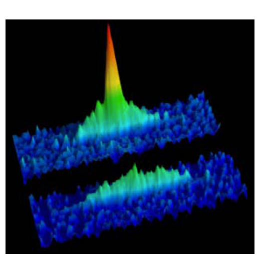
- “what is non-equilibrium about it?”: time independent Hamiltonian => symmetry: $\mathcal{T}_\beta \Phi_\pm(t,\mathbf{x}) = \Phi_\pm^*(-t \pm \mathrm{i}\beta/2, \mathbf{x})$
  - implication 1 (Ward identity): Fluctuation-dissipation relations, e.g. single particle sector:

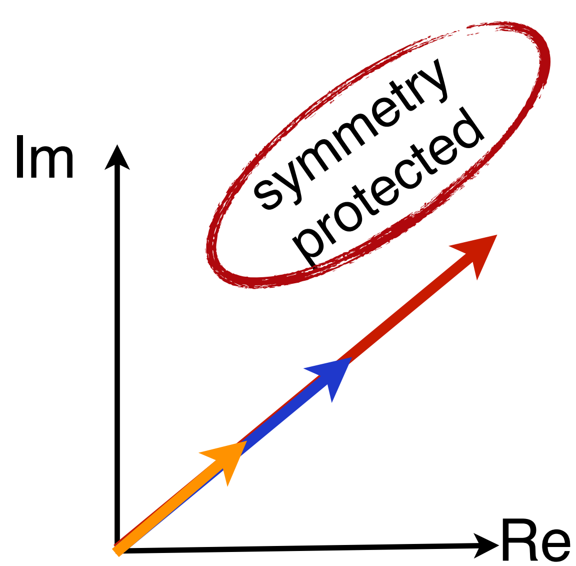
$$G^K(\omega,\mathbf{q}) = (2n_B(\omega/T)+1)[G^R(\omega,\mathbf{q}) - G^A(\omega,\mathbf{q})]$$

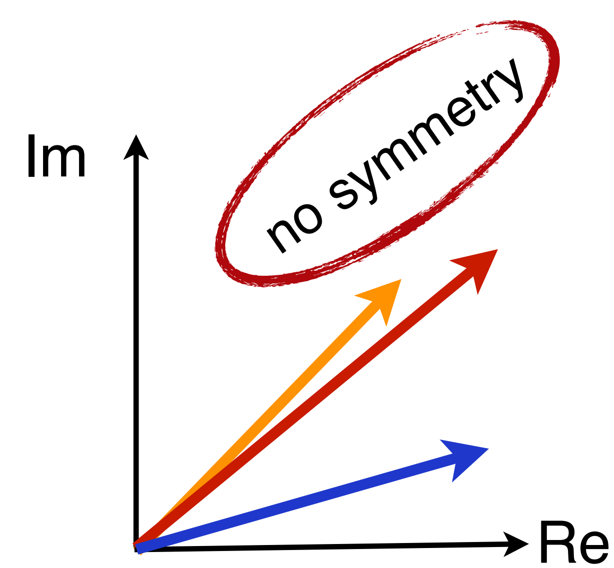
correlations Bose distribution responses

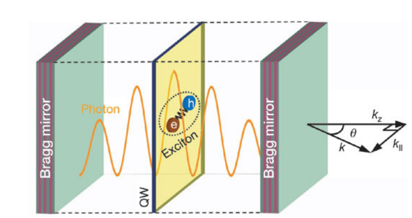
any order <=> detailed balance

- implication 2 (geometric constraint):



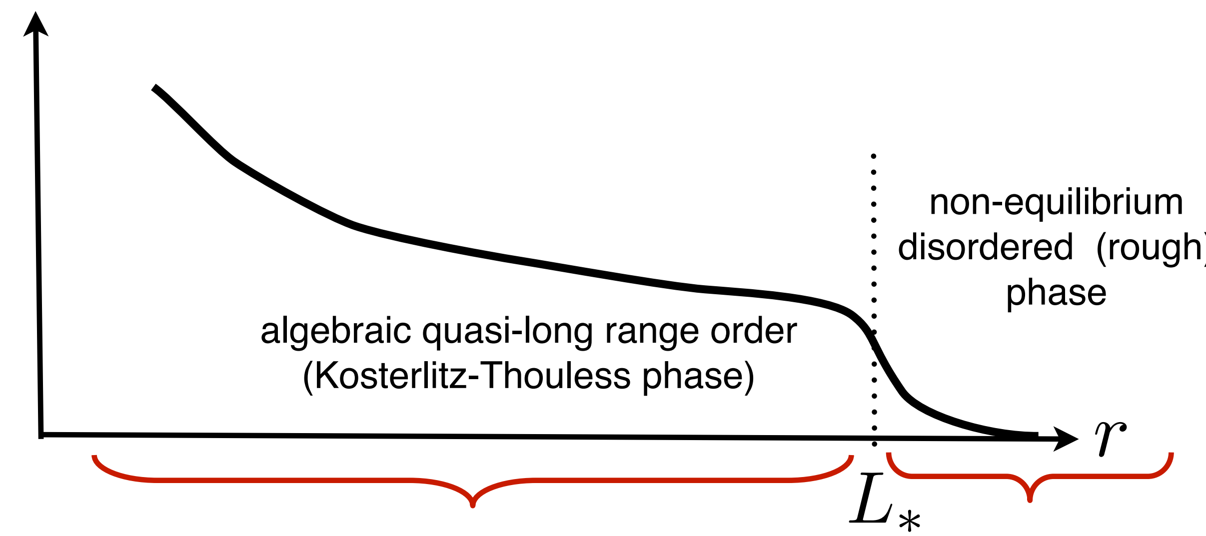






➡ implications?

Application II:
Universal long wavelength behavior in low dimensional Driven Open Quantum Systems



E. Altman, L. Sieberer, L. Chen, SD, J. Toner, PRX (2015)
G. Wachtel, L. Sieberer, SD, E. Altman, arxiv:1604.01042
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L. He, L. Sieberer, SD, in preparation

Microscopic
Quantum Optics

“Thermodynamic”
Many-body physics

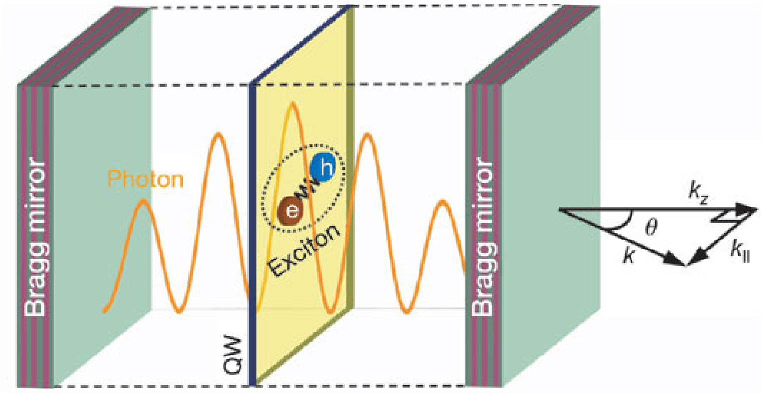
Long wavelength
Statistical mechanics

Program

- driven-dissipative stochastic GPE

$$i\partial_t\phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)\,|\phi|^2\right]\phi + \zeta$$

Exciton-Polaritons



Kasprzak et al., Nature 2006

- decompose into amplitude and phase fluctuations

$$\phi(\mathbf{x},t) = (M_0 + \chi(\mathbf{x},t))e^{i\theta(\mathbf{x},t)}$$

- integrate out fast amplitude fluctuations:

$$\partial_t\theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

phase diffusion   phase nonlinearity   Markov noise

$$\langle\xi(\mathbf{x},t)\xi(\mathbf{x}',t')\rangle = 2\Delta\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

effective noise level

form of the KPZ equation

Kardar, Parisi, Zhang, PRL (1986)

➡ physics of the KPZ equation

➡ implications for low dimensional driven open systems

KPZ equation

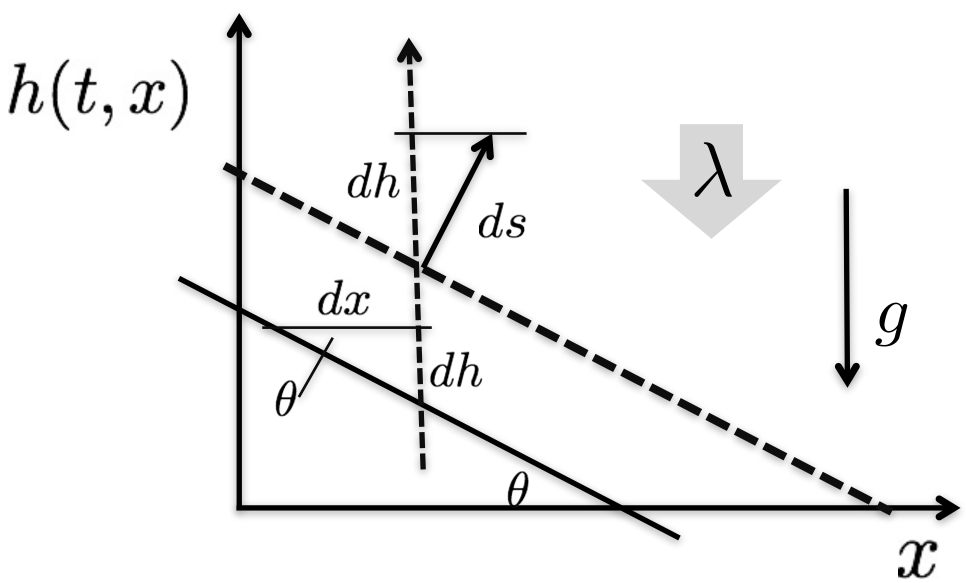
- Point particles: Brownian motion

$$\partial_t n(t,\mathbf{x}) = D\nabla^2 n(t,\mathbf{x}) + \xi(t,\mathbf{x})$$

- Q: analogue of Brownian motion of surfaces?

➡ Qualitatively distinct in the presence of drive (geometric effect)

Kardar, Parisi, and Zhang, PRL (1986)



deposition rate

- growth: $$ds = \lambda dt$$

- geometry: $$dh = \frac{ds}{\sqrt{1+\left(\frac{dh}{dx}\right)^2}} \approx \lambda dt \left(1 - \frac{1}{2}\left(\frac{dh}{dx}\right)^2\right)$$

➡ Brownian motion corrected by terms ~ λ

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \lambda - \frac{\lambda}{2}\left|\nabla h\right|^2 + \xi$$

Properties from a phase analogy

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \lambda - \frac{\lambda}{2}\left|\nabla h\right|^2 + \xi$$

Consider behavior of complex field $\psi(t,\mathbf{x}) = \rho(t,\mathbf{x})e^{i\theta(t,\mathbf{x})}; h \cong \theta$

1) comoving/rotating frame transformation = time-local gauge transformation

$$\psi(t,\mathbf{x}) \to e^{i\lambda t}\psi(t,\mathbf{x}) \quad \text{i.e.} \quad \theta(t,\mathbf{x}) \to \theta(t,\mathbf{x}) \to \theta(t,\mathbf{x}) + \lambda t$$

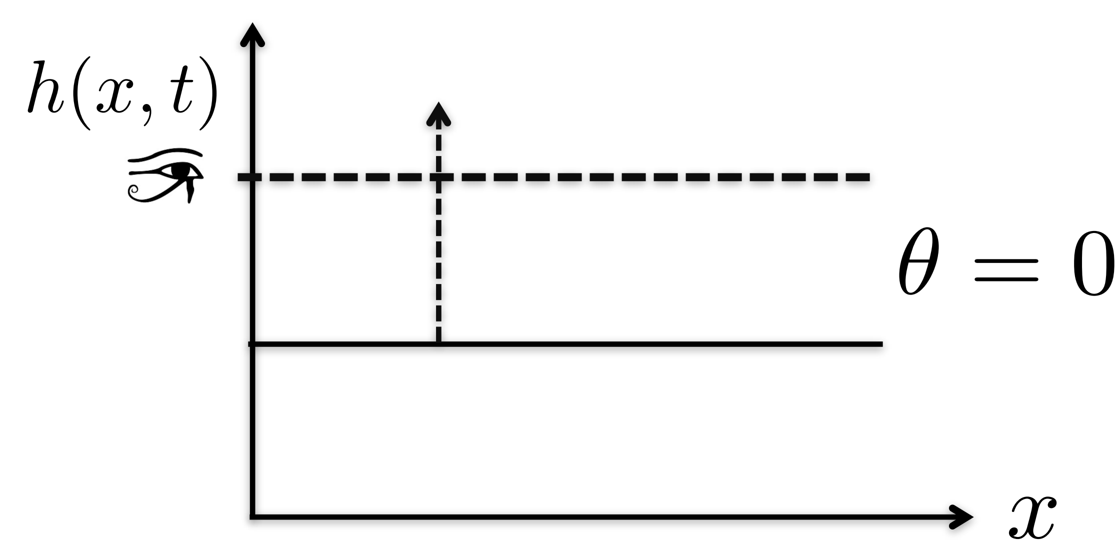
➡ absorb free λ (describes average growth of interface)

$$\frac{\partial h}{\partial t} = D\nabla^2 h - \frac{\lambda}{2}\left|\nabla h\right|^2 + \xi \qquad \text{KPZ equation}$$

➡ λ has a nontrivial effect only under nonequilibrium condition!

indeed $\theta = 0 \Longrightarrow dh = gs = \lambda dt$ => linear equation

balance of forces



Properties from a phase analogy

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \lambda - \frac{\lambda}{2}|\nabla h|^2 + \xi$$

2) scale invariance = global gauge invariance

$$\psi(t,\mathbf{x}) \to e^{i\alpha}\psi(t,\mathbf{x}) \qquad \text{i.e. } \theta(t,\mathbf{x}) \to \theta(t,\mathbf{x}) + \alpha$$

➡ EoM remains gapless: “self-organized criticality”

scaling of correlation functions, e.g.

$$H(t,\mathbf{x}) \equiv \left\langle [h(t,\mathbf{x}) - h(0,0)]^2 \right\rangle = |\mathbf{x}|^{2\chi} f_{\mathrm{KPZ}}\left(\frac{t}{|\mathbf{x}|^z}\right)$$

$$f_{\mathrm{KPZ}}(y \to 0) = \text{constant}$$

$$f_{\mathrm{KPZ}}(y \to \infty) \sim y^{2\chi/z}$$

χ “roughness exponent”, z dynamical exponent

$\chi > 0$: height variance grows with respect to lxl: “rough phase”

$\chi < 0$: height variance shrinks with respect to lxl: “smooth phase”

Properties from a phase analogy

$$\frac{\partial h}{\partial t} = D\nabla^2 h + \lambda - \frac{\lambda}{2}\left|\nabla h\right|^2 + \xi$$

3) Galilean invariance

$$\psi(t,\mathbf{x}) \to e^{i(\frac{1}{2m}|\mathbf{q}_0|^2 t - \mathbf{q}_0\cdot\mathbf{x})}\psi(t,\mathbf{x}-\frac{\mathbf{q}_0}{m}t) \longleftrightarrow$$

$$\theta(t,\mathbf{x}) \to \theta(t,\mathbf{x}-\frac{\mathbf{q}_0}{m}t) + \frac{1}{2m}|\mathbf{q}_0|^2 t - \mathbf{q}_0\cdot\mathbf{x}$$

in notations of KPZ $\left\{\begin{array}{l}\theta = h \\ \dfrac{1}{m} = \lambda\end{array}\right.$ $\Rightarrow$ $h(t,\mathbf{x}) \to h(t,\mathbf{x}-\lambda\mathbf{q}_0 t) + \frac{\lambda}{2}|\mathbf{q}_0|^2 t - \mathbf{q}_0\cdot\mathbf{x}$

➡ A symmetry that connects the dynamical term and the nonlinear term.

$$\frac{\partial h}{\partial t} = D\nabla^2 h - \frac{\lambda}{2}\left|\nabla h\right|^2 + \xi$$

➡ The dynamical exponent is connected with the static (roughness) exponent.

$$z + \chi = 2$$

Exact relation from symmetry!

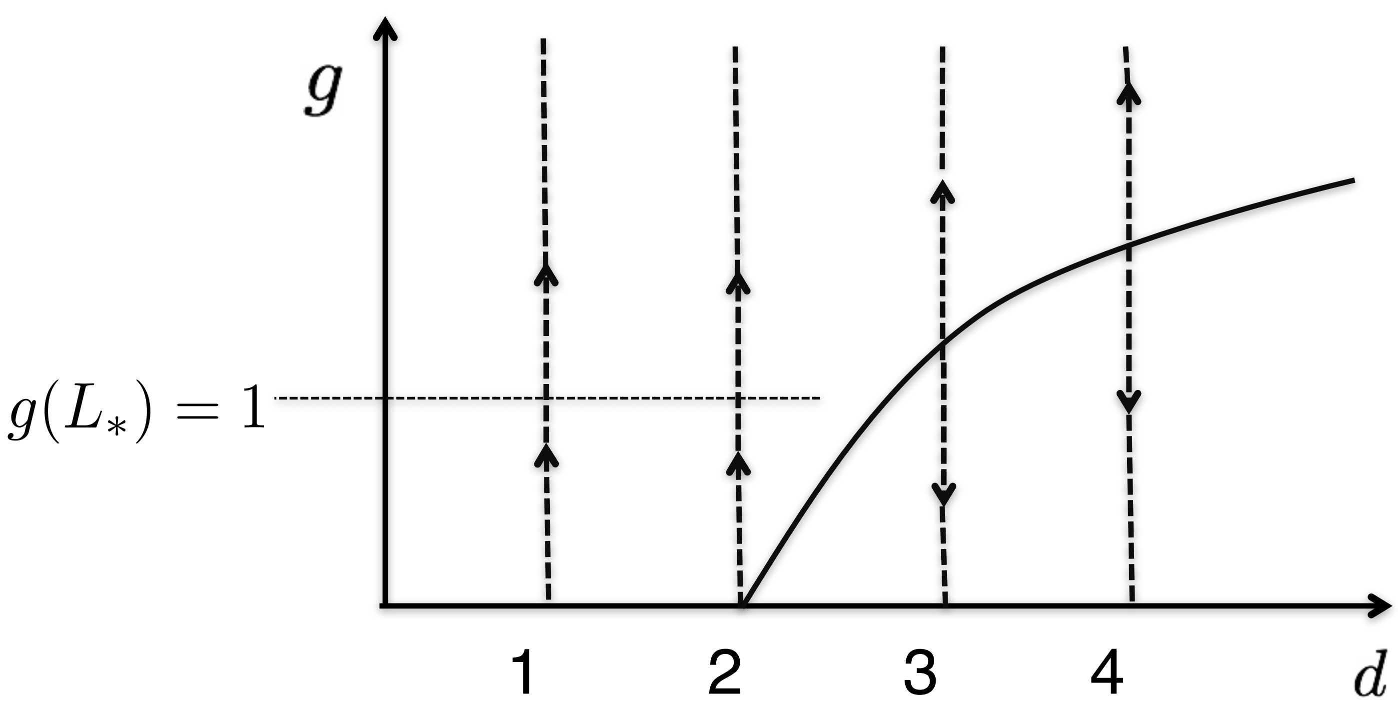
Large scale physics of KPZ equation: RG Approach

- gradually integrate out short scale fluctuations

- RG flow equation (perturbative)

noise level

$$\partial_l g = (2-d)g + k_d g^2 \qquad g = \frac{\lambda^2 \Delta}{D^3}$$



Interpretation:

$g \to \infty$ "rough phase"
strong nonequilibrium KPZ fixed point (not perturbatively accessible)

$g \to 0$ "smooth phase"
effective emergent equilibrium behavior/thermalization

KPZ equation: A paradigm of non-equilibrium stat mech

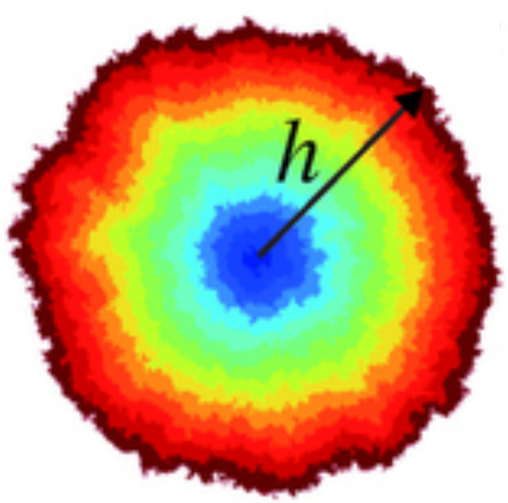
- above and originally: stochastic roughening of surface height $h(\mathbf{x},t)$

$$\partial_t h = D\nabla^2 h + \lambda(\nabla h)^2 + \xi$$

smoothens nonlinear growth noise

Kardar, Parisi, Zhang, PRL (1986)

- but multiple physical contexts



defect growth in liquid crystals

drive: electric field

from Takeuchi et al., Scientific Reports (2011)

bacterial colony growth

drive: sugar

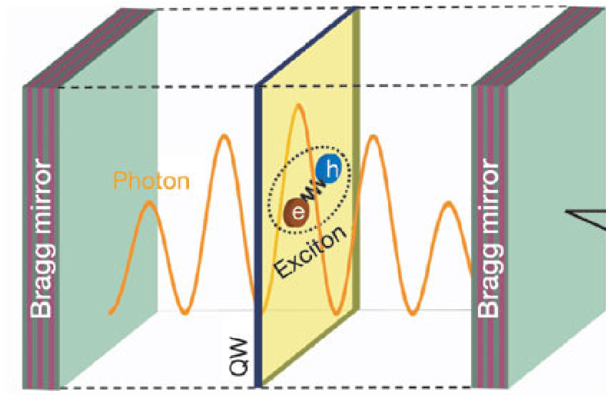
Wakita et al., J. Phys. Jpn. Soc. (1997)

burning paper

drive: oxygen

Maunuksela et al., PRL (1997)

Connection to exciton-polaritons



Kasprzak et al., Nature 2006

- driven-dissipative stochastic GPE

$$i\partial_t \phi = \left[-\frac{\nabla^2}{2m} - \mu + i(\gamma_p - \gamma_l) + (\lambda - i\kappa)|\phi|^2 \right] \phi + \zeta$$

- phase amplitude decomposition $\phi(\mathbf{x},t) = (M_0 + \chi(\mathbf{x},t))e^{i\theta(\mathbf{x},t)}$

$$\partial_t \chi = -2u_d M_0^2 \chi - k_d M_0 |\nabla\theta|^2 - k_c M_0 \nabla^2\theta + \Re(\xi) \quad (1)$$

$$M_0 \partial_t \theta = -2u_c M_0 \chi - k_d M_0 \nabla^2\theta - k_c M_0 |\nabla\theta|^2 + \Im(\xi) \quad (2)$$

- (1) is gapped: linearization justified, adiabatic elimination $\partial_t \chi \overset{!}{=} 0$ fast on scale of θ
- (2) becomes the KPZ equation

$$\Rightarrow \quad \boxed{\partial_t \theta = D\nabla^2\theta + \lambda|\nabla\theta|^2 + \xi}$$

phase diffusion KPZ nonlinearity

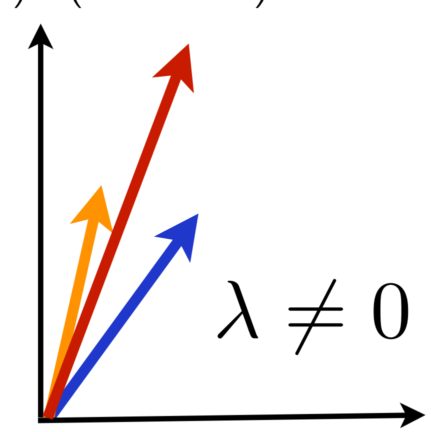
$$D = k_d(1 + R_k R_u) \qquad \lambda = 2k_c\left(\frac{R_u}{R_k} - 1\right)$$

$$R_k = \frac{k_d}{k_c};\, R_u = \frac{u_d}{u_c} \qquad \Delta = \frac{u_d^2 + u_c^2}{2r_d u_d}\sigma$$

$$\langle \xi(t',\mathbf{x}')\xi(t,\mathbf{x})\rangle = 2\Delta\delta(t-t')\delta(\mathbf{x}-\mathbf{x}')$$

equilibrium: $R = R_k = R_u = R_r \Rightarrow \lambda = 0$ protected by symmetry

$\Rightarrow \lambda \neq 0$ signals nonequilibrium



Physical implications: overview

- mapping to KPZ-type equation valid in all dimensions at low noise level / well above threshold
- fundamental difference to classical context: KPZ variable = condensate phase, compact

➡ two complementary approaches:

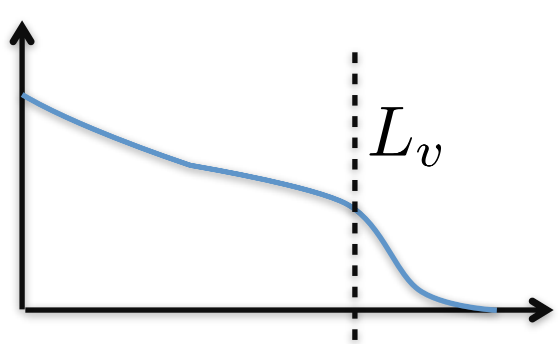
  - neglect compactness, account for KPZ RG flow -> emergent length scale L_*
  - neglect RG flow, account for compactness -> emergent length scale L_v

➡ two non-equilibrium length scales (diverge as $\lambda \to 0$) separating up to three different scaling regimes

➡ 2 dimensions:

$$L_v \ll L_*$$

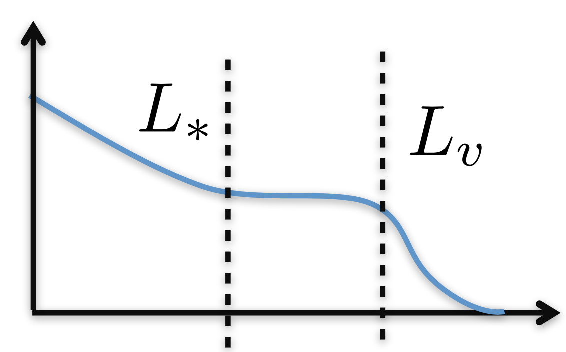
➡ two regimes: vortex proliferation overwrites KPZ scaling



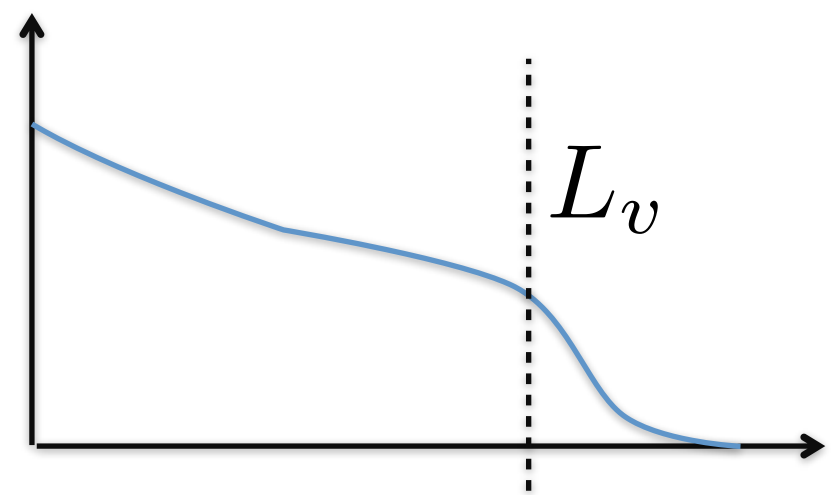
➡ 1 dimension:

$$L_v \gg L_*$$

➡ three regimes: KPZ scaling visible, asymptotically cut off by (space time) vortex proliferation



2 Dimensions



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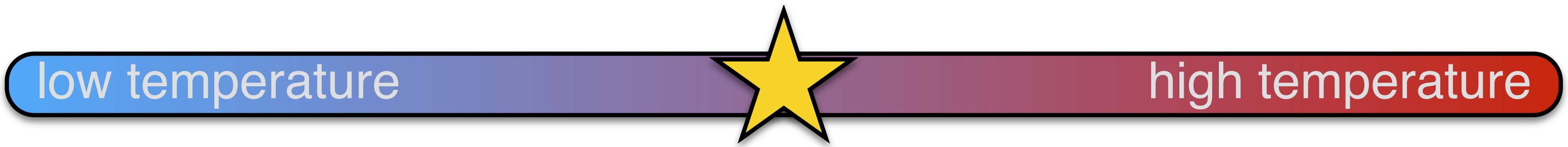
Microscopic Quantum Optics	“Thermodynamic” Many-body physics	Long wavelength Statistical mechanics

A paradigm of equilibrium stat mech: (no) BEC in 2D



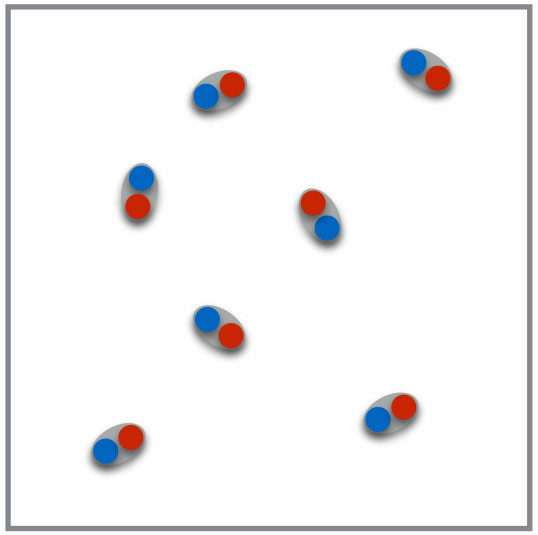
- correlations

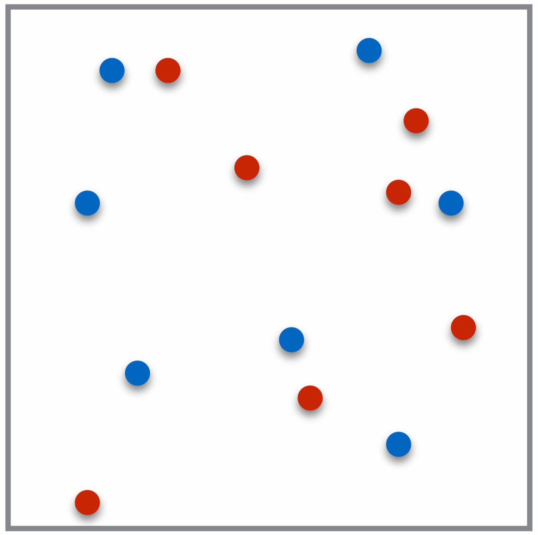
$$\langle \phi(r)\phi^*(0)\rangle \sim r^{-\alpha} \qquad\qquad \sim e^{-r/\xi}$$

- superfluidity

$$\rho_s \neq 0 \qquad\qquad \rho_s = 0$$

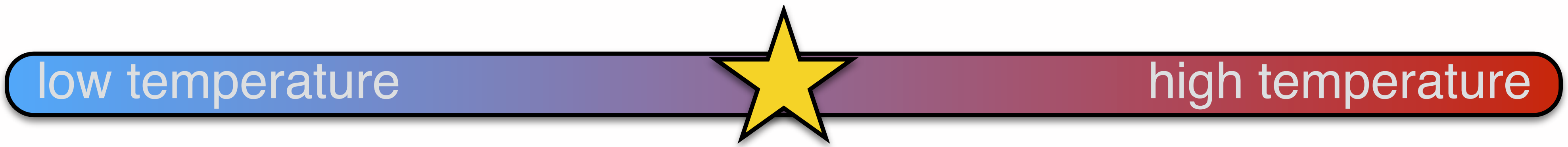
- KT transition: unbinding of vortex-antivortex pairs





... also for driven-dissipative condensates?

Reminder: Algebraic correlations



- correlations

$$\langle \phi(r)\phi^*(0)\rangle \sim r^{-\alpha} \qquad\qquad \sim e^{-r/\xi}$$

- physical reason: gapless spin wave/phonon fluctuations

  - phase-amplitude decomposition

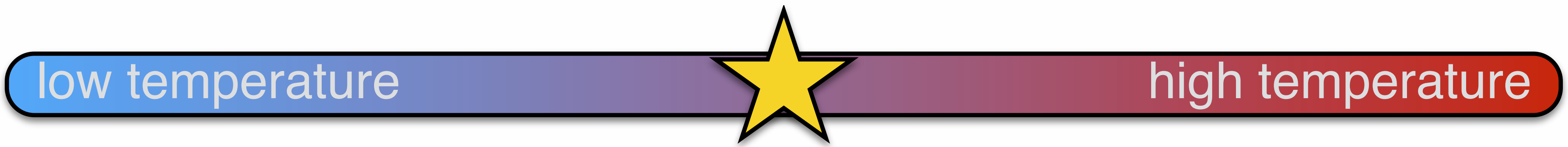
$$\langle \phi(r)\phi^*(0)\rangle \approx n_0 \langle e^{i(\theta(r)-\theta(0))}\rangle \approx n_0 e^{-\langle(\theta(r)-\theta(0))^2\rangle/2}$$

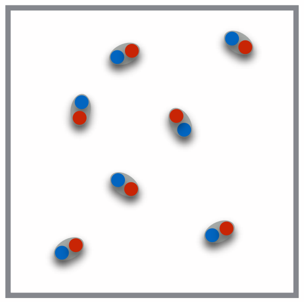
  - phase correlator

$$\langle(\theta(r)-\theta(0))^2\rangle \sim \int d^2q \frac{(e^{iqr}-1)}{q^2} \sim 2\alpha \log(r/a)$$

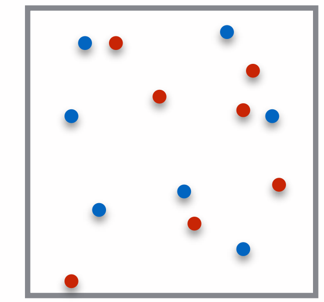
$$S_{SW} = \frac{K}{2}\int d^2x (\nabla\theta)^2$$ spin wave/ phase action

Reminder: KT transition

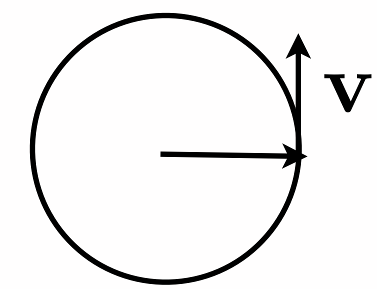




- KT transition: unbinding of vortex-antivortex pairs



- Single vortex picture (KT 1973): balance of energy (deterministic) and entropy (statistic)
  - Low T: vortices and antivortices bound in neutral pairs (irrelevant at long distance)
  - Q: when is it favorable (free energy) minimum to have unbound vortices?



- energy of single free vortex:

vortex current velocity $\mathbf{v} = \dfrac{\mathbf{e}_z \times \mathbf{e}_r}{r}$ $\quad \Rightarrow E = K/2 \int d^2 r \mathbf{v}^2 = \pi K \log(L/a)$

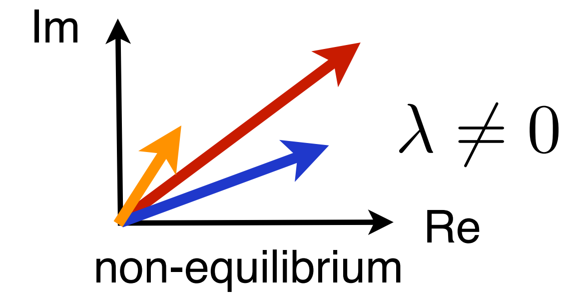
- entropy: sum all equally probable possibility of placing vortices in 2D plane at minimal distance a:

$$S = -k_B \sum_i p_i \log p_i = k_B \log(L/a)^2$$

- free energy $\quad F = E - TS = (K\pi - 2k_B T) \log(L/a)^2$

- vortex proliferation above KT critical temperature $\quad T_{KT} = \dfrac{K\pi}{2k_B}$

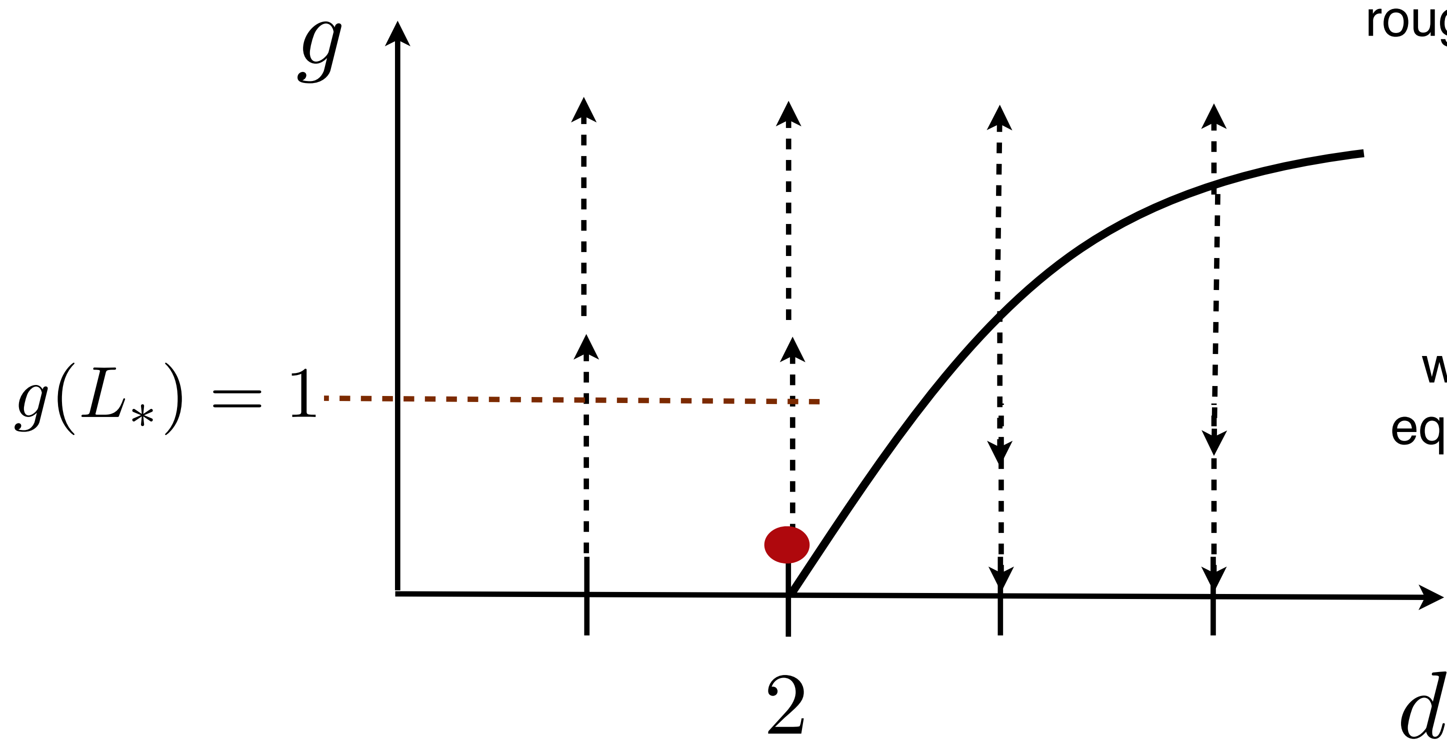
Physical implication I: Smooth KPZ fluctuations



- RG flow of the effective dimensionless KPZ coupling parameter

$$g^2 = \frac{\lambda^2 \Delta}{D^3}$$

strong coupling: disordered / rough non-equilibrium phase



weak coupling: equilibrium phase

- implication: a length scale is generated

$$L_* = a_0 e^{\frac{16\pi}{g^2}}$$

microscopic (healing) length

- exponentially large for
  - weak nonequilibrium λ
  - small noise level Δ

Physical implications I: Absence of quasi-LRO

- long-range behavior of two-point/ spatial coherence function:

$$\langle \phi^*(r)\phi(0)\rangle \approx n_0 e^{-\langle[\theta(\mathbf{x})-\theta(0)]^2\rangle}$$

leading order cumulant expansion

- generated length scale distinguishes two regimes: $L_* = a_0 e^{\frac{16\pi}{g^2}}$

universal equilibrium regime

$$a_0 \ll r \ll L_*$$

Bogoliubov fixed point relevant

$$\langle[\theta(\mathbf{r})-\theta(0)]^2\rangle \sim \log r$$

➡ algebraic decay

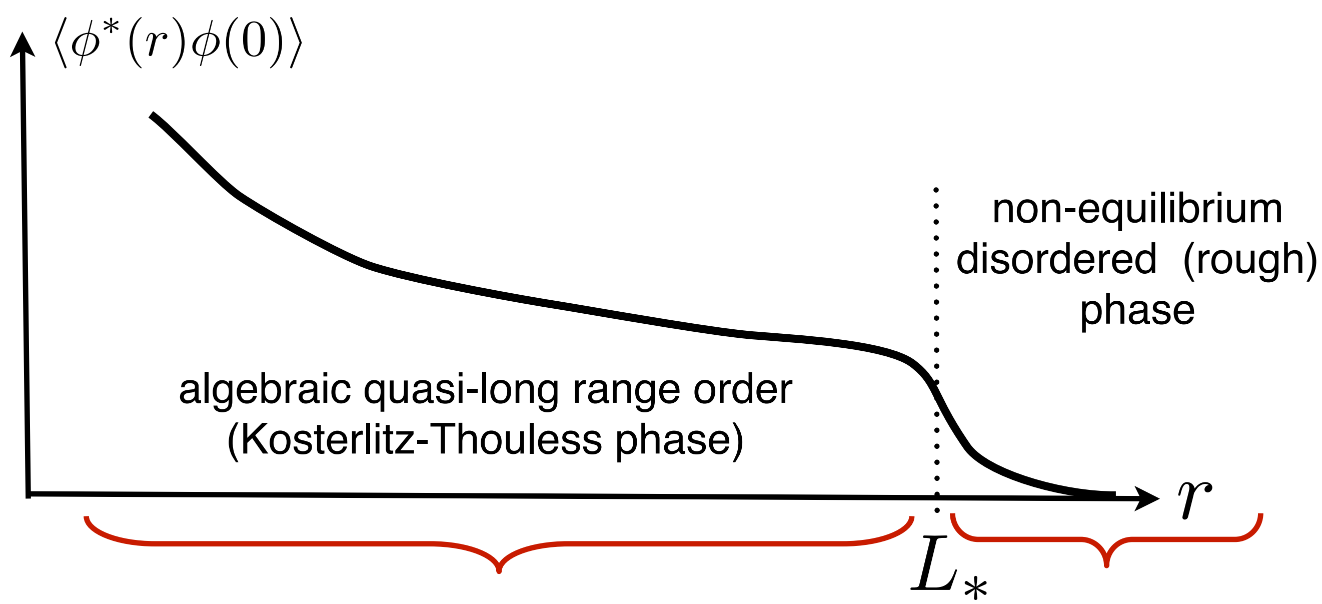
universal non-equilibrium regime

$$r \gg L_*$$

KPZ fixed point relevant

$$\langle[\theta(\mathbf{r})-\theta(0)]^2\rangle \sim r^{2\alpha} \quad \alpha \approx 0.4 \quad (d=2)$$

➡ subexponential decay



➡ algebraic order absent in any two-dimensional driven open system at the largest distances

➡ but crossover scale exponentially large for small deviations from equilibrium

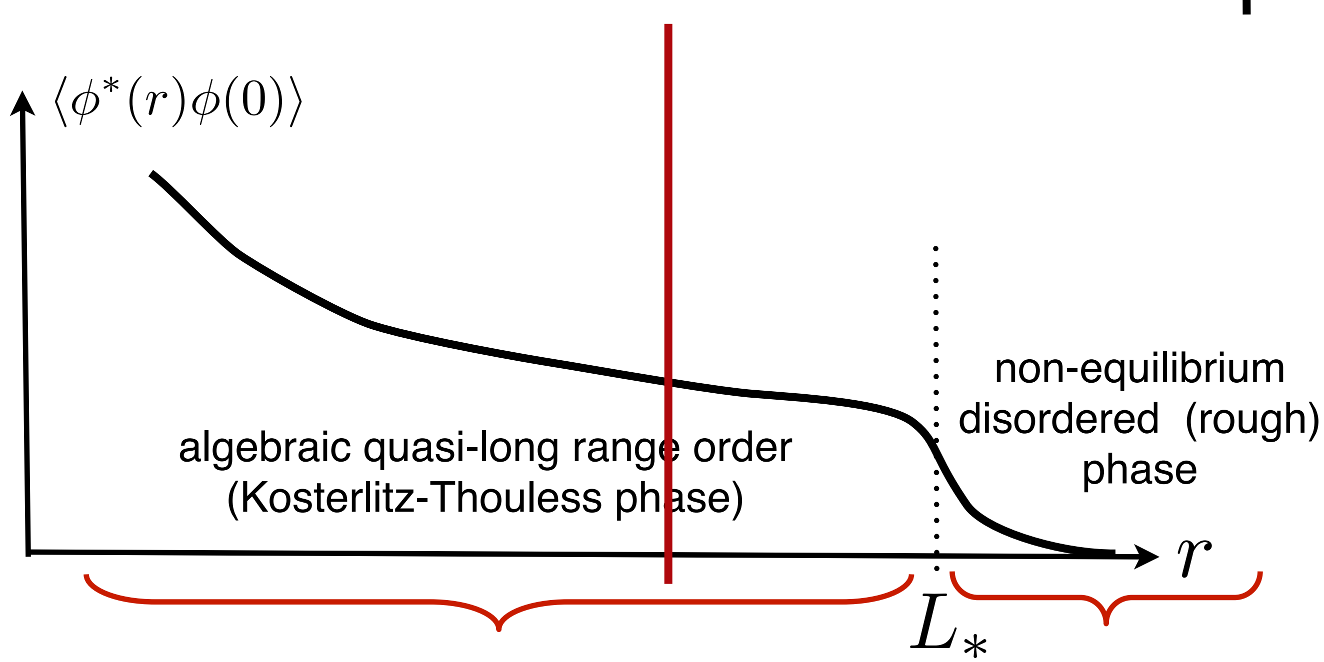
Physical implications I: Absence of quasi-LRO

- long-range behavior of two-point/ spatial coherence function:

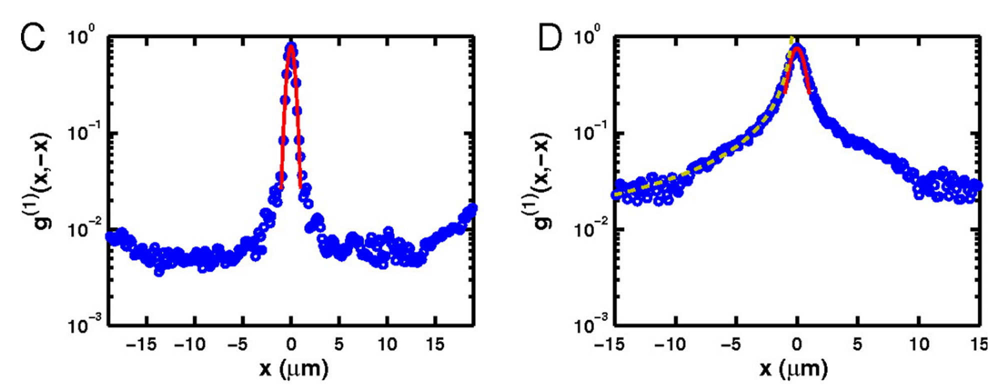
$$\langle \phi^*(r)\phi(0)\rangle \approx n_0 e^{-\langle[\theta(\mathbf{x})-\theta(0)]^2\rangle}$$ leading order cumulant expansion

- generated length scale distinguishes two regimes: $L_* = a_0 e^{\frac{16\pi}{g^2}}$

universal equilibrium regime	universal non-equilibrium regime
$a_0 \ll r \ll L_*$	$r \gg L_*$
Bogoliubov fixed point relevant	KPZ fixed point relevant
$\langle[\theta(\mathbf{r})-\theta(0)]^2\rangle \sim \log r$	$\langle[\theta(\mathbf{r})-\theta(0)]^2\rangle \sim r^{2\alpha}$ $\alpha \approx 0.4 \quad (d=2)$
➡ algebraic decay	➡ subexponential decay



➡ exponentially large crossover scale reconciles with experiments



from Roumpos et al., PNAS (2012)

Physical implications II: Superfluid response

- equilibrium: close connection between correlations and responses
- here: algebraic order decay exponent α_s and superfluid stiffness ρ_s related: $\alpha_s^{-1} = \frac{2\pi}{k_{\rm B} T m^2} \cdot \rho_s$

- superfluid response:
  - additional contribution to microscopic Hamiltonian due to (artificial) gauge field:

$$H_{\rm ext} = \int d\mathbf{x}\, \underbrace{\mathbf{f}(t,\mathbf{x})}_{\text{ext. field}} \cdot \underbrace{\mathbf{j}(t,\mathbf{x})}_{\text{induced current}}$$

EP condensates: J. Keeling, PRL (2011)

- current response χ: $\langle j_i(\omega,\mathbf{q})\rangle = \chi_{ij}(\omega,\mathbf{q}) f_j(\omega,\mathbf{q})$
- isotropy: $\chi_{ij}(\omega,\mathbf{q}) = \underbrace{\chi_l(\omega,\mathbf{q})}_{\text{longitudinal}} P_{ij} + \underbrace{\chi_t(\omega,\mathbf{q})}_{\text{transversal}}(\delta_{ij} - P_{ij})$

$$P = \frac{\mathbf{q}\mathbf{q}^T}{\mathbf{q}^2}$$

projector on longitudinal component (II q)

$$= \chi_t(\omega,\mathbf{q})\delta_{ij} + (\chi_l(\omega,\mathbf{q}) - \chi_t(\omega,\mathbf{q}))P_{ij}$$

- normal (non-superfluid) system: $\chi_{ij} \sim \delta_{ij}$

➡ superfluid response: $\frac{\rho_s}{m} := \lim_{\mathbf{q}\to 0}[\chi_l(0,\mathbf{q}) - \chi_t(0,\mathbf{q})]$

static observable

for open systems: J. Keeling, PRL (2011)

Superfluid response in the driven system

- superfluid response: $$\frac{\rho_s}{m} := \lim_{\mathbf{q}\to 0}[\chi_l(0,\mathbf{q}) - \chi_t(0,\mathbf{q})]$$
- approximation: neglect density fluctuations, but take KPZ non-linear physics into account
- current-current correlator [schematic argument]:

$$\chi_{ij} \sim \langle \partial_i\theta\partial_j\tilde{\theta}\rangle + \langle \partial_i\theta\partial_j\theta\tilde{\theta}\rangle$$

- momentum scaling dimensions:
  $\times k^{-(d+z)+2}$
  - equilibrium: 0 forbidden by symmetry
  - non-equilibrium: $\alpha \approx 0.4$ 0 exact scaling relations of KPZ fixed point
    - genuine non-equilibrium term stabilizes the superfluid response

➡ superfluid response finite! (despite absence of algebraic order).

Superfluid response in the driven system

- superfluid response: $$\frac{\rho_s}{m} := \lim_{\mathbf{q}\to 0}[\chi_l(0,\mathbf{q}) - \chi_t(0,\mathbf{q})]$$
- approximation: neglect density fluctuations, but take KPZ non-linear physics into account
- detailed calculation:

equilibrium system	driven system
$\rho_s = \rho_0$	$\rho_s = Z \cdot \rho_0 + \mathcal{O}(L^{-\alpha})$

$$\boxed{Z = \frac{\ln 2}{8\pi}\, g_*^2}$$

$$g_*^2 = \frac{\lambda_*^2 \Delta_*}{D_*^3}$$

effective KPZ non-linearity at strong coupling fixed point

- factorization: non-universal microscopic, universal long-distance part

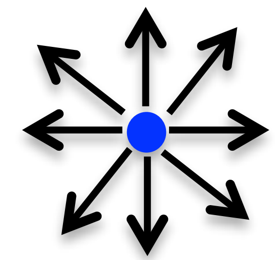
➡ physical observable directly reveals universal KPZ properties!

Physical implications III: Non-equilibrium Kosterlitz-Thouless

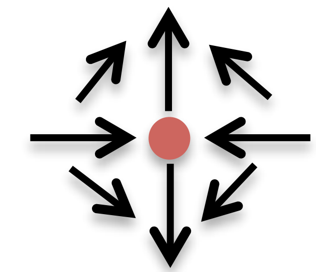
- KPZ equation for phase variable

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

- compact nature of phase allows for vortex defects in 2D!



vortex

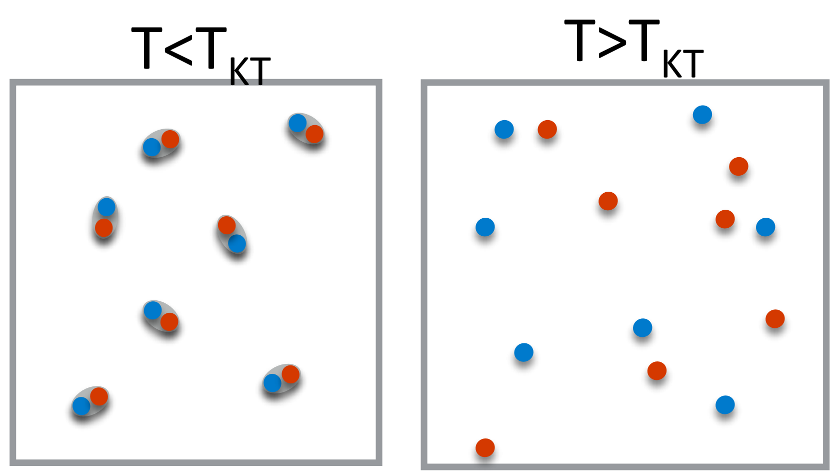


anti-vortex

- in 2D equilibrium: perfect analogy between vortices and electric charges
  - log(r) interactions, $1/(\epsilon r)$ forces
  - dielectric constant ϵ^{-1}= superfluid stiffness

superfluid = dipole gas

$\epsilon^{-1} > 0$



normal fluid = plasma
metallic screening

$\epsilon^{-1} \to 0$

➡ how is this scenario modified in the driven system?

Duality approach

- KPZ equation for phase variable

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

- implementing phase compactness = implementing (local discrete gauge) invariance under

$$\theta_{t,\mathbf{x}} \mapsto \theta_{t,\mathbf{x}} + 2\pi n_{t,\mathbf{x}} \qquad\qquad \theta_{t,\mathbf{x}} \in [0, 2\pi), \quad n_{t,\mathbf{x}} \in \mathbf{Z}$$

- resulting from its origin $\quad \psi_{t,\mathbf{x}} = \sqrt{\rho_{t,\mathbf{x}}}\, e^{i\theta_{t,\mathbf{x}}}$

- deterministic part: lattice regularization

new short distance length scale ->
expect new emergent length scale

$$\partial_t \theta_{\mathbf{x}} = \underbrace{-\sum_{\mathbf{a}} \left[D \sin(\theta_{\mathbf{x}} - \theta_{\mathbf{x}+\mathbf{a}}) + \frac{\lambda}{2}\left(\cos(\theta_{\mathbf{x}} - \theta_{\mathbf{x}+\mathbf{a}}) - 1\right)\right]}_{=: \mathcal{L}[\theta]_{t,\mathbf{x}}} + \eta_{\mathbf{x}}$$

unit lattice direction

deterministic

noise

- NB: lambda = 0: existence of potential

$$\partial_t \theta_{\mathbf{x}} = -\Gamma \frac{\delta \mathcal{H}_{XY}}{\delta \theta_{\mathbf{x}}} + \eta_{\mathbf{x}} \qquad \mathcal{H}_{XY} = K \sum_{\langle \mathbf{x}, \mathbf{x}' \rangle} \cos(\theta_{\mathbf{x}} - \theta_{\mathbf{x}'}) \quad \Rightarrow \mathcal{P}_{\text{Gibbs}} \propto \exp(-\mathcal{H}_{XY}/T), \quad T = \Delta/\Gamma$$

Duality approach

- KPZ equation for phase variable

$$\partial_t \theta = D\nabla^2\theta + \lambda(\nabla\theta)^2 + \xi$$

- implementing phase compactness = implementing (local discrete gauge) invariance under

$$\theta_{t,\mathbf{x}} \mapsto \theta_{t,\mathbf{x}} + 2\pi n_{t,\mathbf{x}} \qquad\qquad \theta_{t,\mathbf{x}} \in [0, 2\pi), \quad n_{t,\mathbf{x}} \in \mathbf{Z}$$

- resulting from its origin $\quad \psi_{t,\mathbf{x}} = \sqrt{\rho_{t,\mathbf{x}}} e^{i\theta_{t,\mathbf{x}}}$

- temporal part: stochastic update

$$\theta_{t+\epsilon,\mathbf{x}} = \theta_{t,\mathbf{x}} + \epsilon \left(\mathcal{L}[\theta]_{t,\mathbf{x}} + \eta_{t,\mathbf{x}}\right) + 2\pi n_{t,\mathbf{x}}$$

chosen to keep $\theta_{t,\mathbf{x}} \in [0, 2\pi)$

  - NB: phase can jump: at this point, continuum limit eps -> 0 ill defined, derivatives discrete

- stochastic difference equation -> discrete dynamical functional integral:

$$\boxed{Z = \sum_{\{\tilde{n}_{t,\mathbf{x}}\}} \int \mathcal{D}[\theta] e^{iS[\theta,\tilde{n}]}}$$

discrete noise -> manifest gauge invariance

$$Z = \int \mathcal{D}[\tilde{\theta}]\mathcal{D}[\theta] e^{iS[\theta,\tilde{\theta}]}$$

vs. continuous variable

$$S = \sum_{t,\mathbf{x}} \tilde{n}_{t,\mathbf{x}} \left[-\Delta_t \theta_{t,\mathbf{x}} + \epsilon \left(\mathcal{L}[\theta]_{t,\mathbf{x}} + i\Delta\tilde{n}_{t,\mathbf{x}}\right)\right]$$

Duality approach

Duality approach

- next steps: sequence of changing the variables
- the action $S = \sum_{t,\mathbf{x}} \tilde{n}_{t,\mathbf{x}} \left[-\Delta_t \theta_{t,\mathbf{x}} + \epsilon \left(\mathcal{L}[\theta]_{t,\mathbf{x}} + i\Delta \tilde{n}_{t,\mathbf{x}}\right)\right]$ is periodic in $\nabla(\theta \pm \epsilon D\tilde{n})$

  ➡ Fourier expansion introducing $\mathbf{j}_\pm$ or $\mathbf{j}, \tilde{\mathbf{j}}$

- parameterization in terms of new fields

$$\begin{pmatrix} \tilde{n} \\ \tilde{\mathbf{j}} \end{pmatrix} = \begin{pmatrix} \Delta_t \\ \nabla \end{pmatrix} \times \begin{pmatrix} \tilde{\phi} \\ -\tilde{\mathbf{A}} \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{z}} \cdot \left(\nabla \times \tilde{\mathbf{A}}\right) \\ -\hat{\mathbf{z}} \times \left(\nabla \tilde{\phi} + \Delta_t \tilde{\mathbf{A}}\right) \end{pmatrix} \qquad \mathbf{j} = -\hat{\mathbf{z}} \times (\nabla \phi + \mathbf{A})$$

due to continuity equation $\Delta_t \tilde{n}_X + \nabla \cdot \tilde{\mathbf{j}}_X = 0$

- new dynamical integral: only discrete variables

$$Z \propto \sum_{\{\phi_X, \tilde{\phi}_X, \mathbf{A}_X, \tilde{\mathbf{A}}_X\}} e^{iS[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}]}$$

Duality approach

- next steps: sequence of changing the variables
  - only discrete variables

$$Z \propto \sum_{\{\phi_X, \tilde{\phi}_X, \mathbf{A}_X, \tilde{\mathbf{A}}_X\}} e^{iS[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}]}$$

  - turn into smooth integration and (a bit of) summation: Poisson formula

$$\sum_{k=-\infty}^{\infty} g(k) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi \, g(\phi) e^{-i2\pi n\phi}$$

  - resulting integral:

$$\boxed{Z \propto \underbrace{\sum_{\substack{\{n_{vX}, \tilde{n}_{vX},\\ \mathbf{J}_{vX}, \tilde{\mathbf{J}}_{vX}\}}}}_{\text{vortex density and current}} \underbrace{\int \mathcal{D}[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}]}_{\text{smooth spin wave fluctuations}} e^{iS[\phi, \tilde{\phi}, \mathbf{A}, \tilde{\mathbf{A}}, n_v, \tilde{n}_v, \mathbf{J}_v, \tilde{\mathbf{J}}_v]}}$$

  - interpretation: study the associated Langevin equations

Electrodynamic Duality

- Langevin equations = Modified noisy Maxwell equations
- formulated in electric and magnetic fields alone:

$$\mathbf{E} = -\nabla\phi - \mathbf{A}, \qquad \tilde{\mathbf{E}} = -\nabla\phi - \partial_t \tilde{\mathbf{A}},$$
$$\mathbf{B} = D\nabla \times \mathbf{A} \qquad \tilde{\mathbf{B}} = \nabla \times \tilde{\mathbf{A}}$$

fixed by gauge invariance

irrotational flow

$$\nabla \cdot \mathbf{E} = 2\pi n_v$$

vortex density & current

modified continuity eq $\partial_t \to 1/D$

$$\nabla \times \mathbf{E} + \frac{1}{D}\mathbf{B} = 0$$

phase dynamics (compact KPZ)

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 2\pi \mathbf{J}_v - \hat{\mathbf{z}} \times \nabla\left(\frac{\lambda}{2}E^2 + \bar{\zeta}\right)$$

$$\nabla \cdot \mathbf{B} = 0$$

KPZ non-linearity and noise

over-damped vortex dynamics (ignoring mag field) $\ddot{\mathbf{r}}_i \to \dot{\mathbf{r}}_i$

$$\frac{d\mathbf{r}_i}{dt} = \mu n_i \mathbf{E}(t, \mathbf{r}_i) + \boldsymbol{\xi}_i$$

- further intuition: obtained heuristically from identification

$$\rho - \bar{\rho} \equiv B\hat{\mathbf{z}} \qquad \hat{\mathbf{z}} \times \nabla\theta \equiv \mathbf{E}$$

and adding vortex sources and currents by hand

Electrodynamic Duality

- check: neglect vortex contributions and “integrate out” gapped magnetic field

irrotational flow

$$\nabla \cdot \mathbf{E} = 2\pi n_v$$

vortex density & current

modified continuity eq $\partial_t \to 1/D$

$$\nabla \times \mathbf{E} + \frac{1}{D}\mathbf{B} = 0$$

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$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 2\pi \mathbf{J}_v - \hat{\mathbf{z}} \times \nabla \left(\frac{\lambda}{2} E^2 + \bar{\zeta} \right)$$

$$\nabla \cdot \mathbf{B} = 0$$

KPZ non-linearity and noise

over-damped vortex dynamics (ignoring mag field) $\ddot{\mathbf{r}}_i \to \dot{\mathbf{r}}_i$

$$\frac{d\mathbf{r}_i}{dt} = \mu n_i \mathbf{E}(t, \mathbf{r}_i) + \boldsymbol{\xi}_i$$

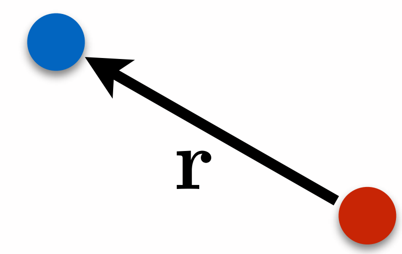
- recover KPZ equation via replacement $\hat{\mathbf{z}} \times \nabla \theta \equiv \mathbf{E}$

$$\frac{\partial \mathbf{E}}{\partial t} = D \nabla^2 \mathbf{E} - \hat{\mathbf{z}} \times \nabla \left(\frac{\lambda}{2} E^2 + \bar{\zeta} \right)$$

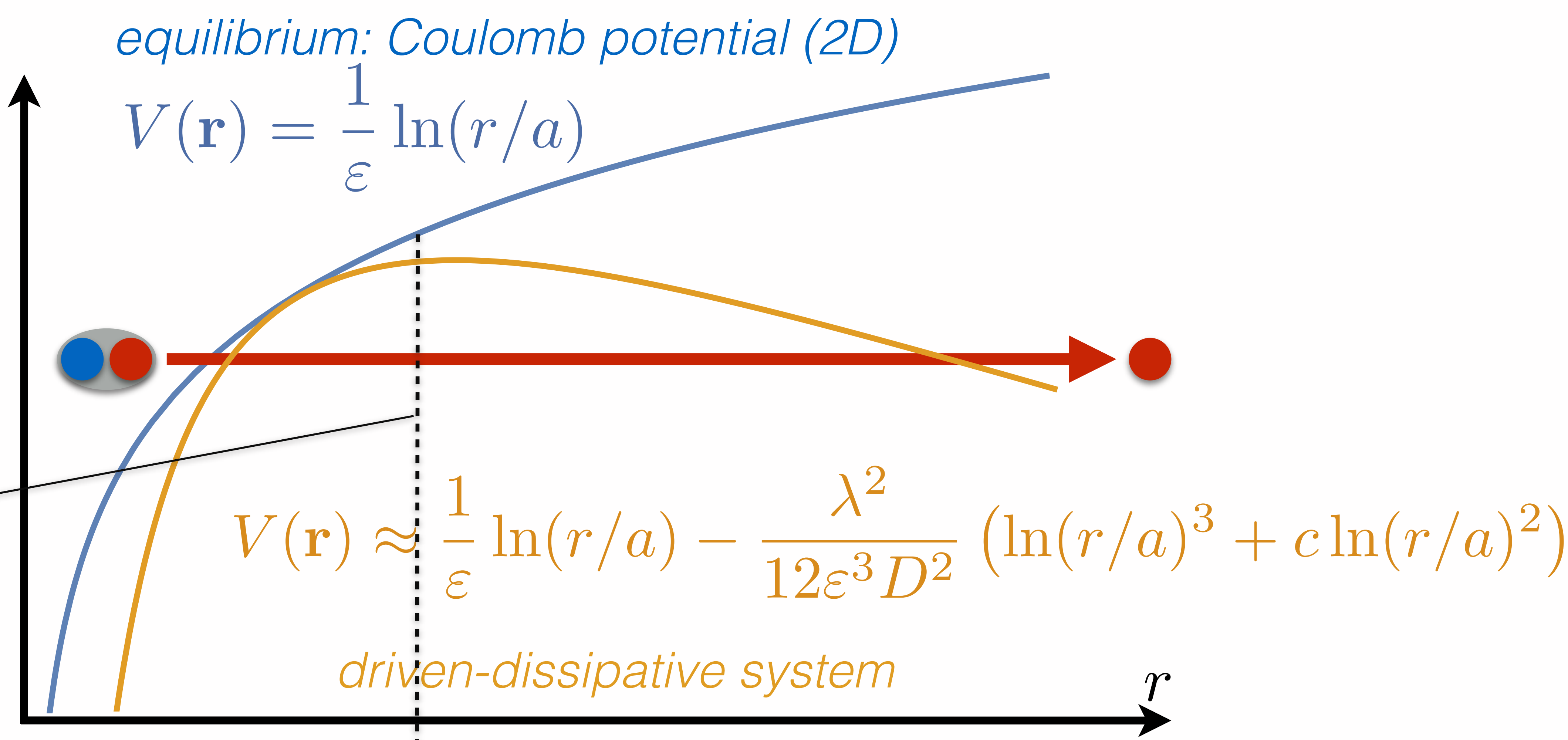
- next: integrate out gapless electric field degrees of freedom = phase fluctuations
    - equilibrium \lambda =0: exactly
    - non-equilibrium: perturbatively in \lambda

A single vortex-antivortex pair

- close to the transition: dilute gas of vortices
- equation of motion for a single vortex-antivortex pair



$$\frac{d\mathbf{r}}{dt} = -\mu \nabla V(r) + \boldsymbol{\xi}$$



length scale:

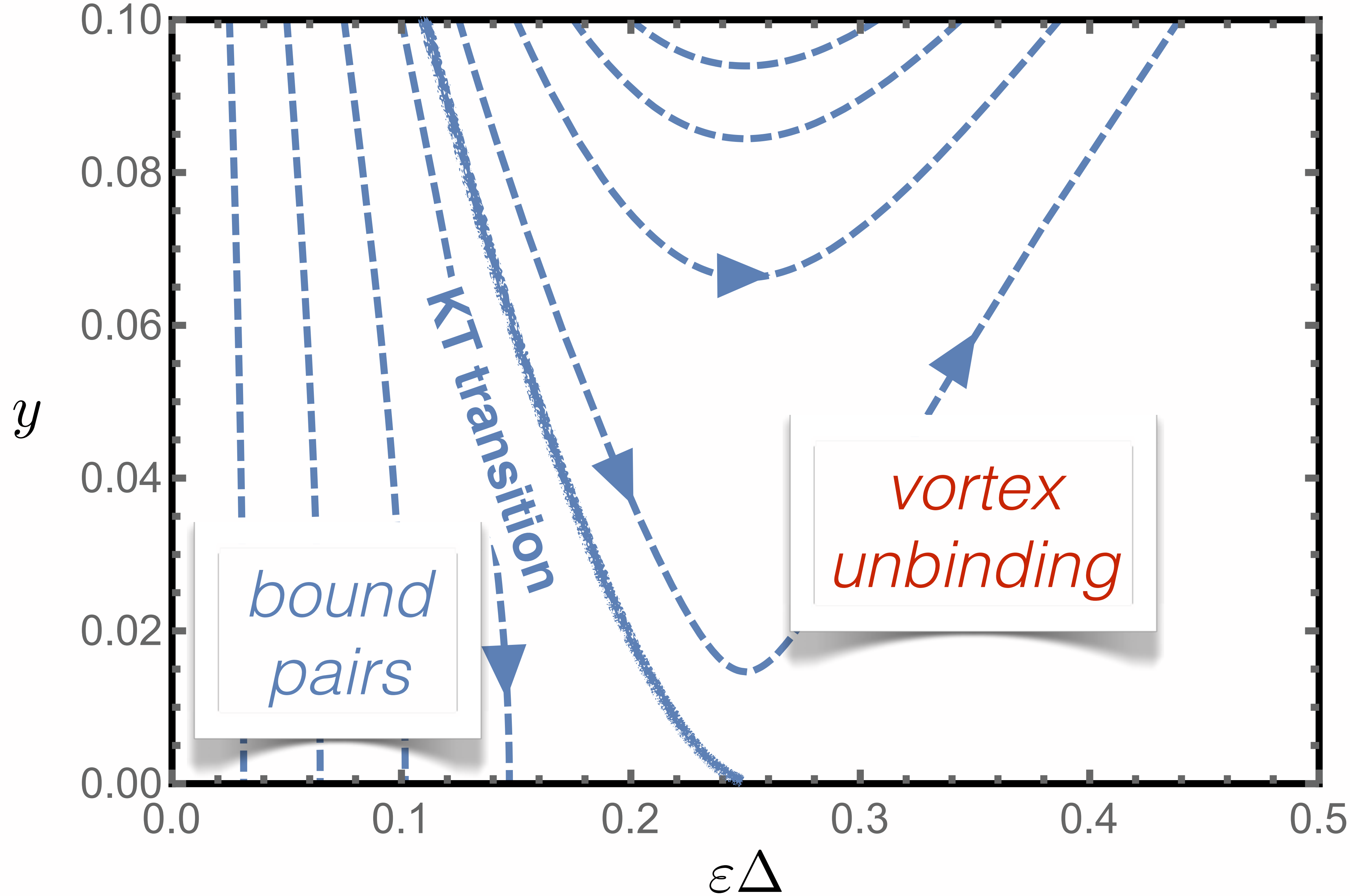
$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

see also: I Aranson et al., PRB (1998) two-vortex problem

noise-activated unbinding for a single pair (at exp small rate)

Modified Kosterlitz-Thouless RG flow

$$\frac{d\varepsilon}{d\ell} = \frac{2\pi^2 y^2}{T} \qquad \frac{dy}{d\ell} = \left[2 - \frac{1}{2\varepsilon T} + \cancel{\frac{\lambda^2}{4\varepsilon^2 D^2}\left(\frac{1}{4} + \ell\right)}\right] y \qquad \frac{dT}{d\ell} = \cancel{\frac{\lambda^2 T}{2\varepsilon^2 D^2}\left(\frac{1}{4} + \ell\right)}$$



equilibrium KT flow

dielectric constant

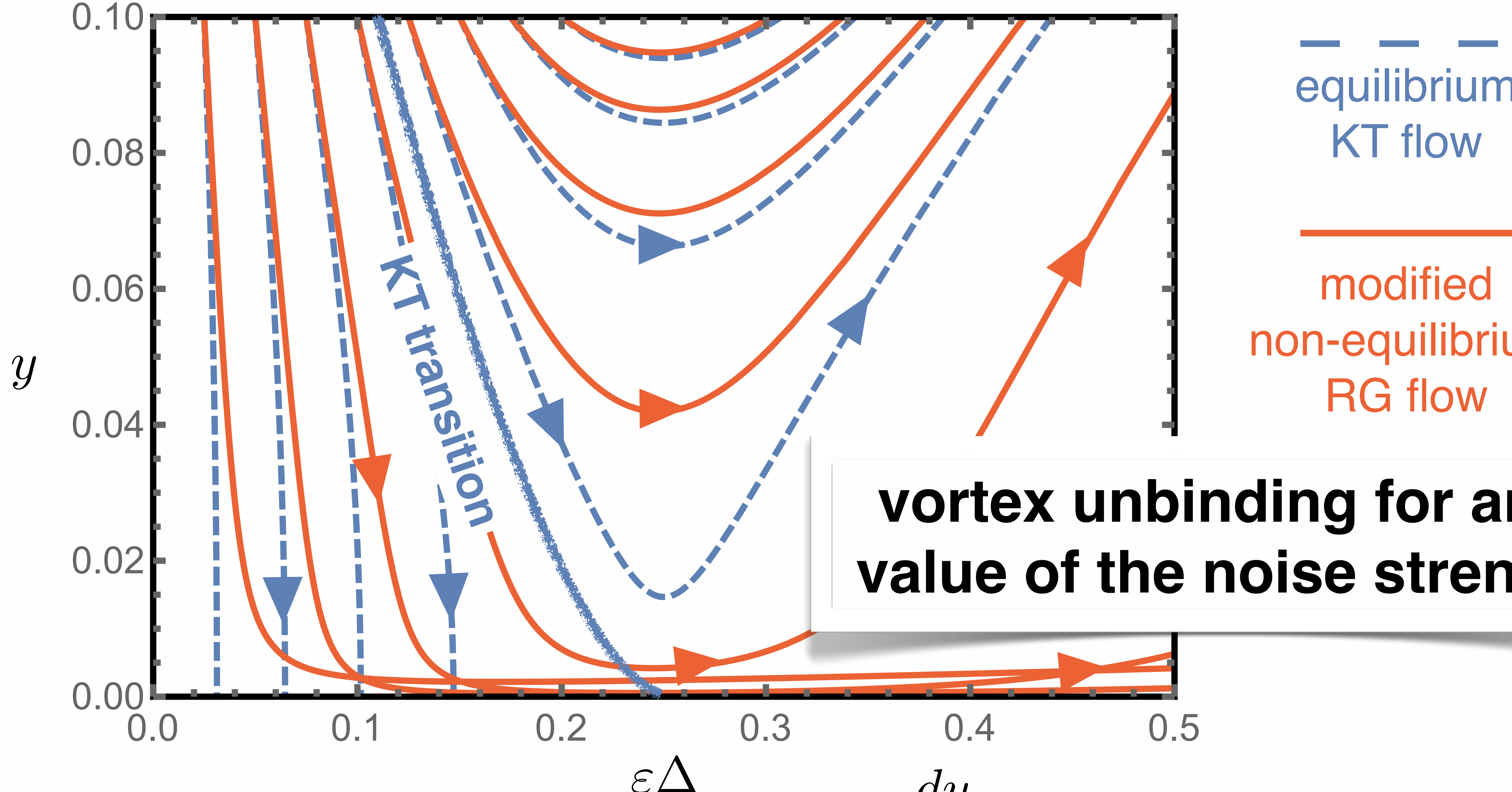
$\epsilon \to \infty$

$\Rightarrow \rho_s \to 0$

superfluid stiffness

Modified Kosterlitz-Thouless RG flow

$$\frac{d\varepsilon}{d\ell} = \frac{2\pi^2 y^2}{T} \qquad \frac{dy}{d\ell} = \left[2 - \frac{1}{2\varepsilon T} + \frac{\lambda^2}{4\varepsilon^2 D^2}\left(\frac{1}{4}+\ell\right)\right] y \qquad \frac{dT}{d\ell} = \frac{\lambda^2 T}{2\varepsilon^2 D^2}\left(\frac{1}{4}+\ell\right)$$



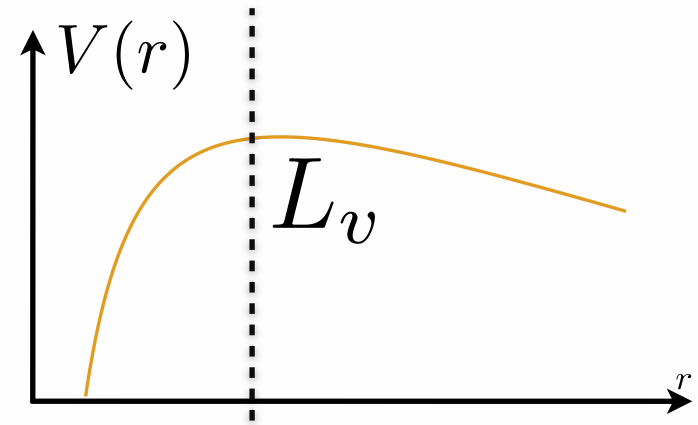
vortex unbinding for any value of the noise strength

$\frac{dy}{d\ell}$ changes sign at a scale L_v

Implication for exciton-polaritons

- for generic parameters, $L_v \ll L_*$ i.e. vortex unbinding overwrites KPZ physics

- vortices: generated at short distance, have to overcome potential barrier by noise activation



- KPZ time: diffusion time to separate to KPZ length:

$$\tau_* = D^{-1}L_*^2 \approx a^2 D^{-1} e^{\frac{16\pi D^3}{\Delta\lambda^2}}$$

- vortex time: time to climb potential wall by noise activation (Arrhenius):

$$\tau_v = \frac{L_v^2}{\mu y^2} e^{-\beta \ln(L_v/a)} \approx \frac{a^2}{\mu y^2} e^{\frac{D}{\lambda}(2+\beta)} \qquad \beta \equiv 1/T \approx D/\Delta$$

- ratio: $\boxed{\tau_*/\tau_v \approx y^2 \frac{\mu}{D} \exp\left[\frac{1}{g^2}\left(16\pi - \frac{\lambda}{D}\right)\right]}$

  - ➡ large exponential factor, λ/D is the small expansion parameter
  - ➡ μ/D relative vortex mobility should be small (but unknown)
  - ➡ vortex fugacity $y = e^{-\beta\epsilon_c}$, ϵ_c the vortex core energy small parameter

Summary: 2D

- two emergent length scales in complementary approaches:

$$L_* = a_0 e^{\frac{16\pi}{g^2}}$$

KPZ length

$$L_v = a_0 e^{\frac{2D}{\lambda}}$$

vortex length

- scaling for the relevant fixed points

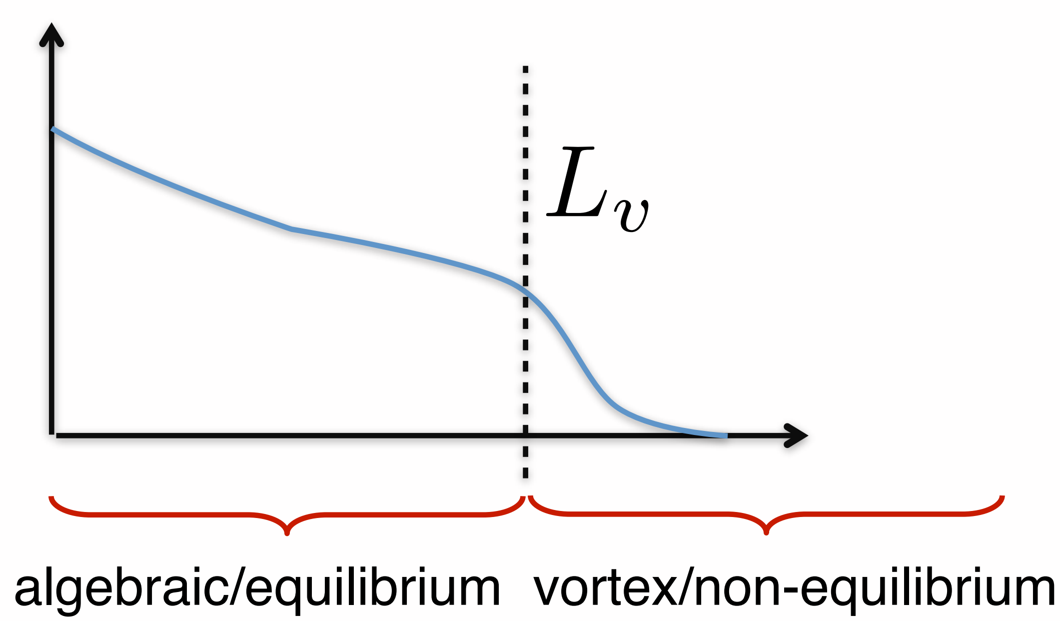
$$\langle \phi^*(r)\phi(0)\rangle \sim e^{-r^{2\chi}}, \quad \chi = 0.4$$

KPZ fixed point

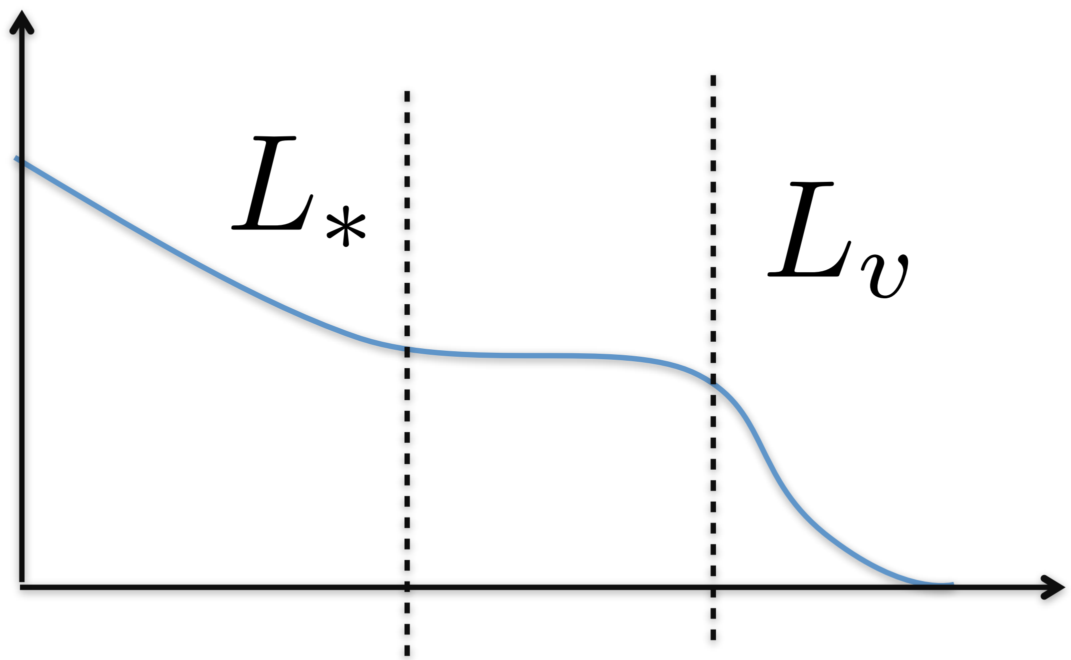
$$\langle \phi^*(r)\phi(0)\rangle \sim e^{-r}$$

free vortex/disordered fixed point

- for exciton-polariton systems, $L_v \ll L_*$



1 Dimension



L. He, L. Sieberer, E. Altman, SD, PRB (2015)
L. He, L. Sieberer, SD, in preparation

Microscopic
Quantum Optics

"Thermodynamic"
Many-body physics

Long wavelength
Statistical mechanics

KPZ exponents

- direct numerical solution of driven-dissipative GPE in one dimension

- observable: phase correlations $$w(L,t) \equiv \left\langle \frac{1}{L}\int_x \theta^2(x,t) - \left(\frac{1}{L}\int_x \theta(x,t)\right)^2 \right\rangle$$

- hosts all critical exponents:

  - stationary limit: static/"roughness" exponent:

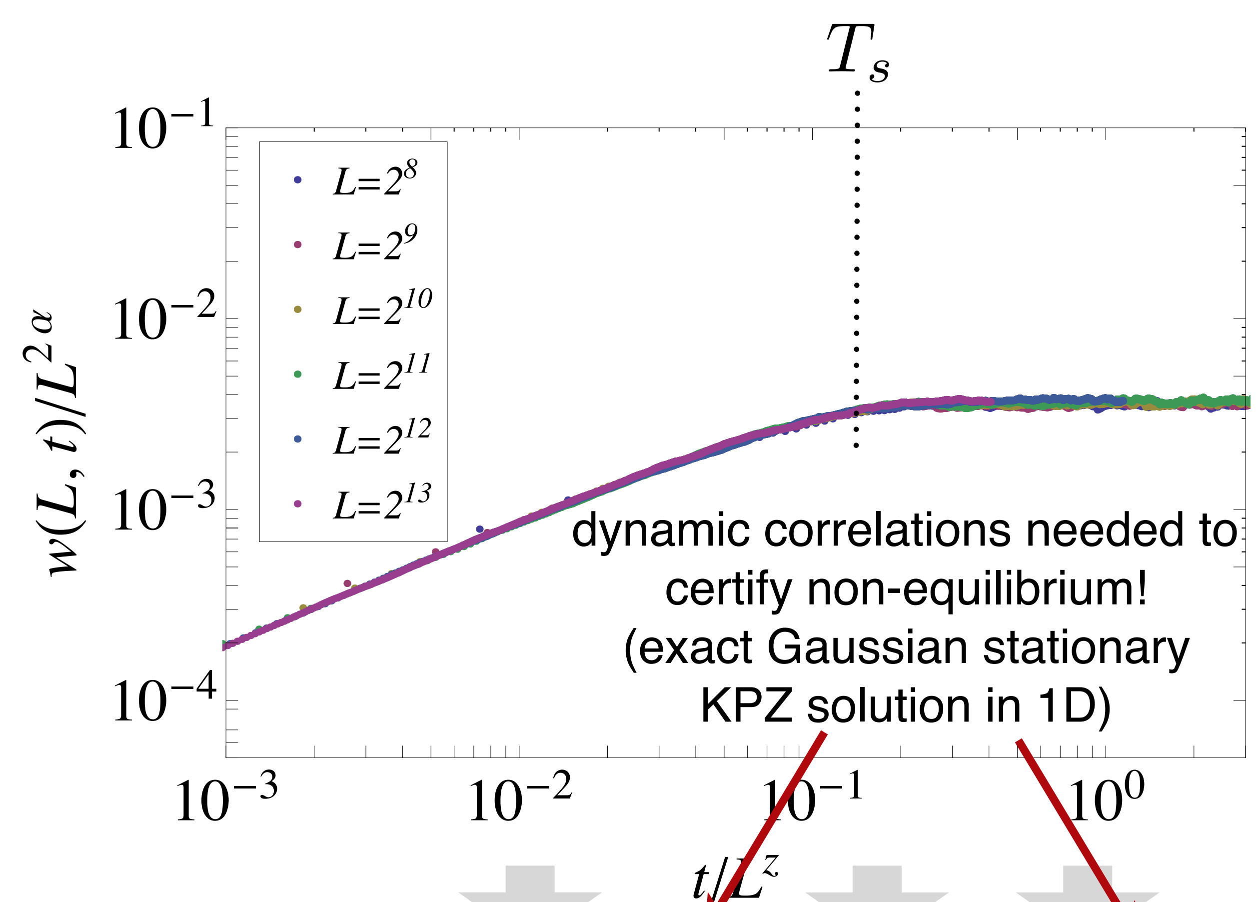
  $$w(L, t \gg L^z) \sim L^{2\alpha}$$

  - time evolution: "growth exponent"

  $$w(L \gg t^{1/z}, t) \sim t^{2\beta}$$

  - crossover timescale: dynamical exponent

  $$T_s \sim L^z \qquad\qquad \beta = \alpha/z$$



$\beta \approx 1/3 \quad z \approx 3/2 \quad \alpha \approx 1/2$

vs. eq.: $\beta = 1/4 \quad z = 2 \quad \alpha = 1/2$

➡ KPZ scaling fully confirmed in phase correlations

L. He, L. Sieberer, E. Altman, SD, PRB (2015)
see also: K. Yi, V. Gladilin, M. Wouters, PRB (2015)

Spatial and temporal coherence function

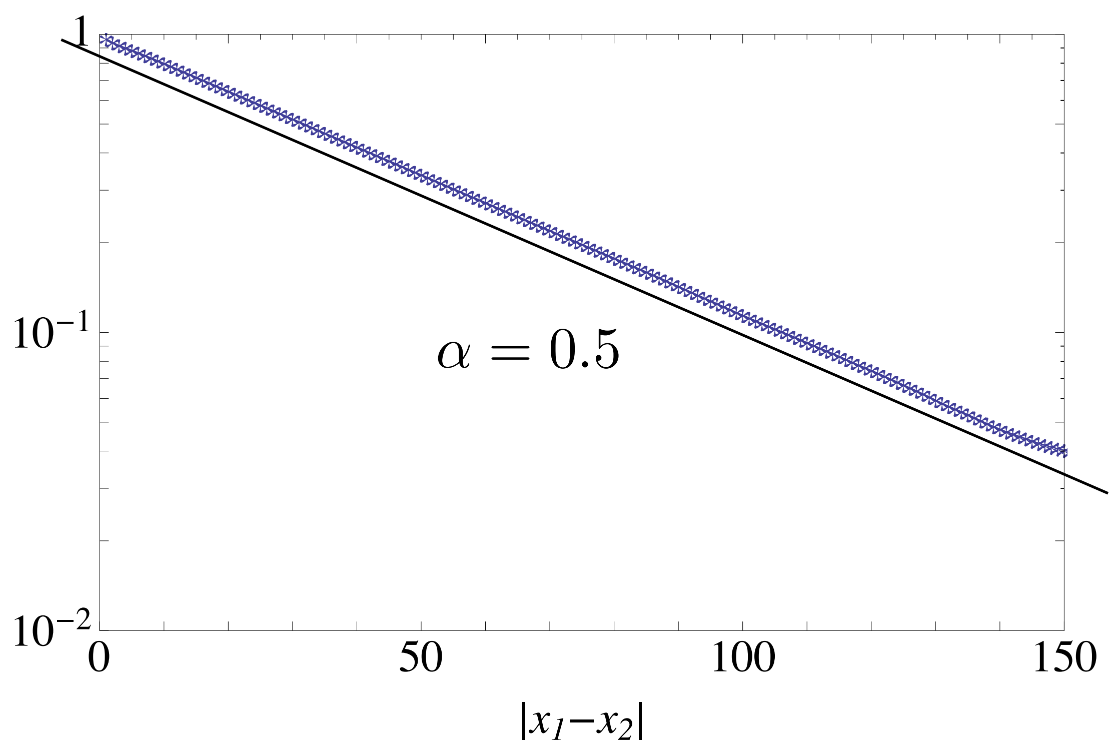
- direct numerical solution of driven-dissipative GPE in one dimension

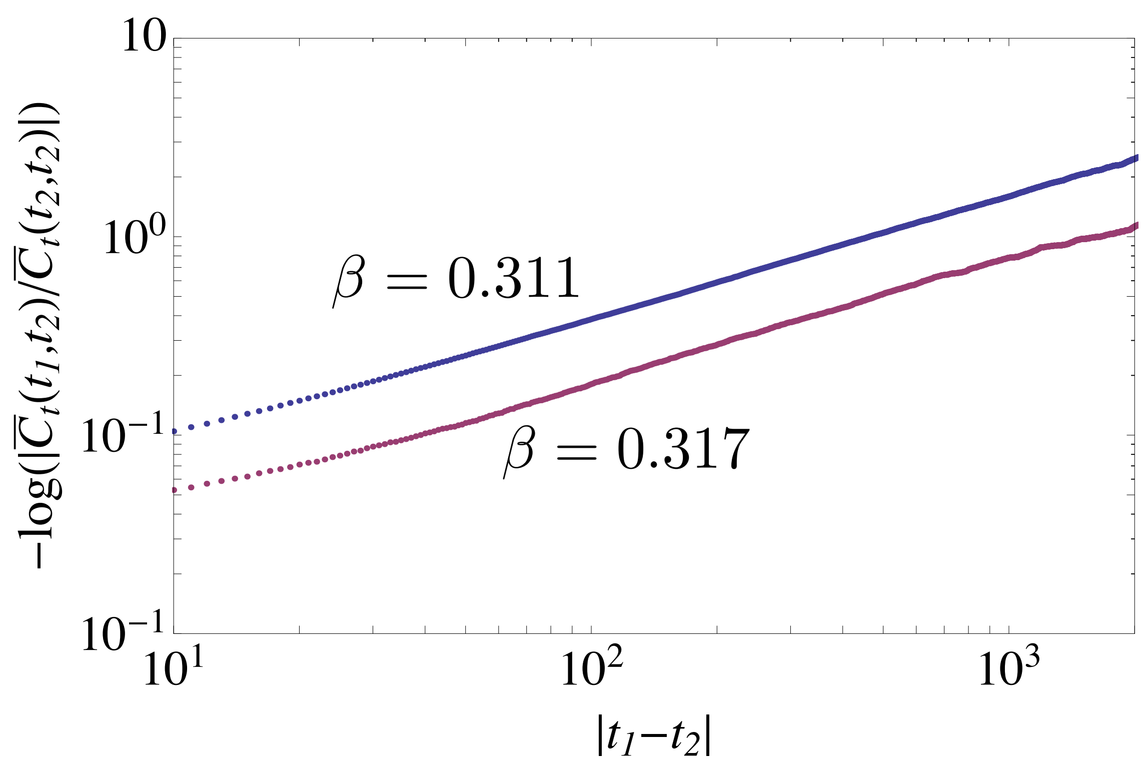
- observable 2: complex field first order spatial and temporal coherence functions

only modulus gauge invariant!

$$\bar{C}_x(x_1,x_2,t) \equiv \frac{1}{L}\int dy \langle\psi^*(x_1+y,t)\psi(x_2+y,t)\rangle \sim e^{-Ar^{2\alpha}}$$

$$\bar{C}_t(t_1,t_2) \equiv \frac{1}{L}\int dx |\langle\psi^*(x,t_1)\psi(x,t_2)\rangle| \sim e^{-B\Delta t^{2\beta}}$$





- KPZ universality: different micr. parameters

➡ KPZ scaling fully confirmed in coherence functions

L. He, L. Sieberer, E. Altman, SD, PRB (2015)
see also: K. Yi, V. Gladilin, M. Wouters, PRB (2015)

Temporal coherence: signatures in experiments

- observability in first order temporal coherence

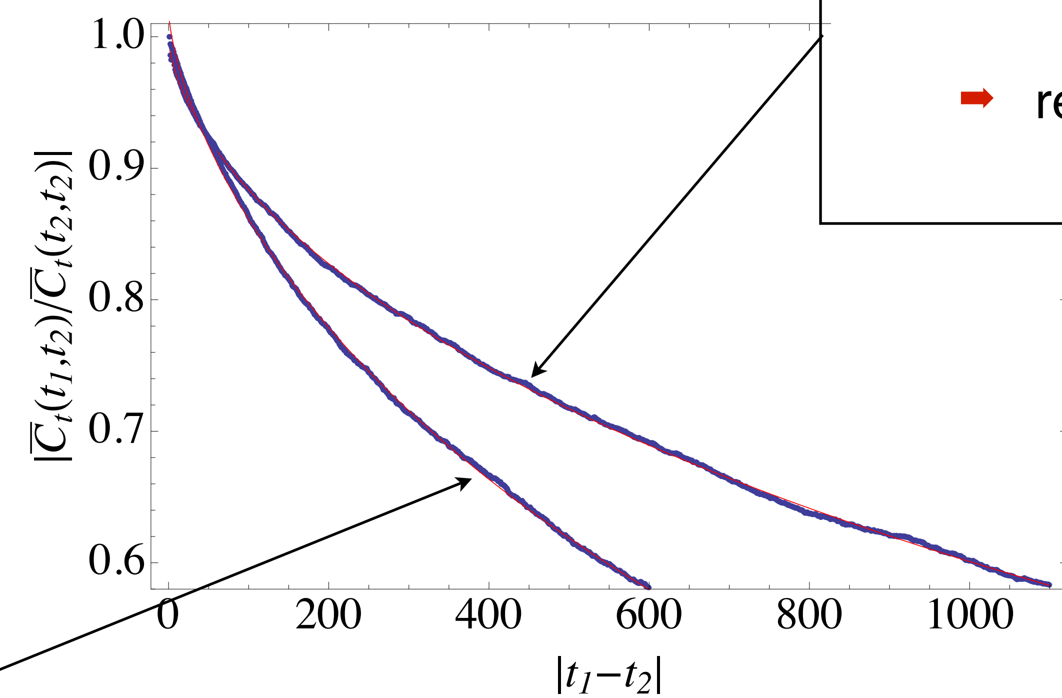
KPZ scaling: fit with $Ae^{-B|t_1-t_2|^{2\beta}}$ ➡ $\beta \approx 1/3$

parameters from Wertz et al. PRL (2012)

$\tau = 30\mathrm{ps}$

➡ required system size

$L = 3 \times 10^3 \mu\mathrm{m}$



realistic system size

$L = 1.5 \times 10^2 \mu\mathrm{m}$

➡ required polariton lifetime

$\tau = 1\mathrm{ps}$

➡ realistic system sizes possible for reduced Q factor

L. He, L. Sieberer, and SD, in preparation.

Appearance of a second scale

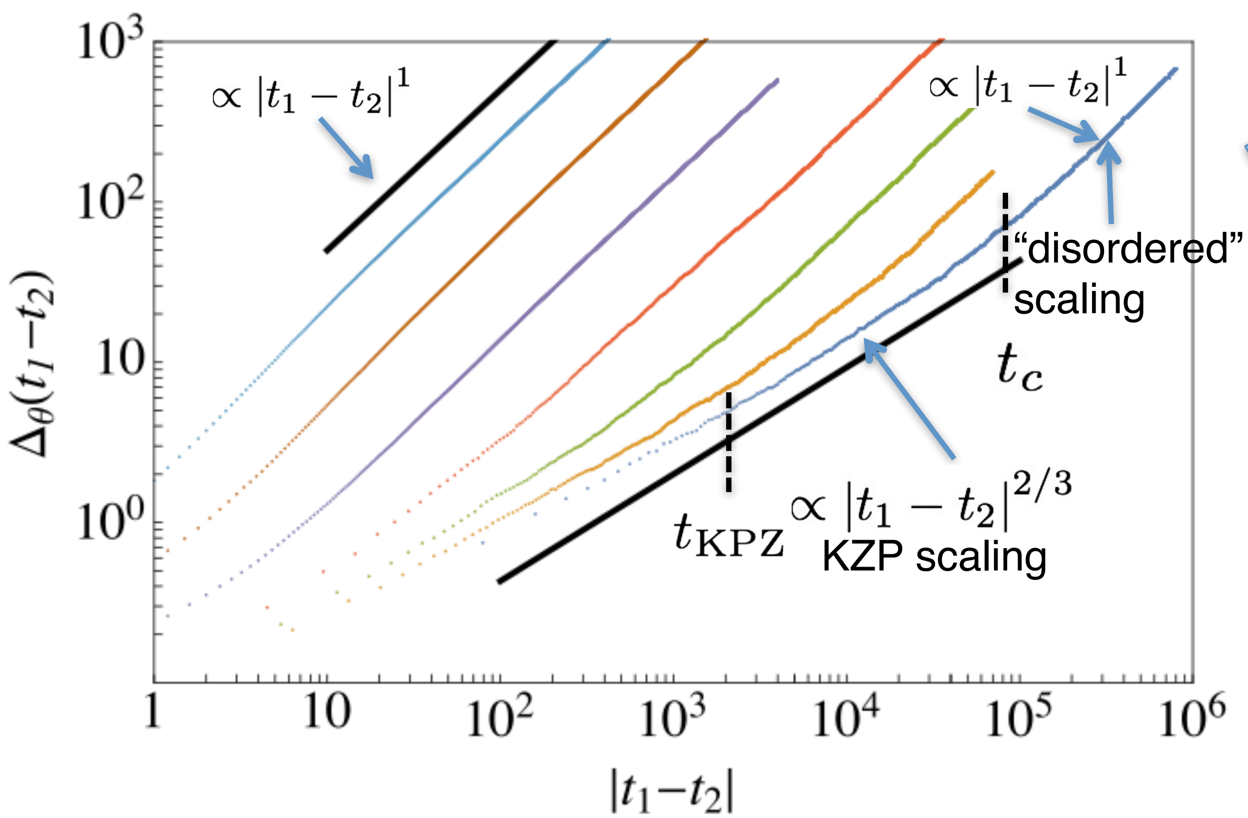
- crossover from KPZ to “thermal” scaling at asymptotic time scales
  - observable: temporal phase fluctuations

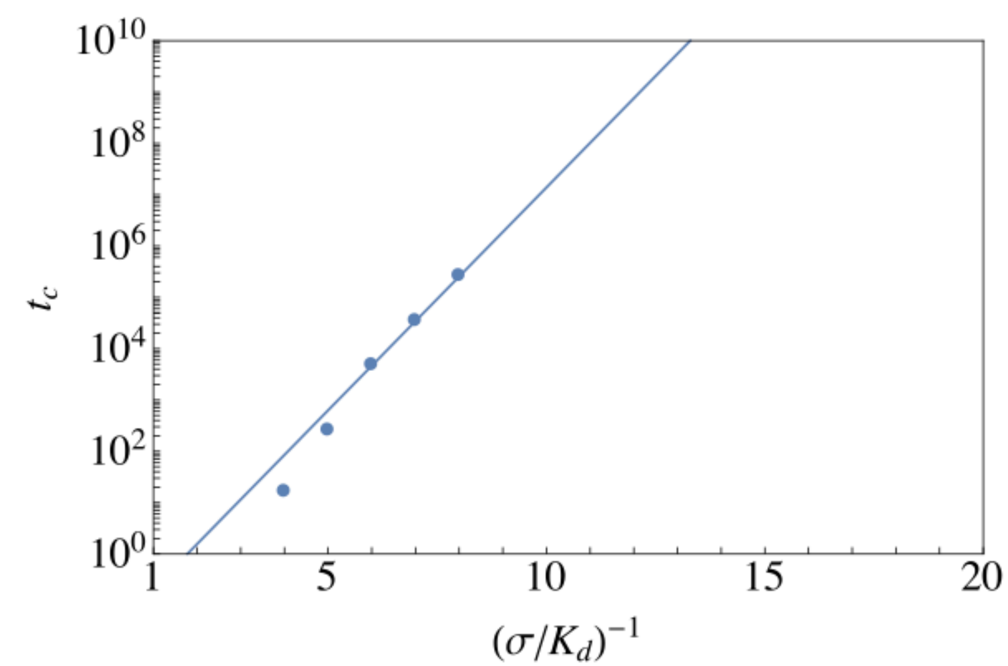
$$\Delta_\theta(t_1 - t_2) \equiv \frac{1}{L}\int dx \left\{ \left\langle [\theta(x,t_1) - \theta(x,t_2)]^2 \right\rangle - \langle \theta(x,t_1) - \theta(x,t_2) \rangle^2 \right\}$$

  - crossover from KPZ to “disordered” scaling -> second crossover scale t_c

$\Delta_\theta(t_1 - t_2) \propto |t_1 - t_2|^{2/3},\ if\ |t_1 - t_2| \ll t_c$ KPZ

$\Delta_\theta(t_1 - t_2) \propto |t_1 - t_2|^{1},\ if\ |t_1 - t_2| \gg t_c$ “disordered”





$t_c \propto e^{A\cdot\sigma^{-1}}$ in weak noise regime

$t_{\mathrm{KPZ}} \propto \sigma^{-2}$ $\Leftarrow$ $\left\{ \begin{array}{l} t_{\mathrm{KPZ}} \propto g^{-4} \\ g \equiv \lambda(\sigma/2D^3)^{1/2} \end{array} \right.$ in weak noise regime

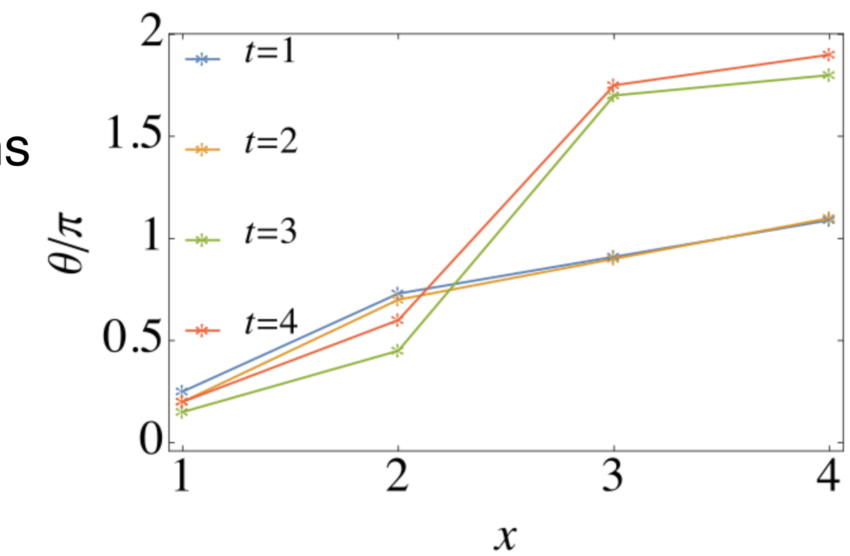
$\boxed{t_c \gg t_{\mathrm{KPZ}} \text{ guaranteed in weak noise regime!}}$

Space-time vortices in 1D XP condensate

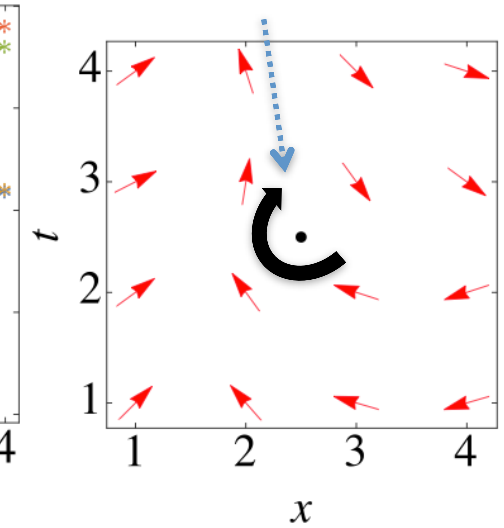
- Physical origin: compactness of phase field

topologically nontrivial phase field configurations on (1+1)D space-time plane, i.e. space-time vortices

spatial phase slip

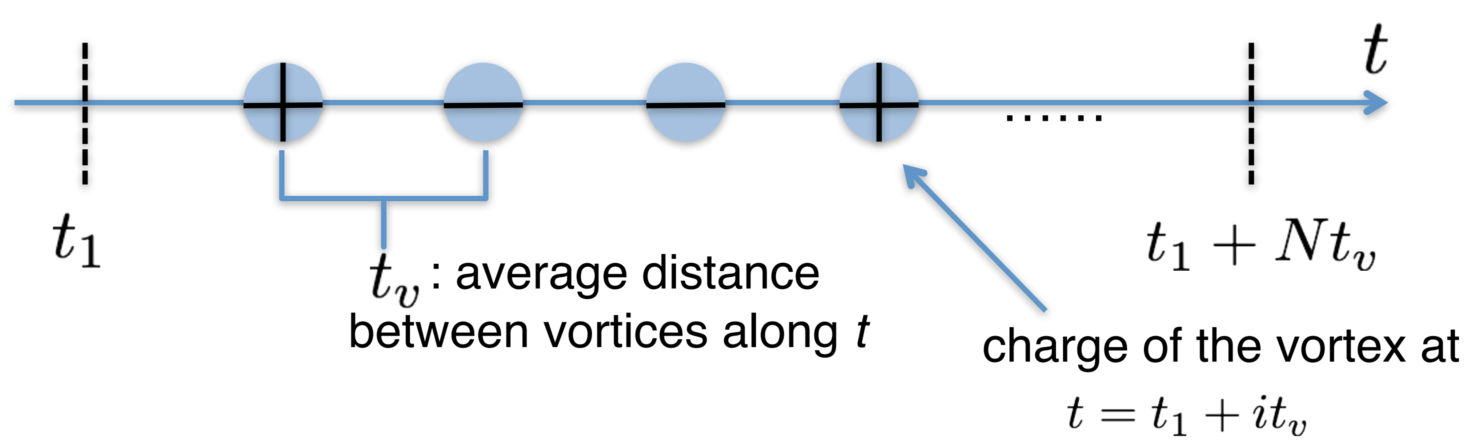


vortex in space-time plane



- Effects of vortices
  - cause large phase fluctuations giving rise to “disordered” scaling $\Delta_\theta(t_1 - t_2) \propto |t_1 - t_2|^1$

minimal model



- uniformly distributed vortices along t with random charges
- phase field value jumps by $\pm\delta\Theta$ whenever a vortex core with charge ± 1 is crossed

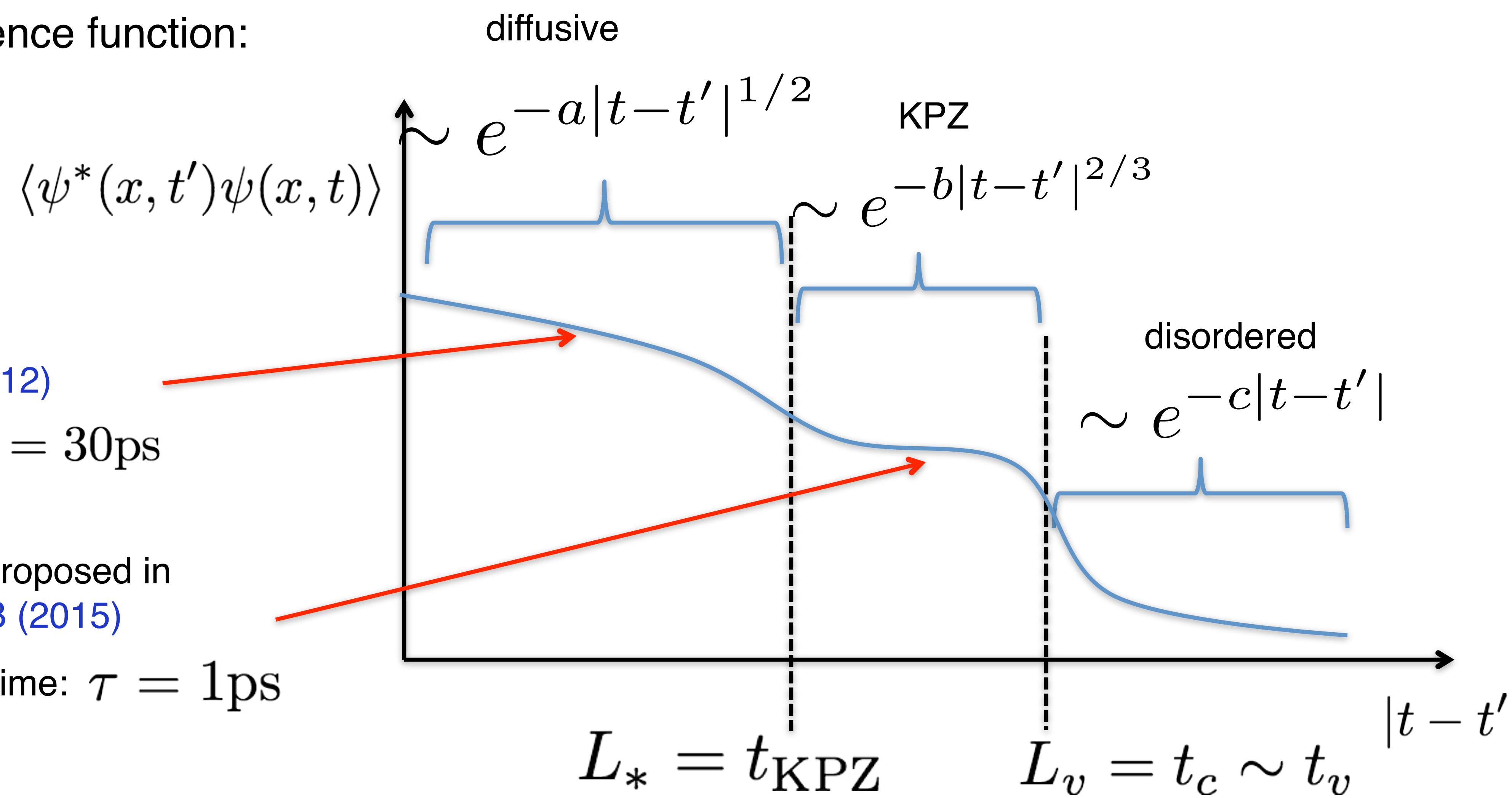
$$\theta(x, t_1 + Nt_v) - \theta(x, t_1) = \delta\Theta \sum_{i=1}^{N} W_i$$

random number ± 1 with equal probability

$$\Delta_\theta(t_1 - t_2) = |t_1 - t_2| \, (\delta\Theta)^2 / t_v$$ “disordered” scaling

Summary: 1D condensates

- temporal coherence function:



parameters from
Wertz et al. PRL (2012)

polariton life time: $\tau = 30\mathrm{ps}$

parameters proposed in
He et al. PRB (2015)

polariton life time: $\tau = 1\mathrm{ps}$

- crossover scales (weak noise)

$$t_{\mathrm{KPZ}} \propto \sigma^{-2}$$

noise level

$$t_c \propto e^{A\cdot\sigma^{-1}}$$

Summary

Review: L. Sieberer, M. Buchhold, SD, *Keldysh Field Theory for Driven Open Quantum Systems*, arxiv (2015)

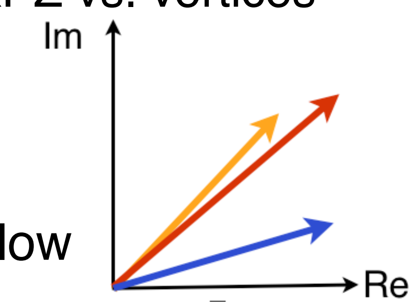
Driven open many-body systems: challenge to theory

microphysics → macrophysics

Many-Body Master Equation → 1-1 mapping → Keldysh functional integral →

Opens up QFT toolbox, here:

- symmetries: eq. vs. non-eq.
- control of IR fluctuations: driven phase transitions
- flexible choice of degrees of freedom: KPZ vs. vortices



- universal macroscopic consequences of microscopic driving:

critical behavior

classical

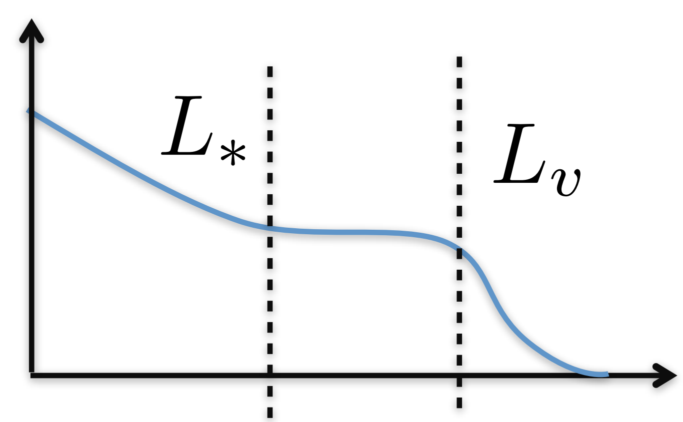
- equilibrium fixed point stable
- modified universality absence of equilibrium symmetry/ detailed balance

quantum

- new non-equilibrium fixed point
- no thermalization/ decoherence
- requires full quantum dynamical field theory

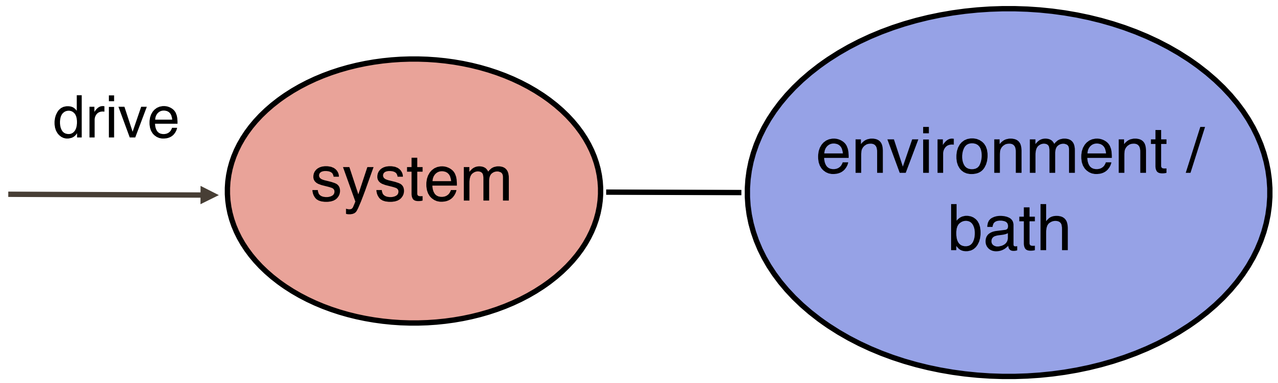
long distance behavior in low dimensions

- mapping to (compact) KPZ
- universal crossovers due to smooth and vortex fluctuations



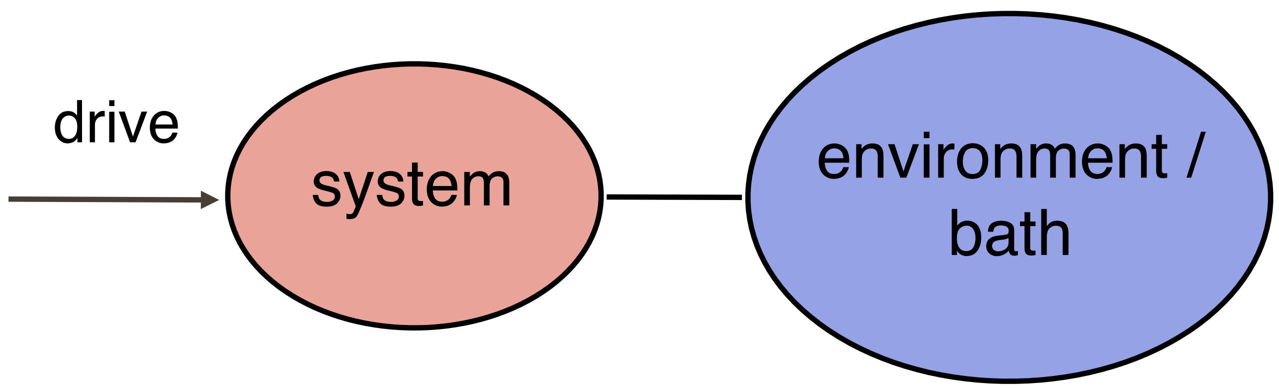
2D: $L_v \ll L_*$ 1D: $L_v \gg L_*$ (time scales)

Reminder: Quantum Master Equation



$$\partial_t \rho = -i[H,\rho] + \kappa \sum_i L_i \rho L_i^\dagger - \tfrac{1}{2}\{L_i^\dagger L_i, \rho\}$$

Brief Reminder: Driven Open Quantum Systems



$$H = \underbrace{H_{\mathrm{S}}}_{\sim \omega_0} + H_B + H_{\mathrm{int}}$$

$$H_B = \int d\omega\, \omega b_\omega^\dagger b_\omega$$

continuum bath of harmonic oscillators

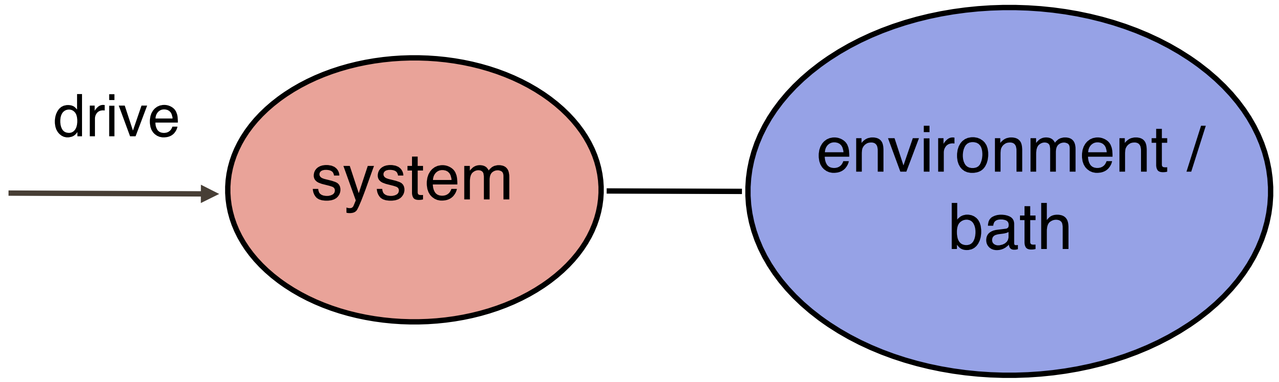
$$H_{\mathrm{int}} = i \int_{\omega_0 - \vartheta}^{\omega_0 + \vartheta} d\omega \kappa(\omega) \left[b_\omega^\dagger L - b_\omega L^\dagger \right]$$

reservoir bandwidth

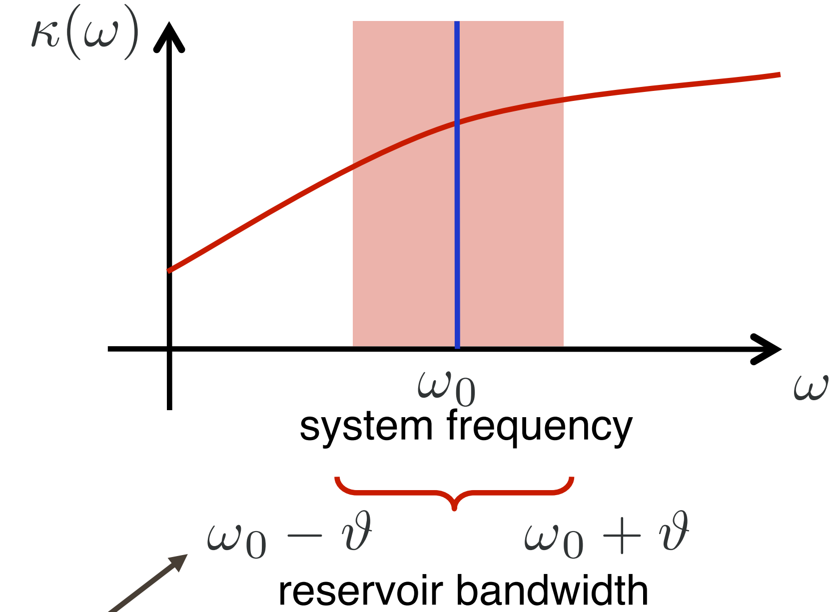
Lindblad / quantum jump operators
polynomial in system operators

linear bath operator coupling to the system

Brief Reminder: Driven Open Quantum Systems



$$\kappa(\omega) \rightarrow \kappa_0$$



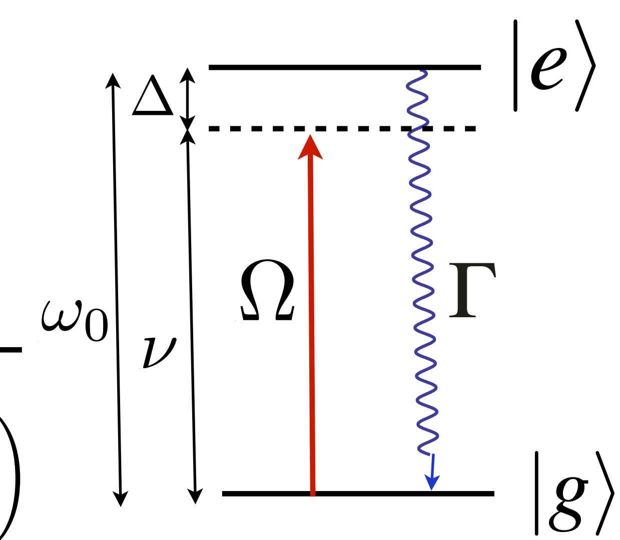
Three approximations (driven system):

(1) Born approximation: $\kappa(\omega)/\omega_0 \ll 1$

(2) Markov approximation: $\kappa(\omega) \approx \text{const.} \Rightarrow \kappa(t-t') \sim \delta(t-t')$

(3) Rotating wave approximation: $\dfrac{\omega_0 - \nu}{\omega_0 + \nu} \ll 1$

$$\omega_0 - \nu = \Delta$$ detuning



in this example:

system Hamiltonian $H = (|e\rangle, |g\rangle) \begin{pmatrix} \Delta & \Omega \\ \Omega & 0 \end{pmatrix} \begin{pmatrix} \langle e| \\ \langle g| \end{pmatrix}$

jump operator $J_\alpha = |g\rangle\langle e| = \sigma^-$