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UNIVERSITY OF INNSBRUCK

Many-Body Physics with Cold Atoms

IQOQI
AUSTRIAN ACADEMY OF SCIENCES

SFB
Coherent Control of Quantum Systems

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Introduction: Cold Atoms vs. Condensed Matter

Q: What can cold atoms add to many body physics?

- New models of own interest
  - Bose-Hubbard model
  - Strongly interacting continuum systems: BCS-BEC Crossover; Efimov effect
  - SU(N) Heisenberg models with variable N
- Quantum Simulation: clean/ controllable realization of model Hamiltonians which are
  - less clear to what extent realized in condensed matter
  - extremely hard to analyze theoretically

  e.g. 2d Fermi-Hubbard model
- Nonequilibrium Physics, time dependence:
  - Condensed matter: thermodynamic equilibrium and ground state physics.
  - Cold atoms: e.g. creation of excited many-body states, study their dynamic behavior
- Scalability:
  - Study crossover from systems with few- to (thermodynamically) many degrees of freedom: Change of concepts?
- Integrate quantum optics concepts and many-body theory, coherent-dissipative manipulation

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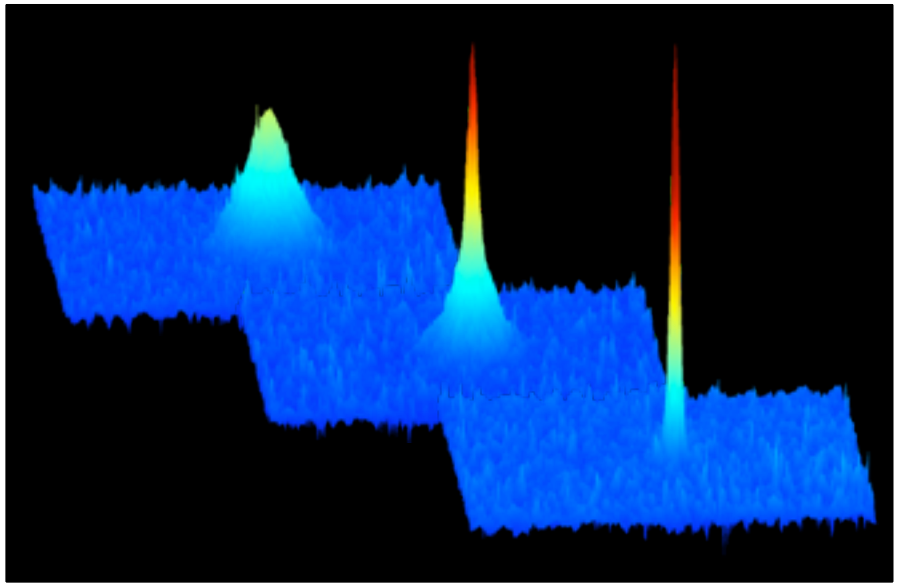
A Brief History of cold atoms and BEC research



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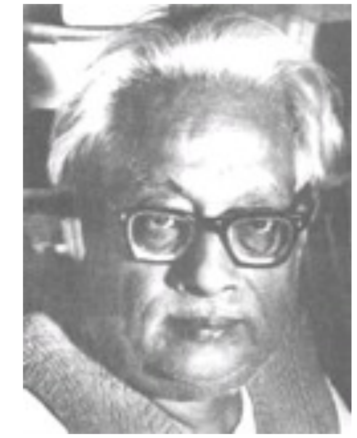


^4He liquified by K. Onnes — 1908

Bose studies statistical Properties of photons — 1924

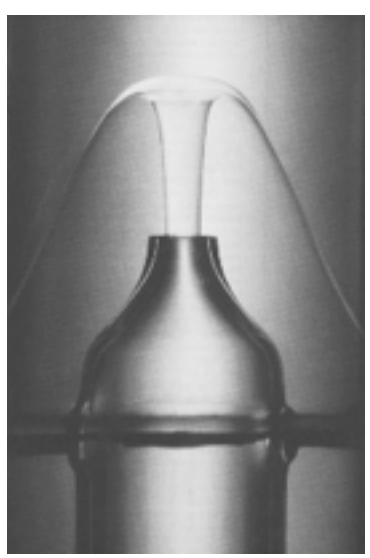
1925 — Bose/Einstein predict condensation of a noninteracting atomic Bose gas

Kapitza / Allen & Misener discover superfluidity of ^4He — 1938

1938 — London makes first connection between ^4He superfluidity and BEC

Allen & Jones discover fountain effect

1941 — Landau produces first self-consistent theory of superfluids



1947 — Bogoliubov produces microscopic theory of weakly interacting Bose gas

1949 — Onsager predicts quantized vortices in liquid ^4He (+ Feynman 1955)

1951 — Landau & Lifshitz / Penrose link BEC and off-diagonal long range order.

A Brief History of cold atoms and BEC research

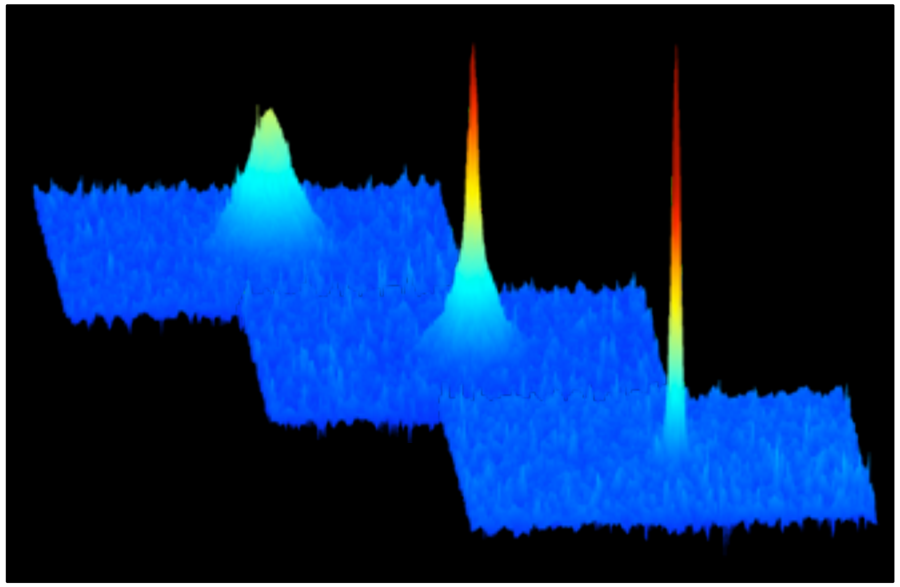
1976 Stwalley and Nosanow (following Hecht, 1959) argue spin polarised ^1H would be Superfluid BEC (Realised MIT 1998).

1985 Neutral Atom traps:
Gaithersburg (Magnetic)
AT&T Bell Labs

1987 Simultaneous Cooling and Trapping in Magneto-Optical Trap (AT&T Bell labs)



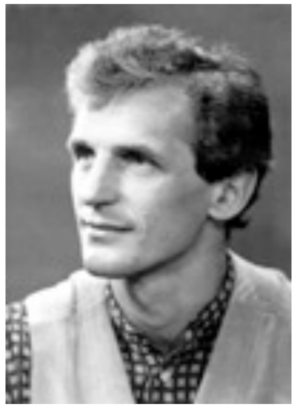
1995 BEC in dilute gases:
JILA (^{87}Rb), MIT (^{23}Na), Rice (^{7}Li)



1997 W. D. Phillips, S. Chu, and C. Cohen-Tannoudji: Nobel Prize for Laser Cooling

2001 E. Cornell, C. Wieman, W. Ketterle: Nobel Prize for BEC in dilute gases





A Brief History of cold atoms and BEC research

		1995	1997	1998	1998	1998	1998	1999	1999	2001	2001	2002	2002	2002
Rb	1995	9	8	3	1	3	5	2	1	1	1	1		1
Na	1995	3												
Li	1995/97	1		1										
H	1998	1												
He*	2001			2										
K	2001							1						
Cs	2002												1	

Innsbruck
05. Oct. 2002

+ Yb (2004), Cr (2005)

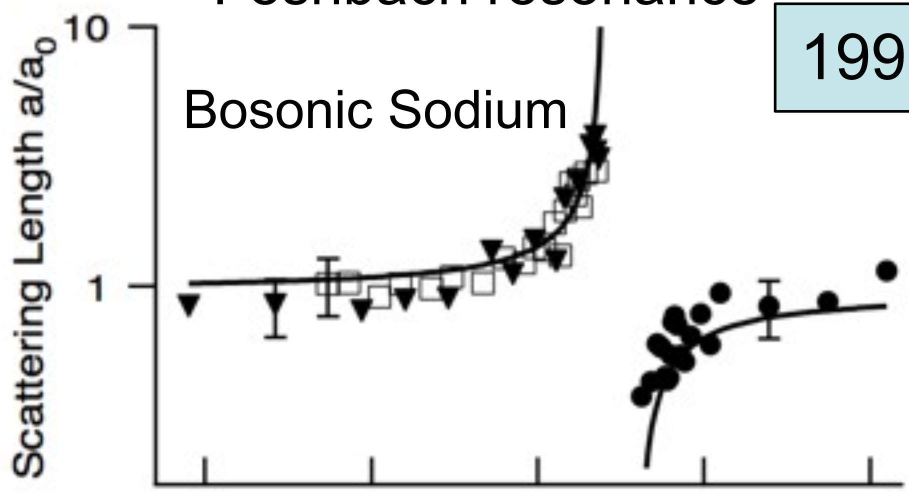
Molecules: Li_2 (2003), K_2 (2003)

Earth alkaline atoms: Ca (2009), Sr (2009)

A Brief History of cold atoms and BEC research

- Further milestone experiments:

Tunable interations:
Feshbach resonance

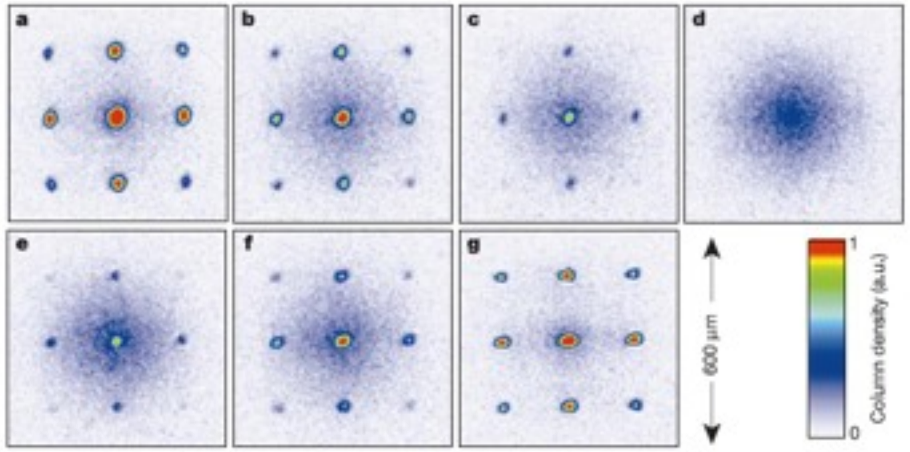


1999

Ketterle Group, MIT

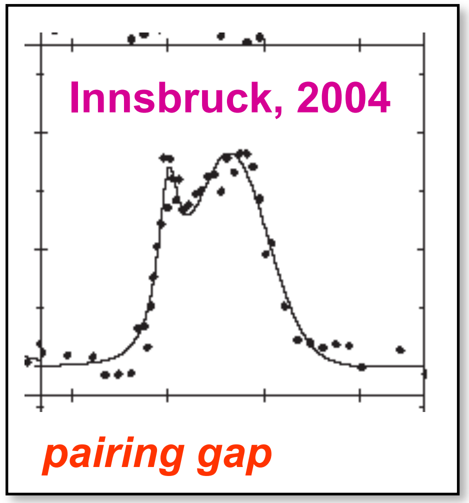
2002

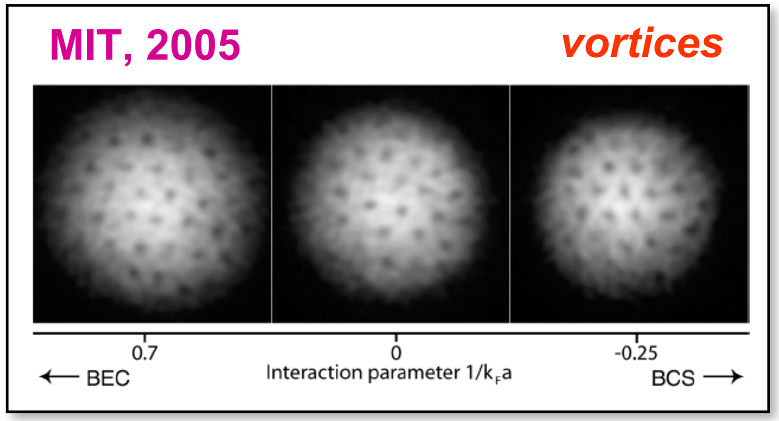
Optical Lattices:
Mott insulator-superfluid
quantum phase transition



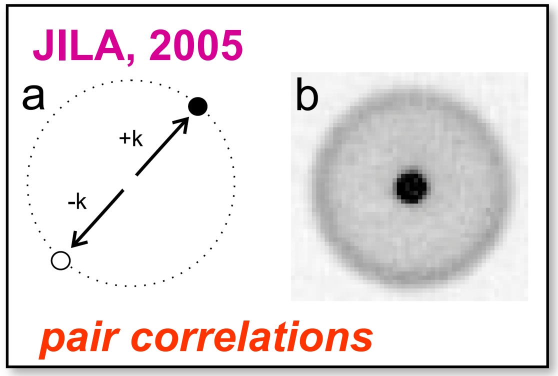
Haensch Group, Munich

Degenerate and strongly interacting fermions





2004



Absolute Scales in Condensed Matter vs. Cold Atoms

- Compare the absolute scales in of liquid ^{4}He and cold atom BEC

  - Liquid ^{4}He "BEC"
    - Typical density 10^{22} cm^{-3}
    - Condensation temperature ~ 2.17 K
    - Strongly Interacting
    - Small Condensate Fraction

  - BEC in dilute gases
    - Typical density 10^{13} – 10^{15} cm^{-3} (cf. density of air: $3\text{x}10^{19}$ cm^{-3})
    - Condensation temperature < 10^{-5} K (cf. CMB radiation: ~ 2.7K)
    - Weakly interacting
    - Much larger condensate fraction

Modelling in Condensed Matter vs. Cold Atoms

- Compare the way of constructing microscopic models in these systems
  - Condensed Matter
    - given chunk of matter
    - have to guess the microscopic Hamiltonian
  - Cold Atoms
    - microscopic parameters from scattering physics (n=T=0)
    - many body physics in separate experiments

➡ Direct experimental access to both micro- and macrophysics

➡ Challenge to theorists to make the connection

➡ Many disciplines and theoretical techniques are involved: Atomic physics, Quantum optics, Condensed matter physics

- Further general properties and features:

Clean system:

- No (uncontrollable) disorder
- Weak dissipation (>1s) (cf. phonons in solid state)

Control:

- tune microscopic parameters,
- modify lattice structure
- choose effective dimensionality

Measurements:

- (quasi-)momentum distribution, noise correlations by releasing atoms.
- Spectroscopy (e.g., lattice modulations / Bragg scattering).

Units

- In these lecture series, we will often work in “natural units”:
  - Quantum mechanical problem:

$$\hbar = 1$$

➡ Measure energy and frequency in the same unit, $E = \hbar\omega$

  - Many-Body problem:

$$k_{\mathrm{B}} = 1$$

➡ Measure energy and temperature in the same unit, $E = \frac{1}{2}k_{\mathrm{B}}T$

  - Nonrelativistic problem: (we often indicate the mass explicitly)

$$2M = 1$$

➡ Measure energy and momentum square in the same unit, $E = \dfrac{\vec{p}^{\,2}}{2M}$

  - Optical problem:

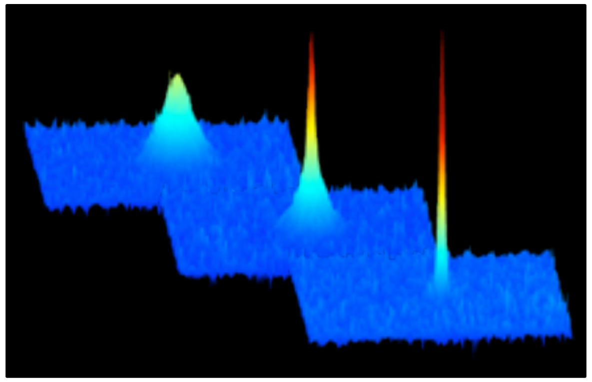
$$c = 1$$

➡ Measure frequency an wave number in the same unit, $\omega = ck$

Lecture Outline

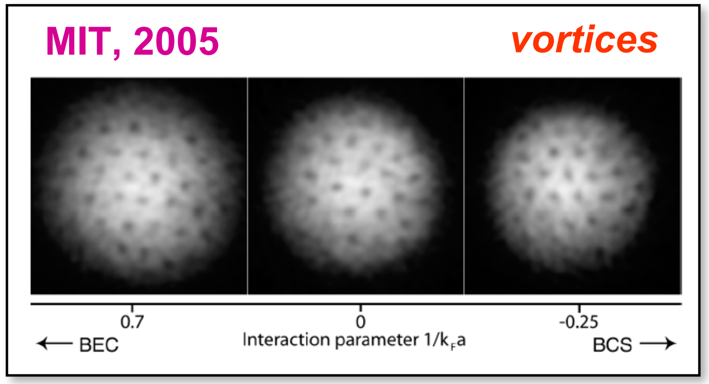
The lecture series consists of two parts:

Part I: Introduction to the theory of ultracold atomic quantum gases



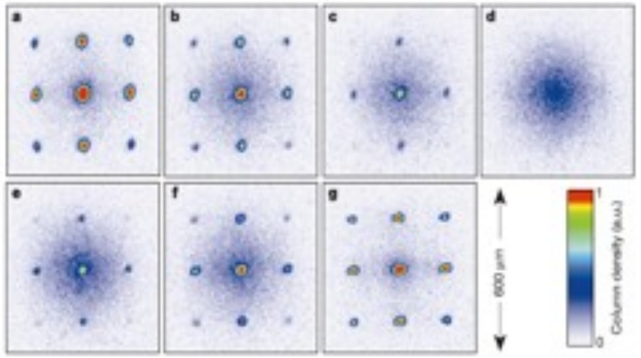
Bose-Einstein Condensation

- Continuum systems:
  - Scales and interactions, Effective theories
  - Weakly interacting Bosons, Bose-Einstein condensation (2nd quantized and functional integral formulation)
  - Weakly interacting Fermions, BCS superfluidity
  - Strong interactions, the BCS-BEC crossover



Fermion Superfluidity

- Lattice systems:
  - Quantum phase transitions
  - Microscopic derivation of the Bose-Hubbard model
  - Phase Diagram of the Bose-Hubbard model, Mott insulator - superfluid transition



Mott insulator - superfluid transition

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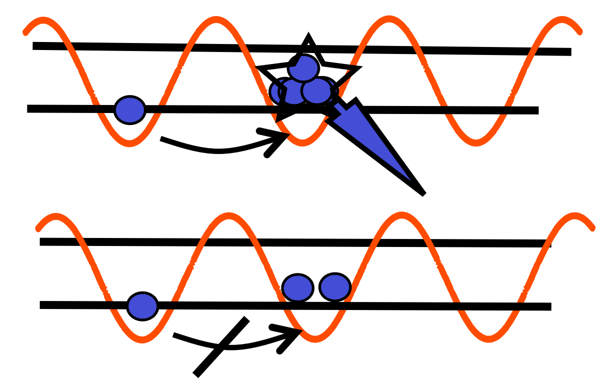
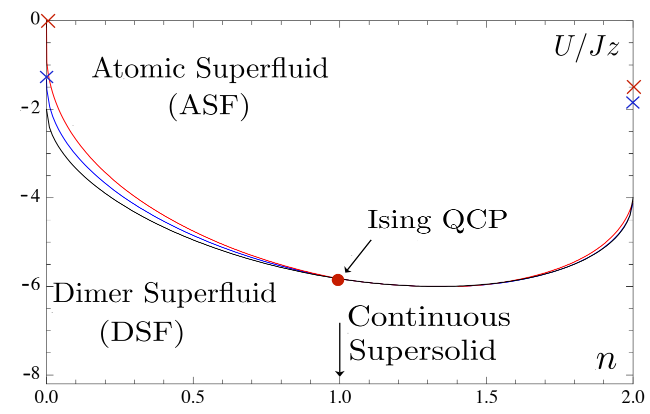
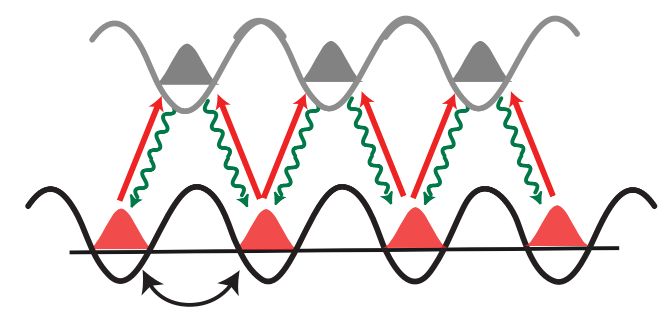


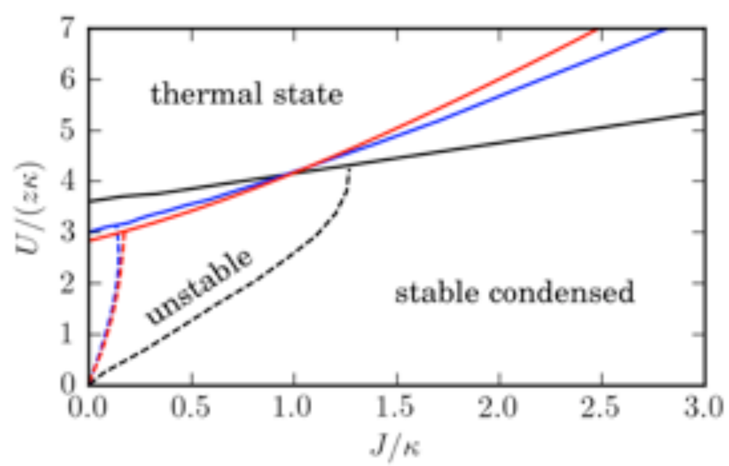
Part II: Many-Body Quantum Optics: Making use of drive and dissipation in cold atomic gases



- Lattice models with 3-body hardcore constraint
  - Microscopic origin: interactions via loss
  - Application I: phase diagram of attractive 3-hardcore bosons
  - Application II: atomic color superfluid for 3-component fermions



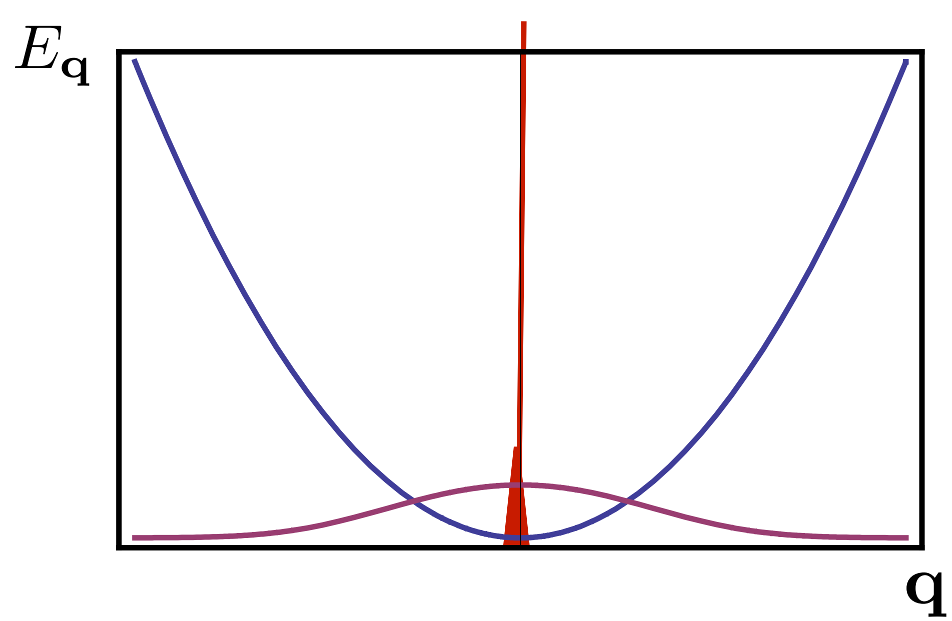
- Quantum state engineering in driven-dissipative many-body systems
  - Proof of principle: driven-dissipative BEC
  - Application I: nonequilibrium phase transition from competing unitary and dissipative dynamics
  - Application II: cooling into antiferromagnetic and d-wave paired states for fermions



Literature

- General BEC/BCS theory:
  - C. J. Pethick, H. Smith, Bose-Einstein Condensation in Dilute Gases, Cambridge University Press (2002)
  - L. Pitaevski and S. Stringari, Bose-Einstein Condensation, Oxford University Press (2003)
- Recent research review articles:
  - I. Bloch, J. Dalibard and W. Zwerger, Many-Body Physics with Cold Atoms, Rev. Mod. Phys. **80**, 885 (2008)
  - S. Giorgini, L. Pitaevski and S. Stringari, Theory of ultracold atomic Fermi gases, Rev. Mod. Phys. **80**, 1215 (2008)
- Many-Body Physics (with and without cold atoms)
  - J. Negele, H. Orland, Quantum Many-particle Systems, Westview Press (1998)
  - A. Altland and B. Simons, Condensed Matter Field Theory, Cambridge University Press (2006)
  - S. Sachdev, Quantum Phase Transitions, Cambridge University Press (1999)
  - K. B. Gubbels and H. C. T. Stoof, Ultracold Quantum Fields, Springer Verlag (2009)

Basic BEC theory



Statistical Mechanics of Noninteracting Bosons

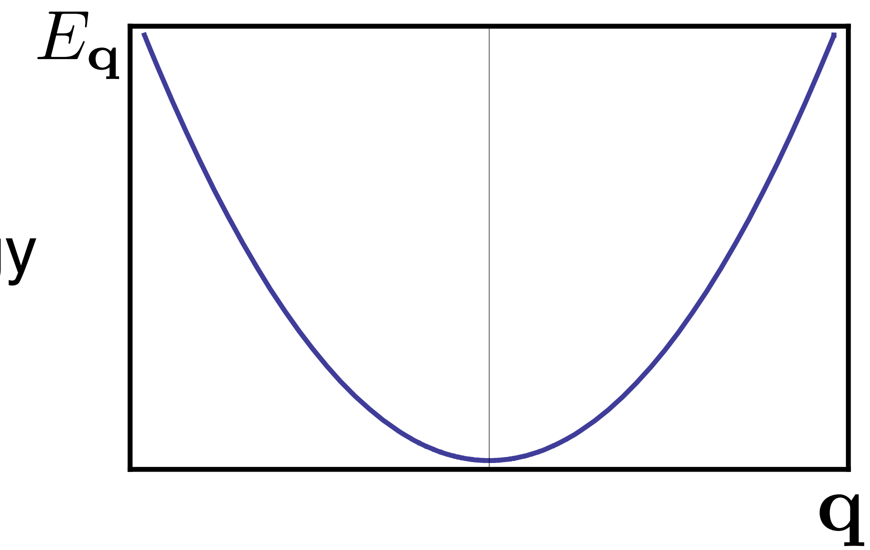
- The Hamiltonian for free particles in occupation number representation ("second quantization")

$$H = \sum_{\mathbf{q}} \frac{\mathbf{q}^2}{2m} a_{\mathbf{q}}^\dagger a_{\mathbf{q}}$$

  - The operator $a_{\mathbf{q}}^\dagger$ creates a particle in the momentum mode **q** : $a_{\mathbf{q}}^{\dagger N}|vac\rangle = \sqrt{(N+1)!}|n_{\mathbf{q}} = N\rangle$. Similarly, $a_{\mathbf{q}}$ annihilates a particle in the mode **q**.
  - The operators satisfy bosonic commutation relations, $[a_{\mathbf{q}}, a_{\mathbf{q}'}^\dagger] = \delta(\mathbf{q}-\mathbf{q}')$ (cf. harmonic oscillator ladder operators, but with an index label **q**).
  - The operator $n_{\mathbf{q}} = a_{\mathbf{q}}^\dagger a_{\mathbf{q}}$ measures the particle number in the momentum mode **q**. The total particle number is $\hat{N} = \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}}$.
  - The dispersion relation is $E_{\mathbf{q}} = \hbar\omega = \frac{\mathbf{q}^2}{2m}$. This is the energy of a single particle in the mode **q**.

- Free particles in occupation number representation = collection of independent harmonic oscillators with energy

$$E_{\mathbf{q}} = \frac{\mathbf{q}^2}{2m}$$



Statistical Mechanics of Noninteracting Bosons

- Statistical properties described by the Free Energy:

$$U = k_B T \log Z, \qquad Z = \mathrm{tr} \exp -\frac{1}{k_B T}(H - \mu \hat{N})$$

temperature

chemical potential

- Trace easily carried out in occupation number basis: H is diagonal:

$$U = k_B T \log \prod_{\mathbf{q}} \sum_{n_{\mathbf{q}}=0}^{\infty} e^{-\frac{1}{k_B T}(\frac{\mathbf{q}^2}{2m}-\mu)n_{\mathbf{q}}} = k_B T \log \prod_{\mathbf{q}} \left(1 - e^{-\frac{1}{k_B T}(\frac{\mathbf{q}^2}{2m}-\mu)}\right)^{-1} = -k_B T \sum_{\mathbf{q}} \log\left(1 - e^{-\frac{1}{k_B T}(\frac{\mathbf{q}^2}{2m}-\mu)}\right)$$

- Consider low temperature behavior of equation of state:

$$N = -\frac{\partial U}{\partial \mu} = \langle \sum_{\mathbf{q}} a^\dagger_{\mathbf{q}} a_{\mathbf{q}} \rangle = \sum_{\mathbf{q}} \langle a^\dagger_{\mathbf{q}} a_{\mathbf{q}} \rangle = \sum_{\mathbf{q}} \frac{1}{e^{\frac{1}{k_B T}(\frac{\mathbf{q}^2}{2m}-\mu)} - 1} \to V \int \frac{d^d q}{(2\pi)^d} \frac{1}{e^{\frac{1}{k_B T}(\frac{\mathbf{q}^2}{2m}-\mu)} - 1}$$

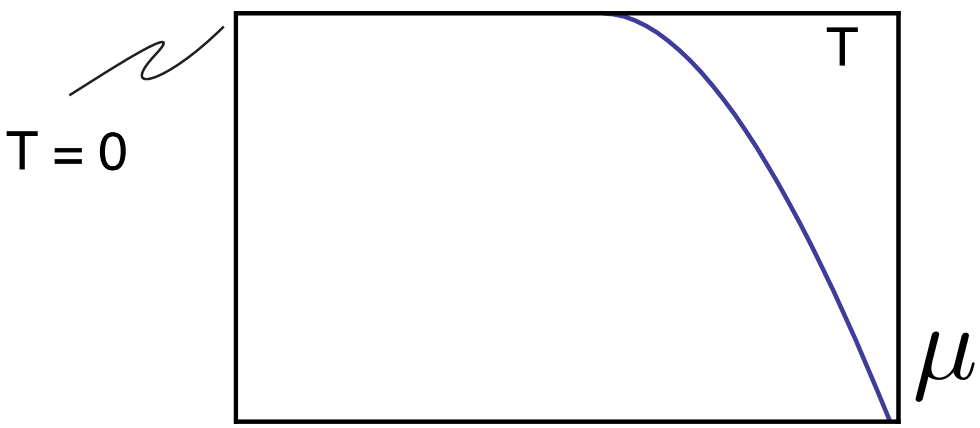
continuum limit

Bose-Einstein distribution

Bose-Einstein Condensation, 3D

- Lower T and study the behavior of μ at fixed n (3D):

$d = 3$



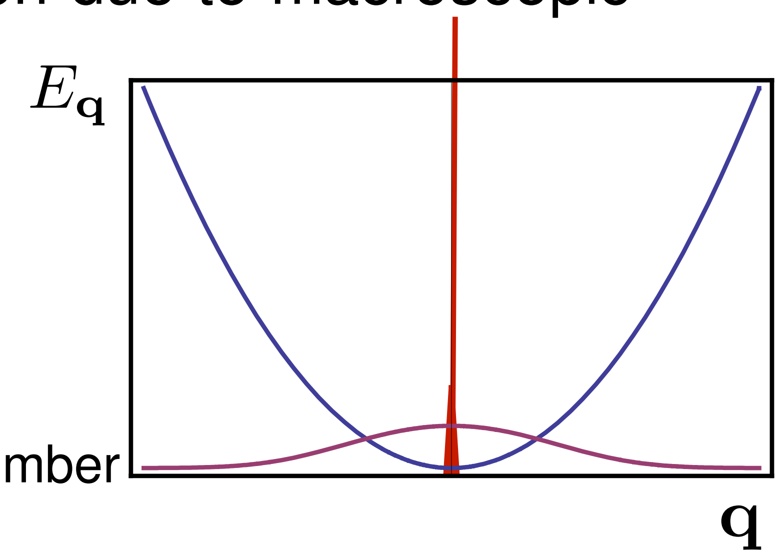
$$n = \frac{N}{V} = \int \frac{d^d q}{(2\pi)^d} \frac{1}{e^{\frac{1}{k_B T}\left(\frac{\mathbf{q}^2}{2m} - \mu\right)} - 1}$$

$$= \lambda_{dB}^{-3}(T) g_{3/2}(e^{\mu/k_B T})$$

Polygamma function

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}$$

- At a finite T, μ hits zero: below this T_c the equation of state has no solution
- Bose and Einstein (1925): Equation below T_c needs modification due to macroscopic occupation of zero mode:



$$n = \langle a_{\mathbf{0}}^\dagger a_{\mathbf{0}} \rangle + \int \frac{d^d q}{(2\pi)^d} \frac{1}{e^{\frac{\mathbf{q}\text{^}\mathbf{2}}{2m k_B T}} - 1}$$

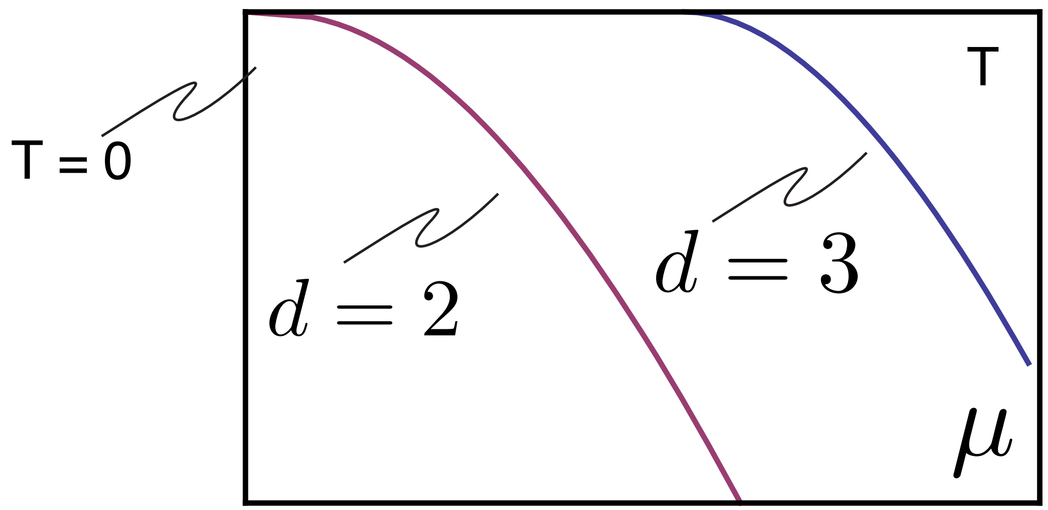
  - plausible: Bosons can populate single quantum state with arbitrary number
  - macroscopic: $N_0 = \langle a_{\mathbf{0}}^\dagger a_{\mathbf{0}} \rangle = \mathcal{O}(N) \propto V$, i.e. extensive
  - critical temperature: determined by

$$n\lambda_{dB}^3 = g_{3/2}(1) = \zeta(3/2) \approx 2.612$$

de Broglie wavelength $\lambda_{dB} = (2\pi\hbar^2/mk_B T)^{1/2} \gtrsim d = n^{-1/3}$ interparticle spacing

Role of Dimension: No BEC in 2D

- Lower T and study the behavior of μ at fixed n (2D):



➡ EoS without condensate can be fulfilled for all T:

➡ No BEC in (homogeneous) 2d space

- Reason: Infrared (low momentum) divergence due to smaller phase space

$$\frac{d^2q}{(2\pi)^2}\langle\hat{n}_{\mathbf{q}}\rangle = \frac{d^2q}{(2\pi)^2}\frac{1}{e^{(\frac{q^2}{2m}-\mu)/k_BT}-1} \propto \frac{dqq}{q^2-\mu}$$

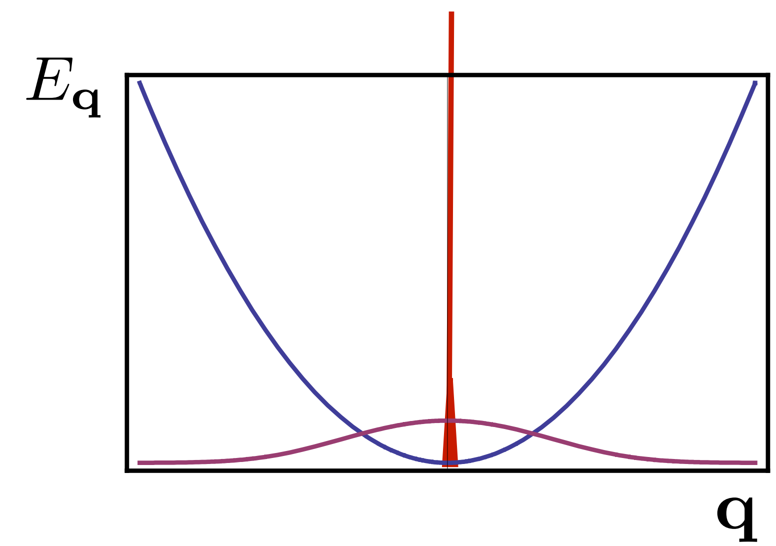
low momenta, small mu

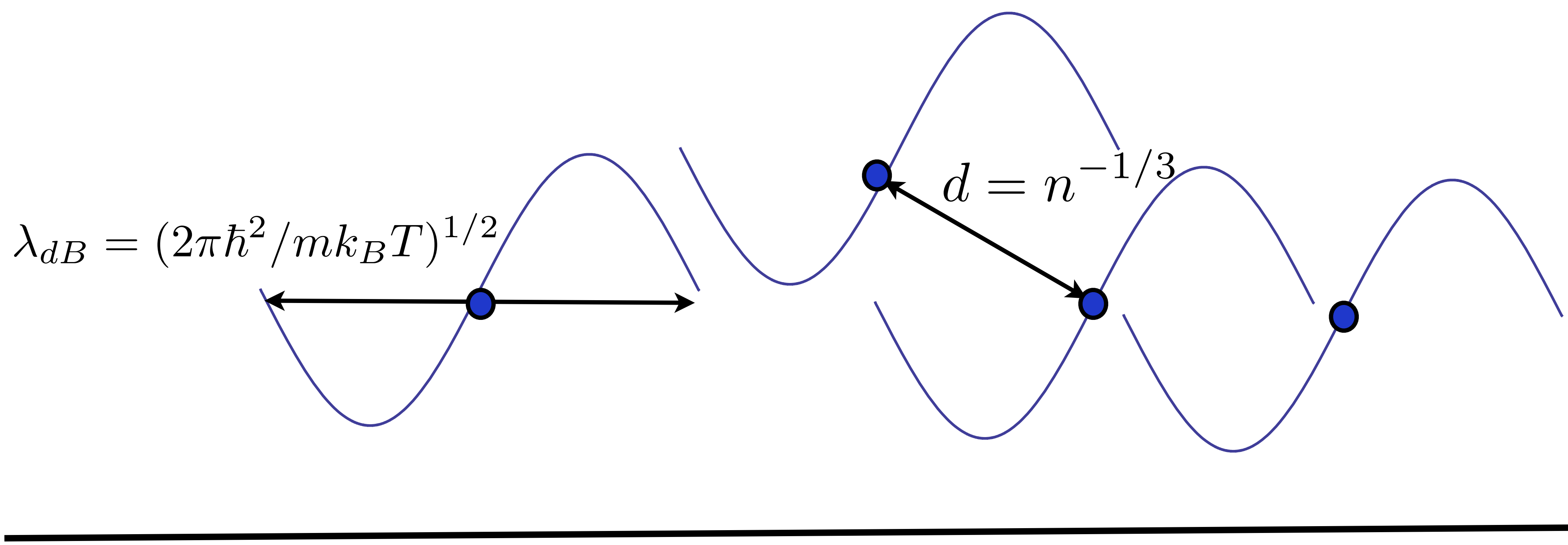
logarithmic divergence

- Remark: More general result
  - No spontaneous breaking of continuous symmetries at finite temperature (Mermin-Wagner Theorem)
  - quasi long range order possible: Kosterlitz-Thouless transition

Summary

- BEC is statistical effect, no interaction needed
- BEC is macroscopic population of a single quantum state, the q=0 mode
- quantum degeneracy condition: $\lambda_{dB} \gtrsim d$
- True BEC at finite temperature only in d=3





- Now: more realistic description of ultracold Bose systems
  - Trapping potential
  - Interactions

Trapping Potential and Weak Interactions

- So far: free particles in homogeneous space, continuum limit :

$$H_0 = H - \mu\hat{N} = \int_{\mathbf{q}} a^\dagger_{\mathbf{q}}(\frac{\mathbf{q}^2}{2m} - \mu)a_{\mathbf{q}} = \int_{\mathbf{x}} a^\dagger_{\mathbf{x}}(-\frac{\triangle}{2m} - \mu)a_{\mathbf{x}}$$

after Fourier transform $\quad a_{\mathbf{q}} = \int_{\mathbf{x}} e^{\mathrm{i}\mathbf{q}\mathbf{x}} a_{\mathbf{x}}; \quad [a_{\mathbf{x}}, a^\dagger_{\mathbf{y}}] = \delta(\mathbf{x} - \mathbf{y})$

- Now we aim at a more realistic description of cold atomic gases:

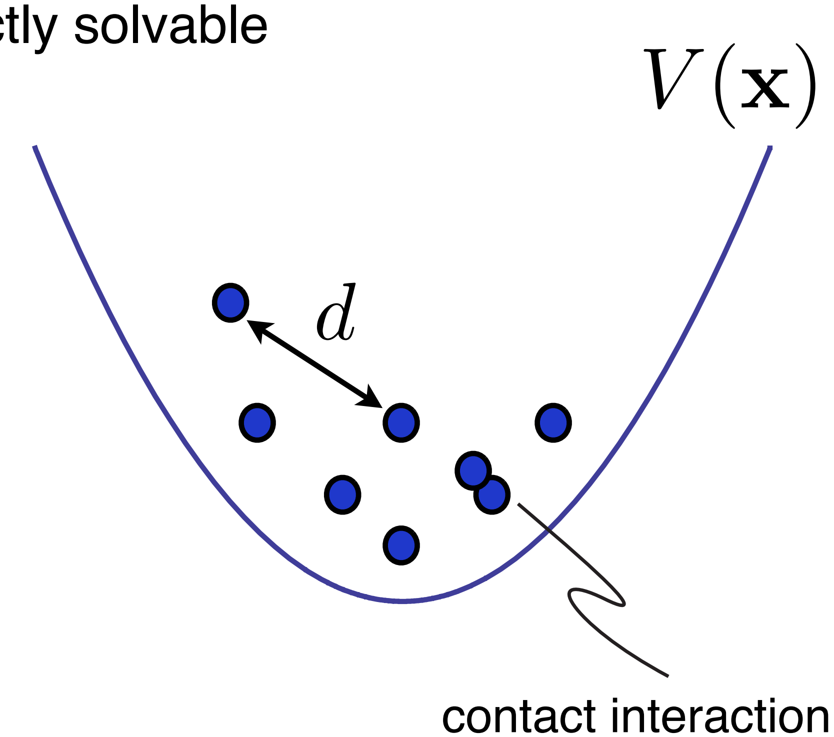
  - Trapping potential: the local density experiences a local potential energy

$$H_{trap} = \int_{\mathbf{x}} V(\mathbf{x})\hat{n}_{\mathbf{x}}, \quad V(\mathbf{x}) = \tfrac{1}{2}m\omega^2\mathbf{x}^2$$

Remark: this is just a collection of local harmonic oscillators: exactly solvable

  - Local two-body interactions:

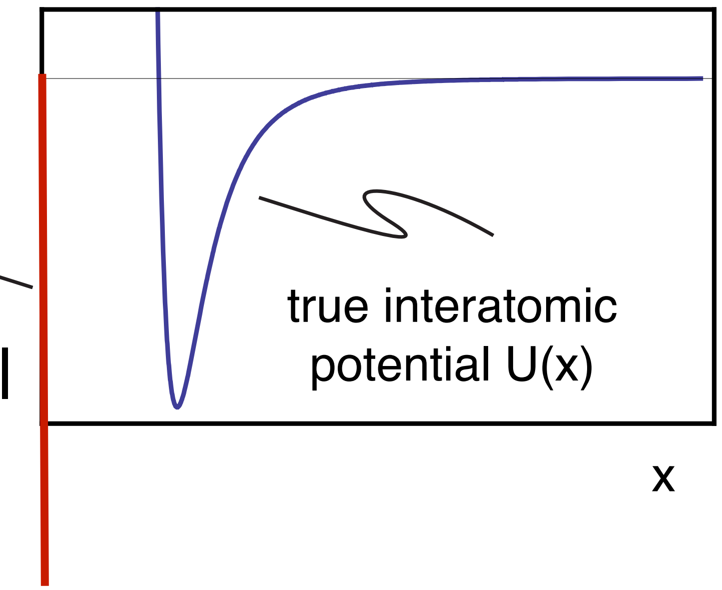
$$H_{int} = \int_{\mathbf{x},\mathbf{y}} g\delta(\mathbf{x} - \mathbf{y})\hat{n}_{\mathbf{x}}\hat{n}_{\mathbf{y}} = g\int_{\mathbf{x}} \hat{n}^2_{\mathbf{x}}$$



- Our workhorse Hamiltonian is

$$H = H_0 + H_{trap} + H_{int}$$

Microscopic Origin of the Interaction Term



- Microscopic scattering physics: Lennard-Jones (LJ) potential
  - $1/r^{12}$ hard core repulsion: repulsion of electron clouds $r_{rep} = \mathcal{O}(a_B)$
  - $1/r^{6}$ attraction: van der Waals (induced dipole-dipole interaction)
    $r_{vdW} = (50...200)a_B$ for alkalis: typ. order of magnitude for interaction length scale

- General properties of LJ type potentials at low energies:
  - isotropic s-wave scattering dominates; the scattered wave function behaves asymptotically as $\psi(\mathbf{x}) \sim a/\mathbf{x}$
  - a is the scattering length. Knowledge of this single parameter is sufficient to describe low energy scattering!
  - within Born approximation, it can be calculated as

$$a_{\text{Born}} \sim \int_{\mathbf{x}} U(\mathbf{x})$$

interatomic potential

➡very different interaction potentials may have the same scattering length!

The Model Hamiltonian as an Effective Theory

$$H = \int_{\mathbf{x}} [a^\dagger_{\mathbf{x}}(- \frac{\triangle}{2m} - \mu + V(x))a_{\mathbf{x}} + g\hat{n}^2_{\mathbf{x}}]$$

- Efficient description by an effective Hamiltonian with few parameters.
- For ultracold bosonic alkali gases, a single parameter, the scattering length a, is sufficient to characterize low energy scattering physics of indistinguishable particles : Effective interaction

$$g = \frac{8\pi\hbar^2}{m}a$$

- A typical order of magnitude for the scattering length is

$$a = \mathcal{O}(r_{vdW}), \ r_{vdW}(50...200)a_B$$

- The validity of the model Hamiltonian is restricted to length scales

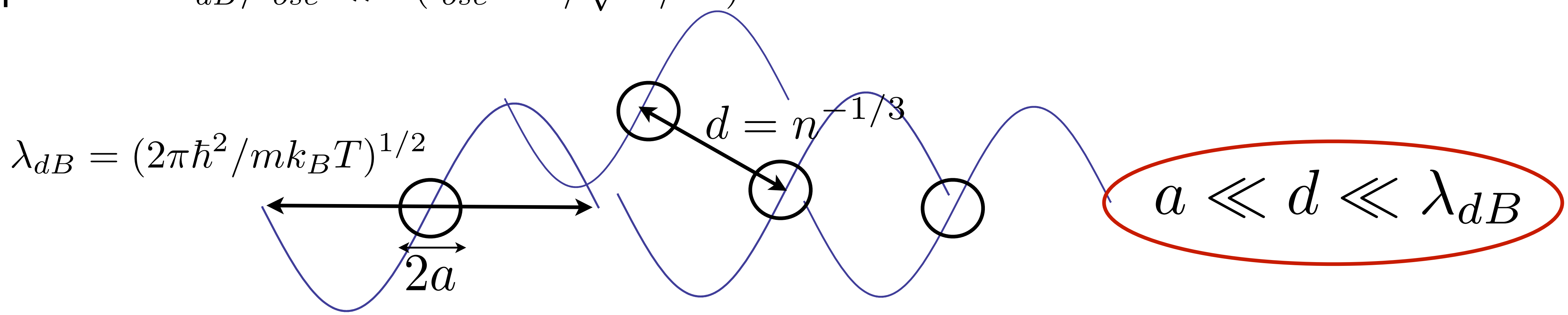
$$l \gg r_{vdW}$$

- For bosons, we must restrict to repulsive interactions a > 0 (else: bosons seek solid ground state, collapse in real space)

- So far: microscopic description; now: many body scales!

Validity of our Hamiltonian: Scales in Cold Dilute Bose Gases

- The effective Hamiltonian is valid because none of many-body length scales can resolve interaction length scale :
- Many-body scales: density and temperature in terms of length scales.
  - diluteness $a/d \ll 1$ $(d = n^{-1/3})$
    - $*$ dilute means weakly interacting: interaction energy $gn \sim a/d \cdot d^{-2}$
    - $*$ clear: three-body interaction terms irrelevant
  - quantum degeneracy: $d/\lambda_{dB} \ll 1$ $(\lambda_{dB} = (2\pi\hbar^2/mk_BT)^{1/2})$
  - trap frequencies: $\lambda_{dB}/l_{osc} \ll 1$ $(l_{osc} = 1/\sqrt{m/2\omega})$



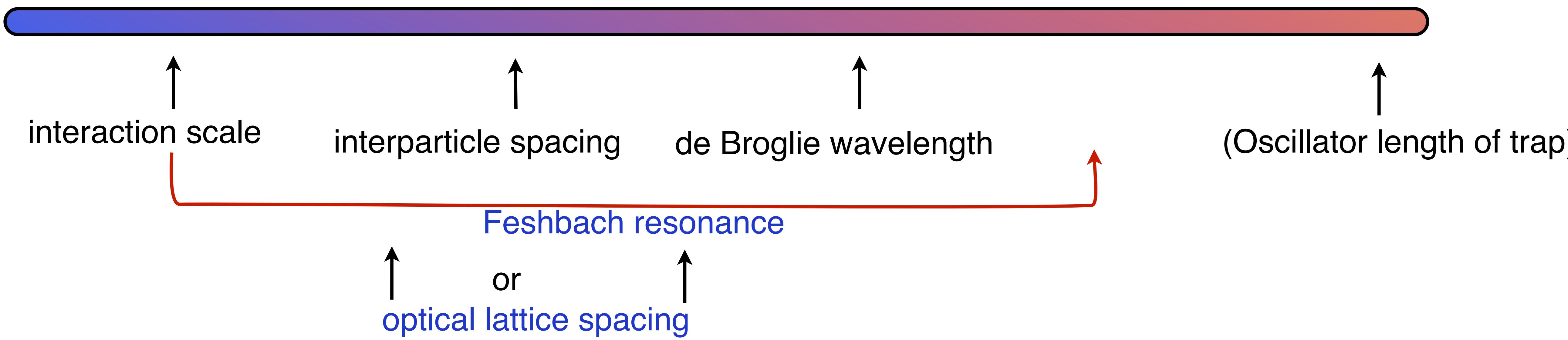
Summary of length scales $\quad a_{\mathrm{B}} = 5.3 \times 10^{-2}\mathrm{nm}$ Bohr radius

length	scattering length a/a_B	interparticle sep. d/a_B	de Broglie w.l. λ_{dB}/a_B	trap size l_{osc}/a_B
	(0.05 ... 0.2)10^3	(0.8 ... 3)10^3	(10 ... 40)10^3	(3 ... 300)10^3

phys. meaning of the ratio: weak interactions/ dilute gases; quantum degeneracy; local density approximation

Violations of the scale hierarchy

- Generic sequence of scales and possible violations:



- With Feshbach resonances, violation of a/d << 1 possible: Dense degenerate system
- With optical lattices, a new length and a new energy scale are introduced:
  - lattice spacing = wavelength of light: high densities (“fillings) become available
  - lattice depth: Kinetic energy is withdrawn more strongly than interaction energy: “strong correlations”

- Both leads to the possibility of “strong interactions/correlations” as we will see

- NB: Despite violation of scale hierarchy for dilute quantum gases, we will be able to give accurate microscopic models

BEC Phenomenology: Gross-Pitaevski Equation

- Heisenberg Equation of motion for the field operator:

$$\partial_t a_{\mathbf{x}}(t) = -\mathrm{i}[H, a_{\mathbf{x}}] = -\mathrm{i}\big(- \frac{\triangle}{2m} - \mu + V(\mathbf{x}) + g a^\dagger_{\mathbf{x}} a_{\mathbf{x}} \big) a_{\mathbf{x}}$$

➡ Nonlinear partial differential operator equation...

- Simplifications:
  - $T = 0$, no interactions: whole density sits in the condensate: $\varphi_0 = \sqrt{N}$
  - $T = 0$, weak interactions $(a/d \ll 1)$: expect $\varphi_0 \approx \sqrt{N}$
  - we can formalize this:

complete set of orthogonal functions

$$a_{\mathbf{x}} = \phi_0(\mathbf{x}) a_{\mathbf{0}} + \sum_{\mathbf{q}\neq 0} \phi_{\mathbf{q}}(\mathbf{x}) a_{\mathbf{q}} = \phi_0(\mathbf{x}) a_{\mathbf{0}} + \delta a_{\mathbf{x}}$$

- macroscopic occupation of zero mode implies: commutator irrelevant, replace with classical c-numbers

$$\langle a^\dagger_{\mathbf{0}} a_{\mathbf{0}} \rangle \approx \langle a_{\mathbf{0}} a^\dagger_{\mathbf{0}} \rangle \approx N \approx \langle a^\dagger_{\mathbf{0}} \rangle \langle a_{\mathbf{0}} \rangle, \text{ thus } a_{\mathbf{0}} \to \sqrt{N}; \;\; \varphi(\mathbf{x}, t) := \phi_0(\mathbf{x}, t) \sqrt{N}$$

- Insert into Heisenberg EoM, keep only terms $\mathcal{O}(\sqrt{N}) \gg 1$

$$\mathrm{i}\partial_t \varphi(\mathbf{x}, t) = \big(- \frac{\hbar^2}{2m} \triangle - \mu + V(\mathbf{x}) + g\varphi^*(\mathbf{x}, t)\varphi(\mathbf{x}, t) \big) \varphi(\mathbf{x}, t)$$

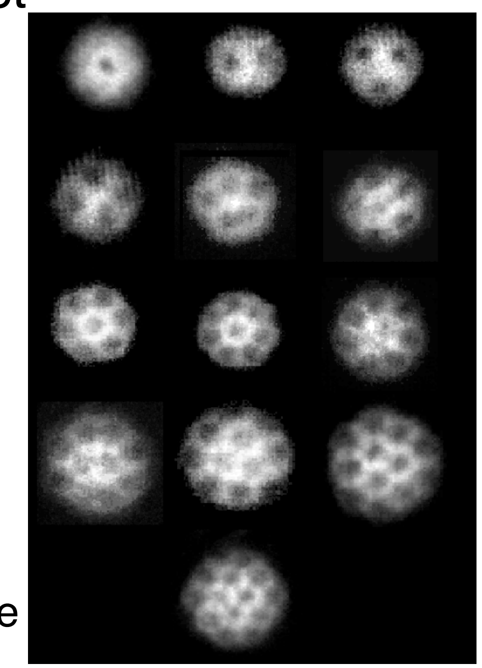
Gross-Pitaevski Equation

Macroscopic wave function

- Gross-Pitaevski Equation :

$$\mathrm{i}\partial_t \varphi(\mathbf{x},t) = \Big(-\frac{\hbar^2}{2m}\triangle - \mu + V(\mathbf{x}) + g\varphi^*(\mathbf{x},t)\varphi(\mathbf{x},t)\Big)\varphi(\mathbf{x},t)$$

- Properties:
  - Classical field equation (cf. classical electrodynamics vs. QED)
  - for g = 0, or single particle: formally recover linear Schrödinger equation -> expect quantum behavior; interpret φ as “macroscopic wave function”
  - however, in general nonlinear -> richer than Schrödinger equation



- Interplay of quantum mechanics and nonlinearity: quantized vortex solutions
  - uniform case V(x) = 0, search static cylinder symmetric symmetric solutions with no z dependence:

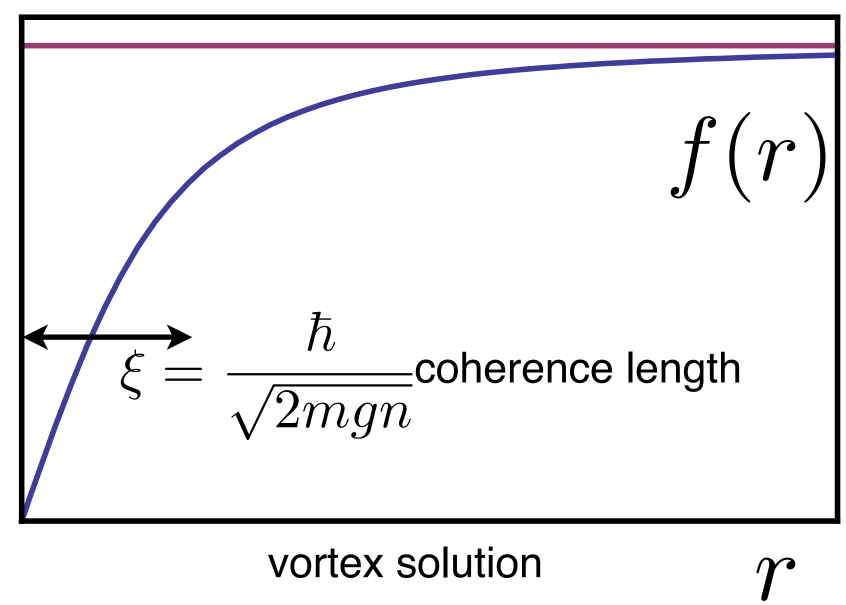
$$\varphi(\mathbf{x},t) = \varphi(r,\phi) = f(r)e^{\mathrm{i}\ell\phi}$$

integer, such that phase returns after 2 pi: unique Wave function

  - GP equation:

$$0 = -\frac{\hbar^2}{2m}\Big(f" + \frac{f'}{r} - \frac{\ell^2 f}{r^2}\Big) - \mu f + g f^3$$

  - large distances: constant solution, determine chemical pot.
  - short distances: condensate amplitude must vanish due to centrifugal barrier, in turn rooted in the quantization of the phase



Quantum Fluctuations: Bogoliubov Theory

- homogeneous case: plane wave expansion of field operator $(\phi_0(\mathbf{x})a_\mathbf{0} = 1/\sqrt{V}\sqrt{N_0} = \sqrt{n_0})$

$$a_\mathbf{x} = \sqrt{n_0} + \sum_{\mathbf{q}\neq 0} e^{\mathrm{i}\mathbf{q}\mathbf{x}} a_\mathbf{q}$$

choice of zero phase without loss of generality

- Grand canonical Hamiltonian. Take the fluctuations into account to leading order:

  - zero order: homogenous mean field reproduced
  - linear terms: vanishes upon proper choice of the chemical potential (equilibrium condition) $\mu = gn_0$
  - quadratic part:

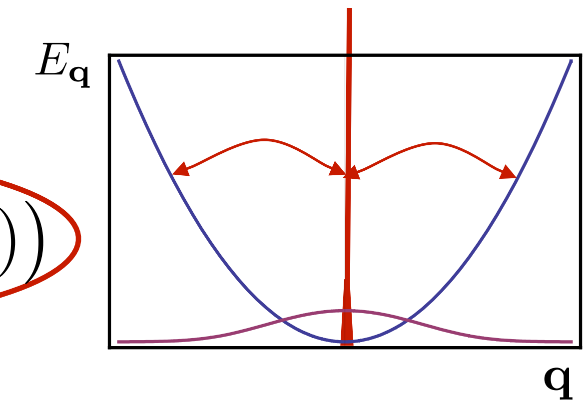
$$H^{(2)}_{Bog} = \sum_{\mathbf{q}\neq 0} \left(a^\dagger_\mathbf{q}\left(\frac{\mathbf{q}^2}{2m} - \mu + 2gn_0\right)a_\mathbf{q} + \frac{gn_0}{2}\left(a^\dagger_\mathbf{q}a^\dagger_{-\mathbf{q}} + a_\mathbf{q}a_{-\mathbf{q}}\right)\right)$$

$$= \frac{1}{2}\sum_{\mathbf{q}\neq 0}(a^\dagger_\mathbf{q}, a_{-\mathbf{q}}) \begin{pmatrix} \frac{\mathbf{q}^2}{2m} + gn_0 & gn_0 \\ gn_0 & \frac{\mathbf{q}^2}{2m} + gn_0 \end{pmatrix} \begin{pmatrix} a_\mathbf{q} \\ a^\dagger_{-\mathbf{q}} \end{pmatrix} + \text{const.}$$

$\mu = gn_0$

Quantum Fluctuations: Bogoliubov Theory

- quadratic Bogoliubov Hamiltonian:



$$H^{(2)}_{Bog} = \sum_{\mathbf{q}\neq 0} \left(a^\dagger_{\mathbf{q}}\left(\frac{\mathbf{q}^2}{2m} - \mu + 2gn_0\right)a_{\mathbf{q}} + \frac{gn_0}{2}\left(a^\dagger_{\mathbf{q}}a^\dagger_{-\mathbf{q}} + a_{\mathbf{q}}a_{-\mathbf{q}}\right)\right)$$

$$= \frac{1}{2}\sum_{\mathbf{q}\neq 0}(a^\dagger_{\mathbf{q}}, a_{-\mathbf{q}}) \begin{pmatrix} \frac{\mathbf{q}^2}{2m} + gn_0 & gn_0 \\ gn_0 & \frac{\mathbf{q}^2}{2m} + gn_0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{q}} \\ a^\dagger_{-\mathbf{q}} \end{pmatrix} + \mathrm{const.}$$

- Discussion:
  - Off-diagonal terms: pairwise creation/annihilation out of the condensate
  - Coupling of modes with opposite momenta
  - Hamiltonian not diagonal in operator space: diagonalize to find spectrum and elementary excitations

- Remarks:
  - An equivalent approach linearizes Heisenberg EoM around homog. GP mean field
  - Validity: The ordering principle is given by the power of the condensate amplitude. Bogoliubov theory is not a systematic perturbation theory in U but becomes good at weak coupling

Bogoliubov Quasiparticles

- Diagonalization in operator space: Bogoliubov transformation

$$\begin{pmatrix} \alpha_{\mathbf{q}} \\ \alpha^{\dagger}_{-\mathbf{q}} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{q}} & v_{\mathbf{q}} \\ v^*_{\mathbf{q}} & u^*_{\mathbf{q}} \end{pmatrix} \begin{pmatrix} a_{\mathbf{q}} \\ a^{\dagger}_{-\mathbf{q}} \end{pmatrix}$$

- The trafo be canonical, $[\alpha_{\mathbf{q}}, \alpha^{\dagger}_{\mathbf{q}'}] = \delta(\mathbf{q} - \mathbf{q}')$, thus $u^2_{\mathbf{q}} - v^2_{\mathbf{q}} = 1$.

- The transformation coefficients are chosen to make any off-diagonal contribution in terms of new operators vanish. They can be chosen real and evaluate to

$$u^2_{\mathbf{q}} = \tfrac{1}{2}(\tfrac{\xi_{\mathbf{q}}}{\epsilon_{\mathbf{q}}} + 1), v^2_{\mathbf{q}} = \tfrac{1}{2}(\tfrac{\xi_{\mathbf{q}}}{\epsilon_{\mathbf{q}}} - 1) \qquad \xi_{\mathbf{q}} = \epsilon_{\mathbf{q}} + gn_0,\ \epsilon_{\mathbf{q}} = \frac{\mathbf{q}^2}{2M}$$

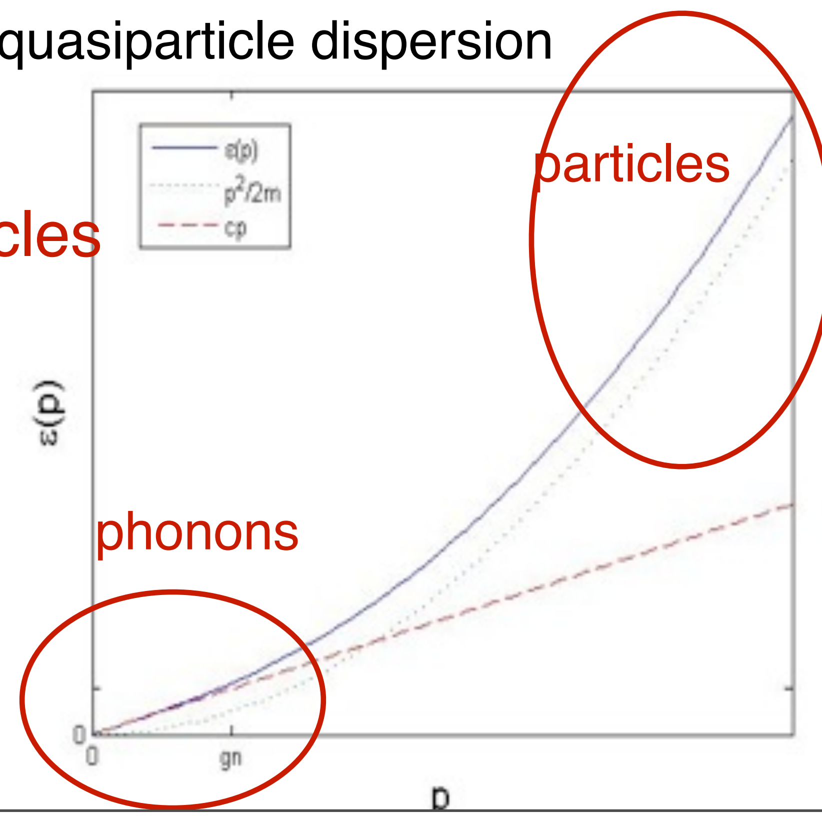
- Rewrite the Hamiltonian with new operators and do normal ordering

$$H = V\frac{gn^2}{2} - \frac{1}{2}\sum_{\mathbf{q}\neq 0}(\epsilon_{\mathbf{q}} + gn_0 - E_{\mathbf{q}}) + \sum_{\mathbf{q}\neq 0} E_{\mathbf{q}}\alpha^{\dagger}_{\mathbf{q}}\alpha_{\mathbf{q}}$$

quasiparticle dispersion

- Result: Collection of harmonic oscillators. New operators: elementary excitations on Bogoliubov ground state: quasiparticles
  - $\alpha^{\dagger}_{\mathbf{q}}$ creates a quasiparticle excitation and $\alpha_{\mathbf{q}}|0_{Bog}\rangle = 0$
  - Their dispersion is $E_{\mathbf{q}} = \sqrt{2gn_0\epsilon_{\mathbf{q}} + \epsilon^2_{\mathbf{q}}} \overset{\mathbf{q}\to 0}{\to} c|\mathbf{q}|, \quad c = \sqrt{\frac{gn_0}{m}}$
  - At low momenta, this is linear and gapless
  - At high momenta, like free particles: quadratic

speed of sound



Energy Correction and Density Depletion

- Physical effects due to quantum fluctuations:
  - Correction to the total energy:

$$E/V = \langle 0_{Bog}|H|0_{Bog}\rangle/V = \frac{gn^2}{2} - \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}(\epsilon_{\mathbf{q}} + gn_0 - E_{\mathbf{q}})$$

interpretation: interaction energy | kinetic energy from modes outside the condensate

- Problem: linear high momentum UV divergence
- Reason: too naive treatment of interactions: formally assumed to be constant up to arbitrarily large momenta
- Cure: UV renormalization of interaction (will perform such program explicitly for fermions later)
- Result: $E/V = \frac{gn^2}{2}(1 + \frac{128}{15\pi^{1/2}}(na^3)^{1/2}) \quad g = \frac{8\pi a}{M}$

- Depletion of the condensate (via normal ordering of the particle number operator):

$$n = n_0 + \int_{\mathbf{q}} \langle a^\dagger_{\mathbf{q}} a_{\mathbf{q}}\rangle = n_0 + \int_{\mathbf{q}} v^2_{\mathbf{q}} \rightarrow \frac{n - n_0}{n} \approx \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}$$

expand in quasiparticle operators

$$na^3 = (a/d)^3$$

interpretation: particles kicked out of condensate

smallness parameter recovered

Phonon Mode and Superfluidity

- Landau criterion of superfluidity: frictionless flow
  - Gedankenexperiment: move an object through a liquid with velocity v.
  - Landau: the creation of an excitation with momentum p and energy $\epsilon_{\mathbf{p}}$ is energetically unfavorable if

$$\boxed{v < v_{\mathrm{c}} = \frac{\epsilon_{\mathbf{p}}}{p}}$$

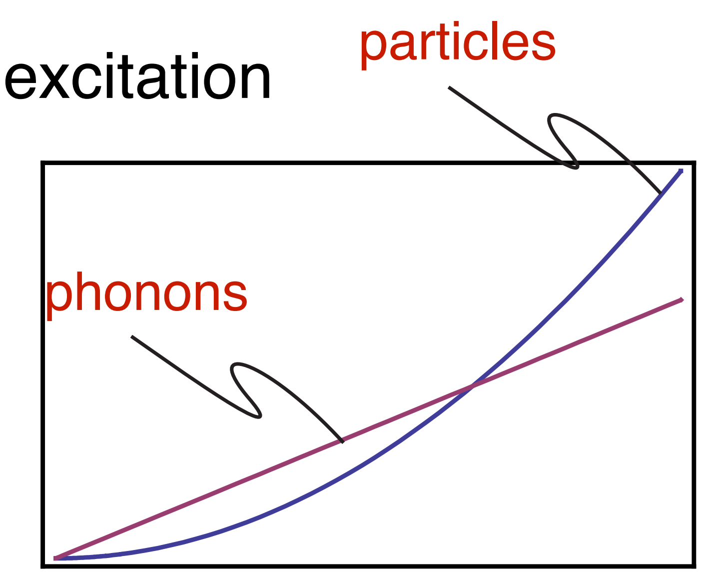
➡in this case, the flow is frictionless, i.e. superfluidity is present

- Weakly interacting Bose gas: Superfluidity through linear phonon excitation

$$\epsilon_{\mathbf{p}} = c|\mathbf{p}|, c = \sqrt{\frac{g n_0}{m}} \rightarrow v_c = c$$

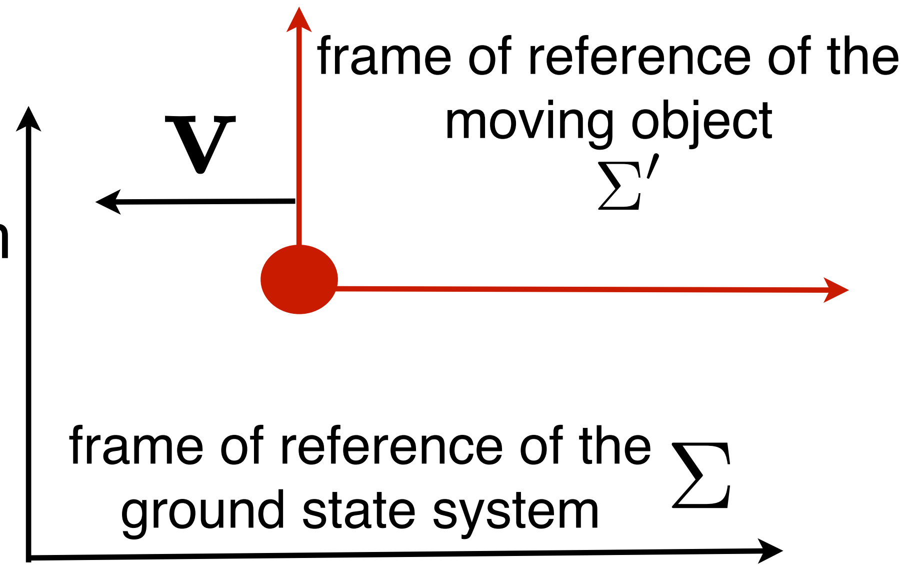
- Free Bose gas: No superfluidity due to soft particle excitations

$$\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} \rightarrow v_c = 0$$



➡ Superfluidity is due to linear spectrum of quasiparticle excitations

Idea of Landau Criterion



- Consider moving object in the liquid ground state of a system
- Question: When is it favorable to create excitations?

  - General transformation of energy and momentum under Galilean boost with velocity $\mathbf{v}$

$$\Sigma: \quad E, \quad \mathbf{p}$$

total system mass

$$\Sigma': \quad E' = E - \mathbf{p}\mathbf{v} + \tfrac{1}{2}M\mathbf{v}^2, \quad \mathbf{p}' = \mathbf{p} - M\mathbf{v}$$

  - Energy and momentum of the ground state in

$$\Sigma: \quad E_0, \quad \mathbf{p}_0 = 0$$

$$\Sigma': \quad E_0' = E_0 + \tfrac{1}{2}M\mathbf{v}^2, \quad \mathbf{p}_0' = -M\mathbf{v}$$

  - Energy and momentum of the ground state plus an excitation with momentum, energy $\mathbf{p}, \epsilon_{\mathbf{p}}$

$$\Sigma: \quad E_{\mathrm{ex}} = E_0 + \epsilon_{\mathbf{p}}, \quad \mathbf{p}_{\mathrm{ex}} = \mathbf{p}$$

$$\Sigma': \quad E'_{\mathrm{ex}} = E_0 + \epsilon_{\mathbf{p}} - \mathbf{p}\mathbf{v} + \tfrac{1}{2}M\mathbf{v}^2, \quad \mathbf{p}'_{\mathrm{ex}} = \mathbf{p} - M\mathbf{v}$$

  - Creation of excitation unfavorable if

$$E'_{\mathrm{ex}} - E'_0 = \epsilon_{\mathbf{p}} - \mathbf{p}\mathbf{v} \geq \epsilon_{\mathbf{p}} - |\mathbf{p}||\mathbf{v}| > 0 \qquad \Rightarrow v < v_c = \frac{\epsilon_{\mathbf{p}}}{p}$$

Summary

- The basic phenomenon of Bose-Einstein condensation is a statistical effect, not driven by interactions.

- Ultracold bosonic quantum gases realize a situation close to that: model Hamiltonians with weak, local, repulsive interactions.

- Important scale hierarchy: $a \ll d \ll \lambda_{dB}$

- In consequence, such systems are well described by relatively simple approximations: at T=0, the physics is well understood in terms of GP equation + quadratic fluctuations.

- GP mean field equation = nonlinear Schrödinger equation for macroscopic wave function (hallmark: quantized vortices).

- Bogoliubov theory encompasses quadratic fluctuations around GP mean field and explains superfluidity through existence of phonon mode.

Mini-Tutorial: Functional Integrals and Effective Action

$$Z = \mathrm{tr} e^{-\beta \hat{H}} = \int \mathcal{D}(\varphi^*, \varphi) e^{-S[\varphi^*, \varphi]}$$

Functional Integrals

- Some familiarity with functional integrals is assumed but we refresh our knowledge:
- Given a grand canonical Hamiltonian , e.g. with general two-body interactions,

$$H - \mu\hat{N} = \sum_{ij}(h_{ij} - \mu\delta_{ij})a_i^\dagger a_j + \sum_{ijkl} V_{ijkl} a_i^\dagger a_j^\dagger a_k a_l$$

- Quantum partition function

$$Z = \mathrm{tr} e^{-\beta(H-\mu\hat{N})} = \sum_{\{\mathrm{n}\}\in \mathrm{Fock\,space}} \langle \mathrm{n}| e^{-\beta(H-\mu\hat{N})} |n\rangle, \quad \beta = 1/k_B T$$

- Here the partition function is represented in Fock space. We now perform a basis change to coherent states leading to the functional integral
- Coherent states (bosons) – eigenstates to the annihilation operators a_i:

$$\begin{aligned} a_i|\phi\rangle &= \phi_i|\phi\rangle, \ \langle\phi|a_i^\dagger = \langle\phi|\phi_i^* \\ |\phi\rangle &= e^{\sum_i \phi_i a_i^\dagger}|\mathrm{vac}\rangle && \text{explicit form} \\ \langle\theta|\phi\rangle &= e^{\sum_i \theta_i^*\phi_i}, \ \langle\phi|\phi\rangle = e^{\sum_i \phi_i^*\phi_i} && \text{overlap and normalization} \\ \mathbf{1}_{\mathrm{Fock}} &= \int \prod_i \frac{d\phi_i^* d\phi_i}{\pi} e^{-\sum_i \phi_i^*\phi_i}|\phi\rangle\langle\phi| && \text{completeness} \end{aligned}$$

- Note: The creation operators do not have eigenstates

Functional Integrals

- Insert identity into quantum partition function

$$\begin{aligned} Z &= \mathrm{tr} e^{-\beta(H-\mu\hat{N})} = \sum\nolimits_{\{\mathrm{n}\}\in \mathrm{Fock\,space}} \langle \mathrm{n}| e^{-\beta(H-\mu\hat{N})}|n\rangle \\ &= \int D(\phi^*,\phi) e^{-\sum_i \phi_i^*\phi_i} \langle\phi|e^{-\beta(H-\mu\hat{N})}|\phi\rangle, \quad D(\phi^*,\phi)\equiv \prod\nolimits_i \frac{d\phi_i^* d\phi_i}{\pi} \end{aligned}$$

- For a normal ordered Hamiltonian $H(a_i^\dagger, a_i) - \mu\hat{N}(a_i^\dagger, a_i)$, we can apply the Feynman strategy of dividing the (imaginary) time interval β into N segments $\Delta\beta = \beta/N$. Note $|\phi_{n=0}\rangle = |\phi_{n=N}\rangle$ – periodic boundary conditions
- Inserting the identity $\mathbf{1}_{\mathrm{Fock}} = \int D(\phi_n^*, \phi_n) e^{-\sum_i \phi_{i,n}^*\phi_{i,n}} |\phi_n\rangle\langle\phi_n|$ after each time step, the respective matrix elements can be calculated explicitly due to the smallness of $\Delta\beta$ with result

$$Z = \int \prod_{n=0}^{N} D(\phi_n^*, \phi_n) e^{-\sum_{n=0}^{N}[\sum_i \phi_{i,n}^*(\phi_{i,n}-\phi_{i,n-1}) + \Delta\beta(H(\phi_{i,n}^*,\phi_{i,n-1}) - \mu N(\phi_{i,n}^*,\phi_{i,n-1}))}$$

  - Here, $H-\mu N$ is a function of classical, but fluctuating variable.
  - The additional index n labels the (imaginary) time evolution. Continuum notation for $N\to\infty$:

$$\Delta\beta\sum_{n=0}^{N} \to \int_0^\beta d\tau, \quad \phi_{i,n}\to\phi_i(\tau), \quad \frac{\phi_{i,n}-\phi_{i,n-1}}{\Delta\beta}\to\partial_\tau\phi_i(\tau), \quad \prod_{n=0}^{N} D(\phi_n^*,\phi_n)\to\mathcal{D}(\phi^*,\phi)$$

Functional Integrals

- Continuum notation for the imaginary time dependence

$$
\begin{aligned}
Z &= \int \mathcal{D}(\phi^*,\phi) e^{-S[\phi^*,\phi]} \\
S[\phi^*,\phi] &= \int_0^\beta d\tau \big[\sum_i \phi_i^*(\tau)(\partial_\tau - \mu)\phi_i(\tau) + H[\phi^*,\phi] \ \big] \\
H[\phi^*,\phi] &= \sum_{ij} h_{ij}\phi_i^*\phi_j + \sum_{ijkl} V_{ijkl}\phi_i^*\phi_j^*\phi_k\phi_l
\end{aligned}
$$

- S is the “classical” or “microscopic” action
- Continuum notation for the discrete indices:
  - interpret indices as spatial indices
  - consider local density-density interactions $V_{ijkl} = v\delta_{ik}\delta_{jl}\delta_{il}$ and h_{ij} such that it describes kinetic energy (1D)

$$
H[\phi^*,\phi] = \sum_{ij} h(\phi_{i+1}^* - \phi_i^*)(\phi_{i+1} - \phi_i) + v\sum_i (\phi_i^*\phi_i)^2
$$

  - the sites $i, i+1$ be separated by a distance a. The continuum limit $a \to 0$ obtains for fixed $\varphi(\tau,x) = a^{-1/2}\phi_i(\tau), 1/2M = ha^2, g/2 = va$, such that

$$
S[\varphi^*,\varphi] \quad = \int_0^\beta d\tau dx[\varphi^*(\tau,x)(\partial_\tau - \tfrac{\triangle}{2M} - \mu)\varphi(\tau,x) + \tfrac{g}{2}(\varphi^*(\tau,x)\varphi(\tau,x))^2]
$$

- S is the “classical” or “microscopic” action of nonrelativistic continuum bosons with local interaction in one spatial dimension

Finite temperatures and Matsubara frequencies

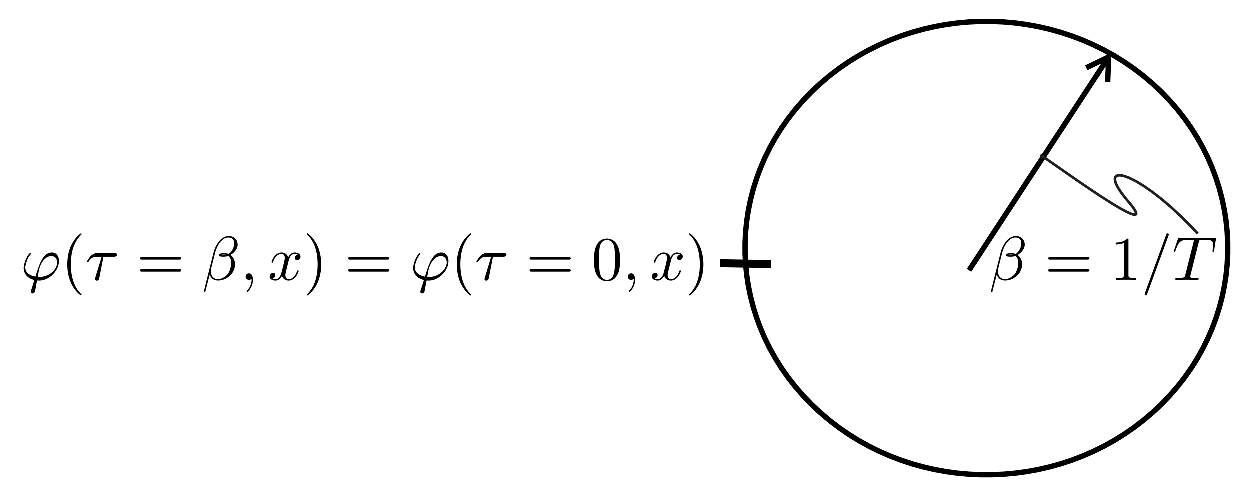
- Unlike the spatial integrations, the imaginary time integrations are restricted to a finite interval
- For bosons, we have used periodic boundary conditions, i.e. $\varphi(\tau = \beta, x) = \varphi(\tau = 0, x)$
- This gives rise to a discreteness in the frequency domain (cf. particle in a box problem) for finite $\beta < \infty\,(T > 0)$: Matsubara modes

$$\begin{aligned} \varphi(\tau, x) &= T \textstyle\sum_n \varphi(\omega_n, x) e^{\mathrm{i}\omega_n \tau}, \; \varphi(\omega_n, x) = \int d\tau \varphi(\tau, x) e^{-\mathrm{i}\omega_n \tau}, \\ \omega_n &= 2\pi n T \end{aligned}$$

- E.g. the free part of the action reads ($\beta^{-1} \int_0^\beta d\tau e^{\mathrm{i}(\omega_n - \omega_m)\tau} = \delta_{\omega_n, \omega_m}$)

$$S[\varphi^*, \varphi] = \sum_n T \int dx [\varphi^*(\omega_n, x)(\mathrm{i}\omega_\mathrm{n} - \frac{\triangle}{2M} - \mu)\varphi(\omega_n, x)]$$

- In the zero temperature limit, the frequency spacing $\Delta\omega = 2\pi T \to 0$ and the Matsubara summation goes over into a continuum Riemann integral, $\sum_n T \to \int \frac{d\omega}{2\pi}$



Matsubara torus

Formula summary

- Functional integral representation of the quantum partition function for ultracold bosons (3D):

$$
\begin{aligned}
Z \quad &= \int \mathcal{D}(\varphi^*, \varphi) e^{-S[\varphi^*,\varphi]} \\
S[\varphi^*, \varphi] \quad &= \int_0^\beta d\tau \left[\left(\int d^3x \varphi^*(\tau, \mathbf{x})(\partial_\tau - \mu)\varphi(\tau, \mathbf{x}) \right) + H[\varphi^*(\tau, \mathbf{x}), \varphi(\tau, \mathbf{x})] \right] \\
&= \int_0^\beta d\tau \int d^3x \left[\varphi^*(\tau, \mathbf{x})(\partial_\tau - \tfrac{\triangle}{2M} - \mu)\varphi(\tau, \mathbf{x}) + \tfrac{g}{2}(\varphi^*(\tau, x)\varphi(\tau, x))^2 \right]
\end{aligned}
$$

- Discussion:
  - The validity of the microscopic action is based on the validity of the microscopic Hamiltonian
  - The boson fields φ^*, φ are commuting but temporally and spatially fluctuating complex numbers
- Fermions: The same form of the microscopic action is obtained when representing the partition function for continuum fermion fields ψ. Important differences are:
  - Additional spin index, for two-component fermions: $\psi(\tau, \mathbf{x}) = (\psi_\uparrow(\tau, \mathbf{x}), \psi_\downarrow(\tau, \mathbf{x}))^T$
  - Due to the anticommutation relations of the fermion operators, the fermionic fields are anticommuting Grassmann numbers,

$$
\{\psi_\sigma(\tau, \mathbf{x}), \psi^*_{\sigma'}(\tau, \mathbf{x})\} = \{\psi_\sigma(\tau, \mathbf{x}), \psi_{\sigma'}(\tau, \mathbf{x})\} = \{\psi^*_\sigma(\tau, \mathbf{x}), \psi^*_{\sigma'}(\tau, \mathbf{x})\} = 0
$$

  - They obey antiperiodic boundary conditions in imaginary time $\psi_\sigma(\tau = \beta, x) = -\psi_\sigma(\tau = 0, x)$, thus $\omega_n = (2n + 1)\pi T$
  - Obviously, these properties will strongly affect the concrete evaluation of the functional integrals

From the Partition Function to the Effective Action

- We have stored the information on the many-body system in the partition function

$$Z \quad = \int \mathcal{D}(\varphi^*, \varphi) e^{-S[\varphi^*, \varphi]}$$

- Of interest are the low order correlation functions such as e.g. $\langle \varphi(\tau, \mathbf{x}) \varphi^*(0, \mathbf{0}) \rangle$
- They can be calculated from introducing (imaginary time and space dependent) artificial sources into the partition function:

$$Z \to Z[j^*(\tau, \mathbf{x}), j(\tau, \mathbf{x})] = \int \mathcal{D}(\varphi^*, \varphi) e^{-S[\varphi^*, \varphi] + \int_{\tau, \mathbf{x}} (j^*(\tau, \mathbf{x}) \varphi(\tau, \mathbf{x}) + j(\tau, \mathbf{x}) \varphi^*(\tau, \mathbf{x}))}$$

- and taking (functional) derivatives evaluated at vanishing sources, e.g.

$$\langle \varphi(\tau, \mathbf{x}) \varphi^*(0, \mathbf{0}) \rangle = \frac{\delta^2 Z}{\delta j^*(\tau.\mathbf{x}) \delta j(0, \mathbf{0})} \Big|_{j = j^* = 0}$$

- Remarks:
  - functional derivatives are defined with $\delta f(x)/\delta f(y) = \delta(x - y)$ plus the standard algebraic rules for differentiation
  - successive differentiation generates the "disconnected correlation functions". The free energy

$$W[j^*, j] = \log Z[j^*, j]$$

  generates the "connected correlation functions"

From the Partition Function to the Effective Action

- The information accessible from $Z[j^*, j]$ or $W[j^*, j]$ via derivatives wrt an unphysical source (j^*, j)
- There is a way to store the information in a more intuitive way: Legendre transform of W(cf. classical mechanics, or thermodynamics)

$$\Gamma[\phi^*, \phi] = -W[j^*, j] + \int j^*\phi + j\phi^*, \quad \phi(\tau, \mathbf{x}) = \frac{\delta W[j^*, j]}{\delta j(\tau, \mathbf{x})} = \langle \varphi(\tau, \mathbf{x}) \rangle$$

- Properties and Discussion:
  - The Legendre transform implements a change of the active variable $(j^*, j) \to (\phi^*, \phi)$
  - The expectation value $\phi = \langle \varphi \rangle$ is called the classical field. For ultracold bosons, it has a direct physical interpretation in terms of the condensate mean field
  - The effective action carries the same information as Z and W, only organized differently. It generates the one-particle irreducible correlation functions

From the Partition Function to the Effective Action

- The effective action has a functional integral representation

$$\boxed{\exp -\Gamma[\phi^*,\phi] = \int \mathcal{D}(\delta\varphi^*,\delta\varphi)\exp -S[\phi^*+\delta\varphi^*,\phi+\delta\varphi], \qquad \frac{\delta\Gamma[\phi^*,\phi]}{\delta\phi(\tau,\mathbf{x})} = j(\tau,\mathbf{x}) = 0}$$

- Discussion
  - NB: Action principle is leveraged over to full quantum status
  - The effective action can be understood “classical action plus fluctuations”. It lends itself to semi-classical approximations (small fluctuations around a mean field)
  - Symmetry principles are leveraged over from the classical action to full quantum status

Effective Action and Propagator

- A useful property: The second derivative of the effective action is the inverse propagator
- Consider a propagation amplitude (in matrix notation)

$$G_{ij} = \langle \delta\varphi_i \delta\varphi_j \rangle = \frac{\delta^2 W}{\delta j_i \delta j_j} = W^{(2)}_{ij}$$

the indices could stand for e.g. $\langle \delta\varphi(\tau, \mathbf{x}) \delta\varphi^*(0, \mathbf{0}) \rangle$

- The above statement is then poperly formulated as the identity

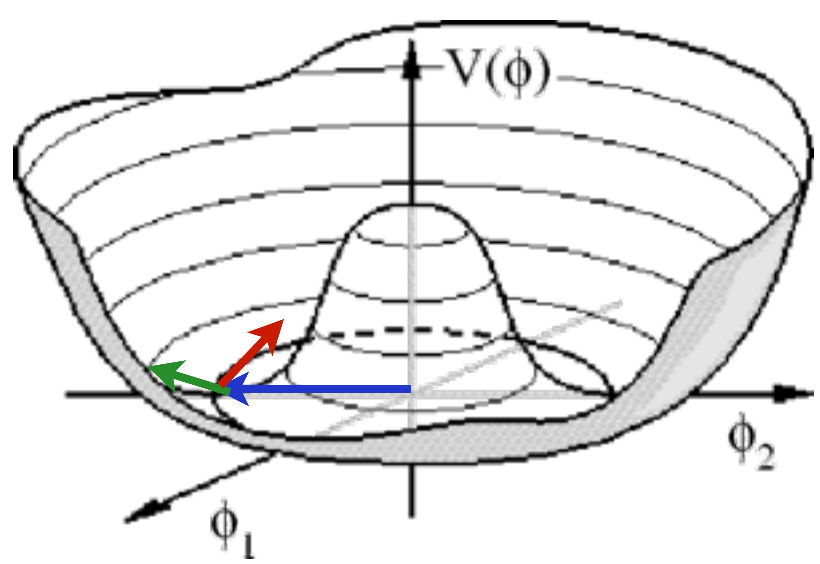
$$\sum_k \Gamma^{(2)}_{ik} G_{kj} = \delta_{ij}$$

- Proof:

$$\sum_k \Gamma^{(2)}_{ik} G_{kj} = \sum_k \frac{\delta^2 \Gamma}{\delta\phi_i \delta\phi_k} \frac{\delta^2 W}{\delta j_k \delta j_j} = \sum_k \frac{\delta^2 \Gamma}{\delta\phi_i \delta\phi_k} \frac{\delta\phi_k}{\delta j_j} = \frac{\delta^2 \Gamma}{\delta\phi_i \delta j_j} = \frac{\delta j_i}{\delta j_j} = \delta_{ij}$$

we have used the chain rule and the field equation for the effective action $\delta\Gamma/\delta\phi_i = j_i$

Functional Integral
for
Weakly Interacting Bosons



Some Introductory Remarks

- This section develops the theory of low temperature weakly interacting boson degrees of freedom from the functional integral point of view. The aim is threefold:

(1) Reproduce the results of Gross-Pitaevski and Bogoliubov theory

(2) Get some familiarity with the functional integral

(3) The emergence of bosonic quasiparticles at low energies is ubiquitous in a large variety of low temperature quantum systems:

- The concept of quasiparticles/ effective low energy theories is particularly accessible in the functional integral formulation
- Examples of emergent bosonic low energy theories:
    - BCS theory (Cooper pairs)
    - Even more explicitly, BCS-BEC crossover (bound states of fermions: molecules)
    - spin waves in magnetic lattice systems
- A rather universal understanding of such theories is crucial, and accessible within the framework developed here

Classical Limit for the Effective Action

- We restore the dimension for the action:

$$[S] = [\tau] \cdot [E] = [\tau] \cdot [\hbar\omega] = [\hbar]$$

- Effective Action:

$$\exp - \frac{\Gamma[\phi^*, \phi]}{\hbar} = \int \mathcal{D}(\delta\varphi^*, \delta\varphi) \exp - \frac{S[\phi^* + \delta\varphi^*, \phi + \delta\varphi]}{\hbar}$$

- In the classical limit $\hbar \to 0$ but fixed Γ, the exponential distribution is sharply peaked around the field configuration for which the classical action in the exponent is minimal.
- This is the case for the "classical" field configuration $\phi(\tau, \mathbf{x})$, determined by the classical action (extremum) variational principle $\delta S / \delta\phi(\tau, \mathbf{x}) = 0$
- With this insight, we can expand the classical action in the functional integral in the fluctuation $(\delta\varphi^*, \delta\varphi)$, and keep only the zero order term
- Thus, in the classical limit we have

$$\Gamma[\phi^*, \phi] = S[\phi^*, \phi]$$

Recovering the Gross-Pitaevski Equation

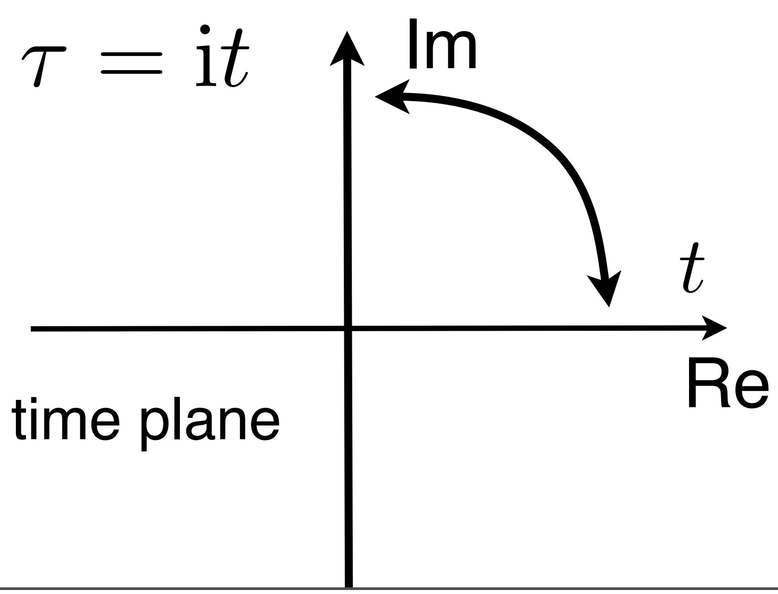
- We consider the imaginary time classical action S and and view it as being analytically continued from the real axis (at $T = 0$ or $\beta \to \infty$):

$$\begin{aligned} \tau \qquad &\to \mathrm{i}t, \quad \phi(\tau, \mathbf{x}) \to \tilde{\phi}(t, \mathbf{x}) \Rightarrow \\ S[\phi^*, \phi] \quad &\to \mathrm{i}S[\tilde{\phi}^*, \tilde{\phi}] = \mathrm{i} \int dt \int d^3x \left[\tilde{\phi}^*(t, \mathbf{x})(-\mathrm{i}\partial_t - \tfrac{\triangle}{2M} - \mu)\tilde{\phi}(t, \mathbf{x}) + \tfrac{g}{2}(\tilde{\phi}^*(t, x)\tilde{\phi}(t, x))^2\right] \end{aligned}$$

- We derive the field equation of motion for the real time classical action $\delta S/\delta\tilde{\phi}^*(t, \mathbf{x}) = 0$

$$\mathrm{i}\partial_t\tilde{\phi}(t, \mathbf{x}) = \big(-\tfrac{1}{2M}\triangle - \mu + g\tilde{\phi}^*(t, \mathbf{x})\tilde{\phi}(t, \mathbf{x})\big)\tilde{\phi}(t, \mathbf{x})$$

- This is precisely the Gross-Pitaevski equation for a weakly interacting Bose-Einstein condensate
- Remark: “classical” refers to the absence of fluctuations. Physically, the global phase coherence implied in this equations is a quantum mechanical effect, with observable consequences: cf. discussion of quantized vortices



Complex time plane

Symmetries of the Microscopic Action

- Gross-Pitaevski action:

$$S[\tilde{\phi}^*, \tilde{\phi}] = \int dt \int d^3x \left[\tilde{\phi}^*(t, \mathbf{x})(-\mathrm{i}\partial_t - \tfrac{\triangle}{2M} - \mu)\tilde{\phi}(t, \mathbf{x}) + \tfrac{g}{2}(\tilde{\phi}^*(t, x)\tilde{\phi}(t, x))^2\right]$$

- Symmetries:
  - Most important is the invariance under global phase rotations $U(1)$. Such transformation is implemented by $\tilde{\phi}(\tau, \mathbf{x}) \to e^{\mathrm{i}\theta}\tilde{\phi}(\tau, \mathbf{x}),\ \tilde{\phi}^*(\tau, \mathbf{x}) \to e^{-\mathrm{i}\theta}\tilde{\phi}^*(\tau, \mathbf{x})$.
  - The associated conserved Noether charge (of the full effective action) is the total particle number $N = \int d^3x \langle \varphi^*(0, \mathbf{x})\varphi(0, \mathbf{x})\rangle$ (NB: needs $U(1)$ symmetry plus linear time derivative)
  - Further continuous "equilibrium" symmetries: time and space translation invariance (energy and momentum conservation) and Galilean invariance (center of mass momentum)

Mean Field Action and Spontaneous Symmetry Breaking

- Homogeneous action: time- and space independent amplitudes ($\int d\tau d^3x = V/T$ -- quantization volume)

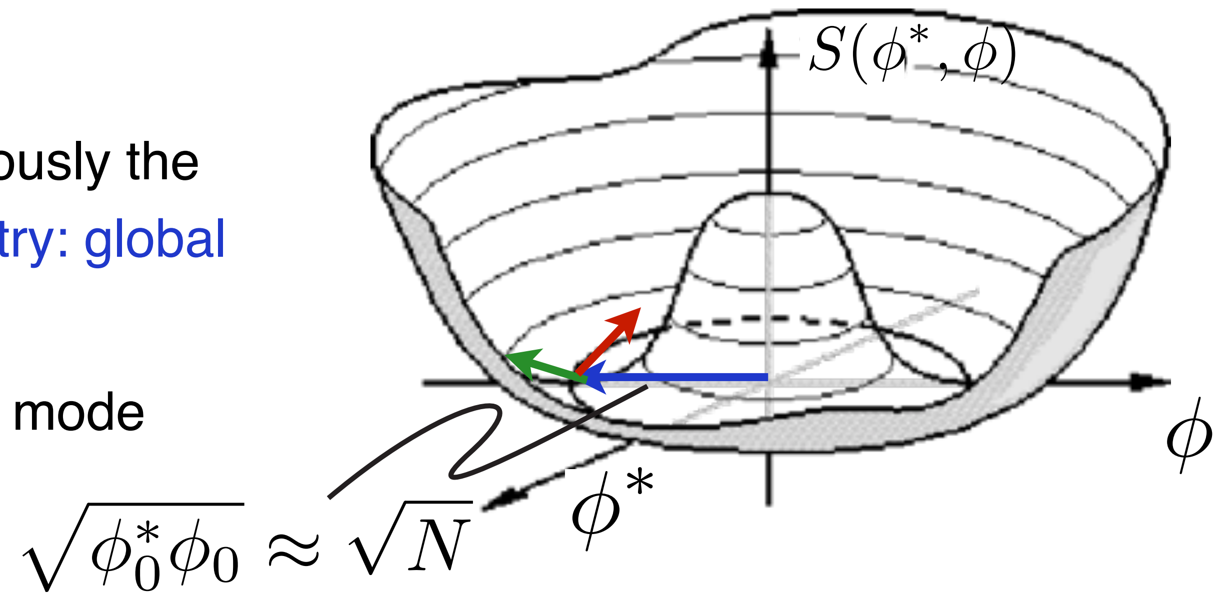
$$S(\phi^*, \phi) = V/T(-\mu\phi^*\phi + \tfrac{g}{2}(\phi^*\phi)^2)$$

- homogeneous GPE or equilibrium condition:

$$0 = \frac{\partial S}{\partial \phi} = \big(-\mu + g\phi^*\phi\big)\phi \qquad \Rightarrow -\mu = -g\phi^*\phi < 0$$

-- negative curvature at origin

- Geometrical interpretation: Mexican hat potential
    - for the ground state, the system chooses spontaneously the direction: spontaneous symmetry breaking (symmetry: global phase rotations U(1))
    - Radial (amplitude) excitations: cost energy, gapped mode
    - angular (phase) excitations: no energy cost due to degeneracy, gapless Goldstone mode



- The radial (amplitude) and angular (phase) excitations can be identified explicitly in the quadratic fluctuations

Quadratic Fluctuations

- We go one step beyond the classical limit and include quadratic fluctuations on top of the mean field
- Expansion of S in powers of $(\delta\varphi^*, \delta\varphi)$ around $(\delta\varphi^*, \delta\varphi) = (0,0)$ yields the approximate effective action (saddle point approximation):

$$\begin{aligned}\Gamma[\phi^*,\phi] &= -\log \int \mathcal{D}(\delta\varphi^*,\delta\varphi) \exp -S[\phi^* + \delta\varphi^*, \phi + \delta\varphi] \\ &\approx S[\phi^*,\phi] - \log \int \mathcal{D}(\delta\varphi^*,\delta\varphi) \exp -\tfrac{1}{2} \int (\delta\varphi, \delta\varphi^*) S^{(2)}[\phi^*,\phi] \begin{pmatrix} \delta\varphi \\ \delta\varphi^* \end{pmatrix}\end{aligned}$$

Here, we have used the field equation $\delta S/\delta(\delta\varphi) = \delta S/\delta\phi = 0$

- We restrict to the homogeneous case $\phi(\tau, \mathbf{x}) = \phi_0$

Quadratic Fluctuations

- Homogenous case:
  - The field equation (in general: GPE) reduces to an equilibrium condition determining the chemical potential (see above)

$$0 = \frac{\delta S}{\delta \phi}\Big|_{\mathrm{hom.}} = (-\mu + g\phi_0^*\phi_0)\phi_0^*$$

  For small enough T and g, $|\phi_0| \neq 0$ such that at equilibrium

$$\mu_0 = g\phi_0^*\phi_0.$$

  Coincides with Bogoliubov Theory
  - It is favorable to work in frequency and momentum space. There, the exponent reads explicitly ($Q = (\omega_n, \mathbf{q}), \int_Q = \sum_n T \int \frac{d^3q}{(2\pi)^3}$):

$$S^{(2)}(\phi^*, \phi) = \Big(\frac{\vec{\delta}}{\delta(\delta\varphi)(-Q)}, \frac{\vec{\delta}}{\delta(\delta\varphi^*)(Q)}\Big) S \begin{pmatrix} \frac{\overleftarrow{\delta}}{\delta(\delta\varphi)(K)} \\ \frac{\overleftarrow{\delta}}{\delta(\delta\varphi^*)(-K)} \end{pmatrix} \propto \delta(K-Q) \quad \text{-- diagonal in momentum space}$$

$$\Longrightarrow \quad \frac{1}{2}\int_Q \big(\delta\varphi(-Q), \delta\varphi^*(Q)\big) \begin{pmatrix} g\phi_0^{*2} & -\mathrm{i}\omega_{\mathrm{n}} + \frac{\mathbf{q}^2}{2\mathrm{M}} - \mu + 2g\phi_0^*\phi_0 \\ \mathrm{i}\omega_{\mathrm{n}} + \frac{\mathbf{q}^2}{2\mathrm{M}} - \mu + 2g\phi_0^*\phi_0 & g\phi_0^2 \end{pmatrix} \begin{pmatrix} \delta\varphi(Q) \\ \delta\varphi^*(-Q) \end{pmatrix}$$

  - NB: The frequency-independent part of the matrix coincides with Bogoliubov theory
  - This is a quadratic form leading to a Gaussian functional integral. A technique to evaluate it is discussed below

Phase and Amplitude Fluctuations

- We analyze the quadratic action for the boson fluctuations, using $-\mu = g\phi^*\phi$

$$S_F[\delta\varphi^*, \delta\varphi] = \frac{1}{2}\int_Q \left(\delta\varphi(-Q), \delta\varphi^*(Q)\right) \left(\begin{array}{cc} g\phi_0^{*2} & -\mathrm{i}\omega_\mathrm{n} + \epsilon_\mathbf{q} + g\phi_0^*\phi_0 \\ \mathrm{i}\omega_\mathrm{n} + \epsilon_\mathbf{q} + g\phi_0^*\phi_0 & g\phi_0^2 \end{array} \right) \left(\begin{array}{c} \delta\varphi(Q) \\ \delta\varphi^*(-Q) \end{array} \right)$$

- We perform a change of basis (real and imaginary parts),

$$\delta\varphi_1(Q) = (\delta\varphi^*(-Q) + \delta\varphi(Q))/\sqrt{2}, \quad \delta\varphi_2(Q) = \mathrm{i}(\delta\varphi^*(Q) - \delta\varphi(-Q))/\sqrt{2}$$

- The action in the new coordinates reads ($\rho_0 = \phi_0^*\phi_0$ and we choose ϕ real without loss of generality)

$$S_F[\delta\varphi_1, \delta\varphi_2] = \frac{1}{2}\int_Q \left(\delta\varphi_1(-Q), \delta\varphi_2(Q)\right) \left(\begin{array}{cc} \epsilon_\mathbf{q} + 2g\rho_0 & -\omega_\mathrm{n} \\ \omega_\mathrm{n} & \epsilon_\mathbf{q} \end{array} \right) \left(\begin{array}{c} \delta\varphi_1(Q) \\ \delta\varphi_2(-Q) \end{array} \right)$$

Phase and Amplitude Fluctuations

- Action in terms of real fields:

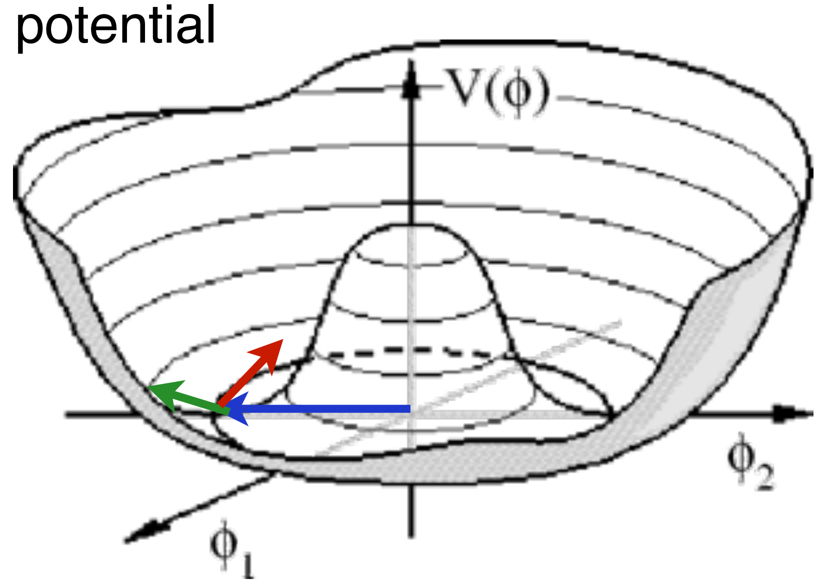
$$S_F[\delta\varphi_1, \delta\varphi_2] = \frac{1}{2}\int_Q (\delta\varphi_1(-Q), \delta\varphi_2(Q)) \begin{pmatrix} \epsilon_{\mathbf{q}} + 2g\rho_0 & -\omega_{\mathrm{n}} \\ \omega_{\mathrm{n}} & \epsilon_{\mathbf{q}} \end{pmatrix} \begin{pmatrix} \delta\varphi_1(Q) \\ \delta\varphi_2(-Q) \end{pmatrix}$$

- Discussion:
  - Real part corresponds to amplitude fluctuations (see figure) and is gapped (massive) with $2g\rho_0$
  - Imaginary part corresponds to phase fluctuations and is gapless (massless)
  - The dispersion relation obtains from the poles of the propagator G, or the zeroes of $S^{(2)} = G^{-1}$ (analytically continued to real continuous frequencies $E = \mathrm{i}\omega_n$)

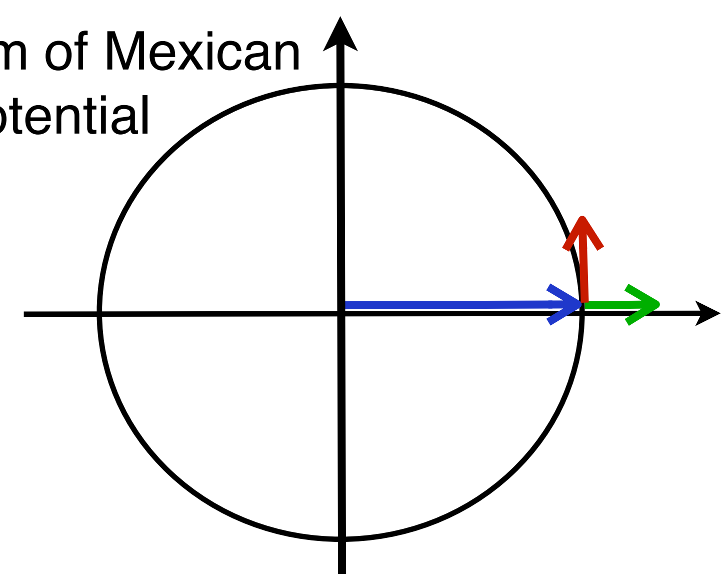
$$\det G^{-1}(\mathrm{E} = \mathrm{i}\omega, \mathbf{q}) \stackrel{!}{=} 0 \quad \Rightarrow E_{\mathbf{q}} = \sqrt{\epsilon_{\mathbf{q}}(\epsilon_{\mathbf{q}} + 2g\rho_0)}$$

reproduces the Bogoliubov excitation spectrum with $E_{\mathbf{q}} \approx c|\mathbf{q}|,\ c = \sqrt{g\rho_0/M}$ for $\mathbf{q} \to 0$

Mexican hat potential



Bottom of Mexican hat potential



Phase-Only Action

- If we are interested in the physics at very low momenta/energies $E \ll g\rho_0$, we can integrate out the amplitude mode:

$$\int \mathcal{D}\delta\varphi_1 \mathcal{D}\delta\varphi_2 \exp -S_F[\delta\varphi_1, \delta\varphi_2] = \mathcal{N} \int \mathcal{D}\delta\varphi_2 \exp -S_{ph}[\delta\varphi_2]$$

- This is done by completing the square in the exponent. The phase-only action reads

$$S_{ph}[\delta\varphi_2] = \tfrac{1}{4g\rho_0} \int_Q \delta\varphi_2(Q)(\omega^2 + c^2\mathbf{q}^2)\delta\varphi_2(-Q)$$

- Discussion:
- This action has a relativistic dispersion $E^2 = c^2\mathbf{q}^2$ (analytic continuation), with speed of sound $c = \sqrt{g\rho_0/M}$
- The speed of sound is also obtained from the limit $\mathbf{q} \to 0$ on the full dispersion relation. Now, we have found the interpretation of the linear dispersion in terms of phase fluctuations

Goldstone's Theorem

- The gapless nature of low energy excitation is not accidental but an exact property of the theory
- This is expressed in Goldstone's theorem: Assume a theory which is invariant under continuous global symmetry transformations. If the symmetry is spontaneously broken, then there are gapless excitations, the Goldstone modes.
- Discussion:
  - Does not prove the existence of symmetry breaking, but makes statement in case of
  - For an $O(N)$ symmetry, there are $N-1$ Goldstone modes
  - In our case, we have $U(1) \simeq O(2)$ and thus one Goldstone mode (phase)
  - The proof is straightforward using the effective action formalism

Proof of Goldstone's Theorem

- The exact field equation for the full effective action is $\delta\Gamma/\delta\phi(\tau,\mathbf{x}) = \delta\Gamma/\delta\phi^*(\tau,\mathbf{x}) = 0$. In the homogeneous limit, simplification $\partial\Gamma/\partial\phi = \partial\Gamma/\partial\phi^* = 0$
- Explicitly for $U(1)$ invariance: In the homogeneous limit, dependence of Γ restricted to the invariant $\rho = \phi^*\phi : \Gamma = \Gamma[\rho]$
- Field parametrization: $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$, $\phi^* = (\phi_1 - \mathrm{i}\phi_2)/\sqrt{2}$. I.e. $\rho = \frac{1}{2}(\phi_1^2 + \phi_2^2)$
- Equilibrium: choose real expectation value such that $\phi_1 = \phi_{1,0} + \delta\phi_1, \phi_2 = \delta\phi_2$. Spontaneous symmetry breaking: $\phi_{1,0} \neq 0$
- Consider field equation for ϕ_1,

$$0 = \frac{\partial\Gamma}{\partial\phi_1}\Big|_{\mathrm{eq}} = \phi_{1,0}\frac{\partial\Gamma}{\partial\rho}\Big|_{\mathrm{eq}} \quad \Rightarrow \quad \frac{\partial\Gamma}{\partial\rho}\Big|_{\mathrm{eq}} = 0$$

- consider mass term of ϕ_2:

$$m_2^2 \equiv \frac{\partial^2\Gamma}{\partial\phi_2^2}\Big|_{\mathrm{eq}} = \left(\frac{\partial^2\rho}{\partial\phi_2^2}\frac{\partial\Gamma}{\partial\rho} + \left(\frac{\partial\rho}{\partial\phi_2}\right)^2\frac{\partial^2\Gamma}{\partial\rho^2}\right)\Big|_{\mathrm{eq}} = \frac{\partial\Gamma}{\partial\rho}\Big|_{\mathrm{eq}} = 0$$

  I.e. for a symmetry broken spontaneously in the real direction, the imaginary part of the field is massless
- Goldstone's theorem is a relation of first and second derivatives of the homogenous part of the full effective action

Parenthesis: Multidimensional Gaussian Integrals

- The equilibrium functional integral is defined as the continuum limit of a multidimensional integral over an exponential distribution
- An analytically tractable class of such integrals are of the Gaussian type, i.e. the exponent is quadratic in the integration variable
- Consider a multidimensional Gaussian integral for a real vector variable $\mathbf{m}$ and positive semidefinite matrix $\mathbf{A}$

$$Z_G = \int \prod_{i=1}^{N} dm_i \exp -\tfrac{1}{2}\mathbf{m}^{\mathbf{T}}\mathbf{A}\mathbf{m} = \int \prod_{i=1}^{N} dm_i \exp -\tfrac{1}{2}\mathbf{m}'^{\mathbf{T}}\mathbf{D}\mathbf{m}' = \prod_{i=1}^{N} (2\pi\lambda_i^{-1})^{1/2} = \det[\mathbf{A}/(2\pi)]^{-1/2}$$

- Remarks:
  - In the second equality, we have performed a diagonalization to the matrix $\mathbf{D} = \lambda_i \delta_{ij}\ \mathbf{m}' = \mathbf{U}\mathbf{m}, \mathbf{D} = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}$
  - We can then do each one dimensional Gaussian integral separately
  - And use the invariance of the determinant under choice of basis

Parenthesis: Multidimensional Gaussian Integrals

- We note the important relation for a Gaussian free energy (using invariance of tr under choice of basis),

$$\boxed{W_G = \log Z_G = \log\det[\mathbf{A}/(2\pi)]^{-1/2} = -\frac{1}{2}\mathrm{tr}\log\mathbf{A} + \mathrm{const.}}$$

- For a fermionic Gaussian free energy (Grassmann variables), one obtains in contrast

$$\boxed{W_G^{(F)} = \log Z_G^{(F)} = \log\det[\mathbf{A}/(2\pi)]^{+1/2} = +\frac{1}{2}\mathrm{tr}\log\mathbf{A} + \mathrm{const.}}$$

Saddle Point Effective Potential

- We con now formulate the homogenous saddle point approximation for the effective action. Due to homogeneity, we switch to the effective potential

$$\mathcal{U}[\phi^*,\phi] \equiv \frac{\Gamma[\phi^*,\phi]}{V/T}$$

with quantization volume $V/T = \int d\tau \int d^3x$

- We obtain, with abbreviation $\rho_0 = \phi_0^*\phi_0$ (we can choose ϕ_0 real so also $\phi_0^*\phi_0^* = \phi_0\phi_0 = \rho_0$)

$$\begin{aligned}
\mathcal{U}[\phi_0^*,\phi_0;\mu;T] &\approx -\mu\rho_0 + \tfrac{g}{2}{\rho_0}^2 + \tfrac{1}{2}\mathrm{tr}\log S^{(2)} \\
&= -\mu\rho_0 + \tfrac{g}{2}\rho_0^2 + T\textstyle\sum_n \int \frac{d^3q}{(2\pi)^3}\log\det_{2\times2}\begin{pmatrix} g\rho_0 & -\mathrm{i}\omega_n + \epsilon_{\mathbf{q}} - \mu + 2g\rho_0 \\ \mathrm{i}\omega_n + \epsilon_{\mathbf{q}} - \mu + 2g\rho_0 & g\rho_0 \end{pmatrix} \\
&= -\mu\rho_0 + \tfrac{g}{2}\rho_0^2 + T\int \frac{d^3q}{(2\pi)^3}\log\left(e^{\sqrt{(\epsilon_{\mathbf{q}}-\mu+2g\rho_0)^2-(g\rho_0)^2})/2T} - e^{-\sqrt{(\epsilon_{\mathbf{q}}-\mu+2g\rho_0)^2-(g\rho_0)^2})/2T}\right) + \mathrm{const}
\end{aligned}$$

tr runs over all indices: frequency/ momentum and the indices of the 2×2 matrix accounting for the complex boson. We use $2\log\sinh x = \sum_n \log(1 + x^2/(n\pi)^2)$, $\sinh x = (\exp x - \exp(-x))/2$. The (infinite) constant is irrelevant for thermodynamics.

Equation of State

- Effective Potential:

$$\mathcal{U}[\phi_0^*, \phi_0; \mu; T] \quad = -\mu\rho_0 + \frac{g}{2}\rho_0^2 + T\int \frac{d^3q}{(2\pi)^3} \log\left(e^{\sqrt{(\epsilon_{\mathbf{q}}-\mu+2g\rho_0)^2-(g\rho_0)^2})/2T} - e^{-\sqrt{(\epsilon_{\mathbf{q}}-\mu+2g\rho_0)^2-(g\rho_0)^2})/2T}\right)$$

- Discussion:
  - High temperature limit $T \to \infty$: $\rho_0 = 0, \mu < 0$. $\mathcal{U}[0,0;\mu;T] \approx T\int \frac{d^3q}{(2\pi)^3} \log\left(1 - e^{-(\epsilon_{\mathbf{q}}-\mu)/T}\right) + \text{const.}'$
  - Zero temperature limit $T \to 0$: $\mathcal{U}[\phi_0^*, \phi_0; \mu; T] \approx \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} \sqrt{(\epsilon_{\mathbf{q}} - \mu + 2g\rho_0)^2 - (g\rho_0)^2})$We can make connection to Bogoliubov theory (next slide)
  - NB: Though it seems that this theory interpolates well between low and high temperatures, it becomes problematic close to the finite temperature BEC phase transition

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- The equation of state, i.e. the explicit expression for the density, can be obtained from the effective potential
- Using the functional integral representation,

$$n = \lim_{\tau\to 0, \mathbf{x}\to 0} \langle \hat{\phi}^*(\tau, \mathbf{x})\hat{\phi}(0, \mathbf{0})\rangle = -\partial\mathcal{U}/\partial\mu, \;\; \hat{\phi}(\tau, \mathbf{x}) = \phi(\tau, \mathbf{x}) + \delta\varphi(\tau, \mathbf{x})$$

NB: This relation follows also using thermodynamics

Recovering the Bogoliubov Theory

- We calculate the equation of state explicitly
- Within the saddle point approximation and the equilibrium value $\mu = g\rho_0$, we find

$$n = \phi_0^*\phi_0 + \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{2g\rho_0}{\sqrt{2g\rho_0(\epsilon_{\mathbf{q}}+g\rho_0)}}$$

- Discussion:
  - Diverges linearly at high momenta
  - This is because the functional integral does not respect normal ordering. Note, for bosonic fields ϕ and operators a, $\langle\phi^*\phi\rangle = \frac{1}{2}\langle\phi^*\phi + \phi\phi^*\rangle = \frac{1}{2}\langle a^\dagger a + a a^\dagger\rangle = \langle a^\dagger a\rangle + \frac{1}{2}$. For each momentum mode:

$$\begin{aligned} n &= \phi_0^*\phi_0 + \int \frac{d^3q}{(2\pi)^3}\langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}}\rangle = \phi_0^*\phi_0 + \int \frac{d^3q}{(2\pi)^3}\left(\langle\delta\varphi_{\mathbf{q}}^*\delta\varphi_{\mathbf{q}}\rangle - \frac{1}{2}\right) \\ &= \phi_0^*\phi_0 + \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\left(\frac{2g\rho_0}{\sqrt{2g\rho_0(\epsilon_{\mathbf{q}}+g\rho_0)}} - 1\right) = \phi_0^*\phi_0 + \int \frac{d^3q}{(2\pi)^3}v_{\mathbf{q}}^2 \end{aligned}$$

The Bogoliubov result is reproduced

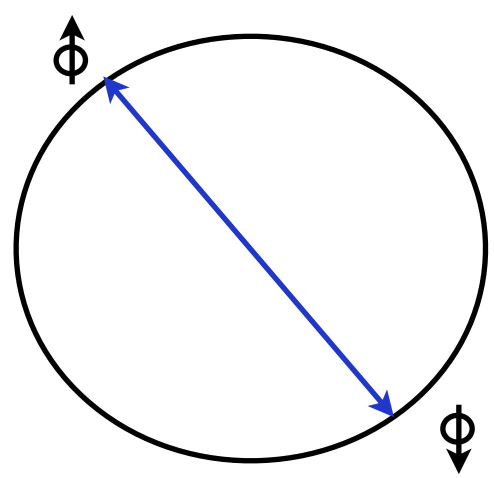
Summary: Weakly interacting Bosons

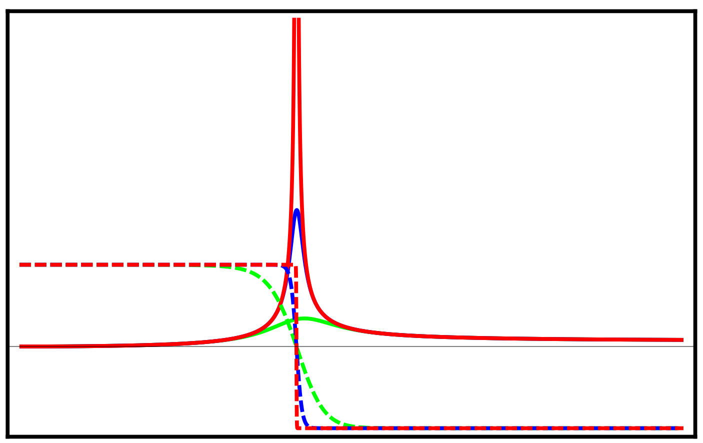
- We have applied the effective action to weakly interacting bosons
- Important concepts:
  - Classical Limit and Gross-Pitaevski Theory
  - Spontaneous symmetry breaking
  - Phase/ amplitude fluctuations and Bogoliubov theory
  - Goldstone's theorem
- These concepts can be applied to many other physical situations.
- We now use the effective action formalism to analyze weakly interacting fermions

Weakly Interacting Fermions

$$\langle \psi_\uparrow \rangle = \langle \psi_\downarrow \rangle = 0$$

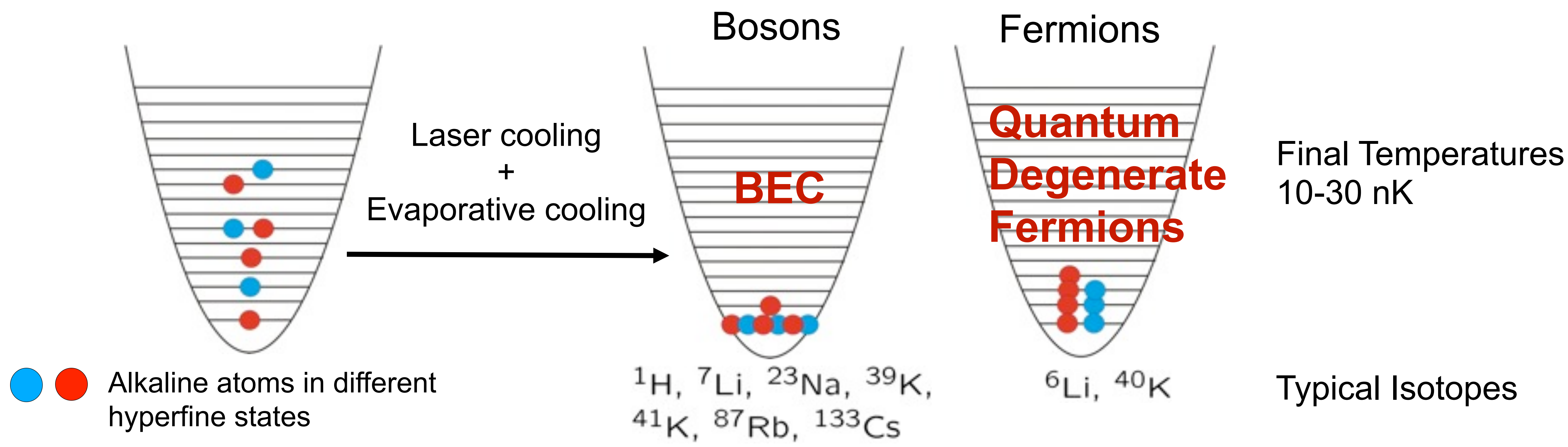
$$\langle \psi_\uparrow \psi_\downarrow \rangle \neq 0$$





Degenerate Bose vs. Fermi Gases

- Bosons: commutator $[b(\mathbf{x}), b^\dagger(\mathbf{y})] = \delta(\mathbf{x}-\mathbf{y}),\ [b(\mathbf{x}), b(\mathbf{y})] = [b^\dagger(\mathbf{x}), b^\dagger(\mathbf{y})] = 0$
- Bose-Einstein distribution: $n_{\mathbf{q}} = (\exp(\frac{\epsilon_{\mathbf{q}}-\mu}{T}) - 1)^{-1}$, $\epsilon_{\mathbf{q}} = \frac{\mathbf{q}^2}{2M}$, $\mu \leq 0$
- Fermions: Pauli principle $c_\sigma^{\dagger\,2} = 0$, anticommutator $\{c_\sigma(\mathbf{x}), c^\dagger_{\sigma'}(\mathbf{y})\} = \delta_{\sigma,\sigma'}\delta(\mathbf{x}-\mathbf{y}),\ \{c_\sigma(\mathbf{x}), c_{\sigma'}(\mathbf{y})\} = \{c^\dagger_\sigma(\mathbf{x}), c^\dagger_{\sigma'}(\mathbf{y})\} = 0$
- Fermi-Dirac distribution: $n_{\mathbf{q}} = (\exp(\frac{\epsilon_{\mathbf{q}}-\mu}{T}) + 1)^{-1}$, $\epsilon_{\mathbf{q}} = \frac{\mathbf{q}^2}{2M}$, $\mu > 0$
- real space picture:



- Alkaline atoms (bosons and fermions) can be prepared in the quantum degenerate regime, using laser cooling and evaporative cooling
- The fermion spin is realized with two hyperfine states

Free Fermions and Fermi Momentum

- Collection of some useful formulae and abbreviations for 3D two-component fermions:
- The equation of state for free fermions at zero temperature:

$$n = \underbrace{2}_{\text{two spin states}} \int \frac{d^3q}{(2\pi)^3} (\exp(\tfrac{\epsilon_{\mathbf{q}}-\mu}{T} + 1)^{-1} \xrightarrow{T\to 0} 2\int \frac{d^3q}{(2\pi)^3} \theta(\epsilon_{\mathbf{q}} - \mu) = \frac{(2M\mu)^{3/2}}{3\pi^2} \equiv \frac{k_{\mathrm{F}}^3}{3\pi^2}$$

- The Fermi momentum k_F is defined as the momentum scale associated to the chemical potential of free fermions at T = 0

$$k_{\mathrm{F}} \equiv (2M\mu_{T=0}^{(\mathrm{free})})^{1/2}$$

- The Fermi momentum is a measure for the total density of a fermion system:
  - It is a measure for the mean interparticle spacing $d = (3\pi^2)^{1/3} k_{\mathrm{F}}^{-1}$
  - It is temperature and interaction independent
  - In contrast, the chemical potential is a function of temperature and interactions

- The associated energy and temperature scales are the Fermi energy and the Fermi temperature

$$\epsilon_{\mathrm{F}} = \frac{k_{\mathrm{F}}^2}{2M}, \quad T_{\mathrm{F}} = \frac{\epsilon_{\mathrm{F}}}{k_{\mathrm{B}}}$$

Physical Picture for Weakly Attractive Fermions

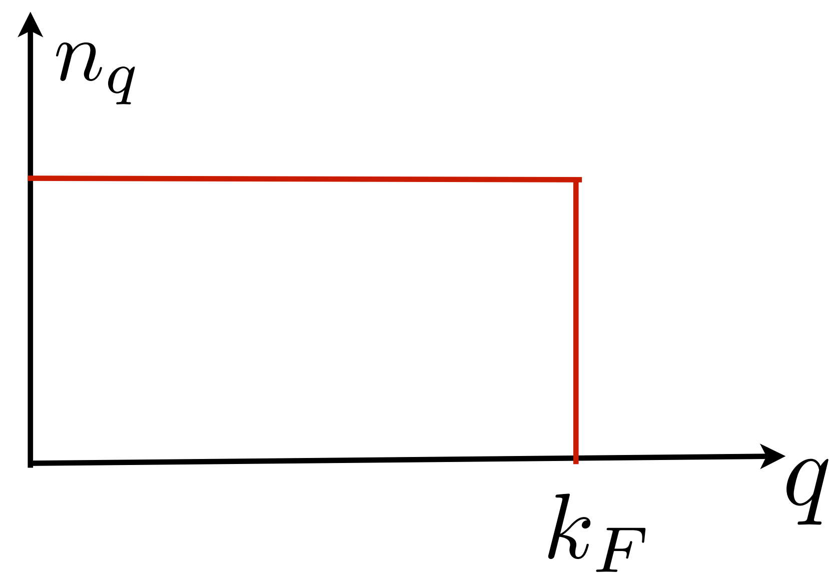
- The low temperature physics of fermions is governed by the Pauli principle

  (1) Expression of a Fermi sphere in momentum space

  (2) Absence of fermion condensation: $\langle\psi_\sigma\rangle = 0 \quad \sigma = \uparrow, \downarrow$

  (3) Local s-wave interactions of fermions are only possible for more than one spin state (ultracold atoms: hyperfine states)

Fermi distribution at low T



- Now we allow for weak 2-body s-wave attraction between 2 spin states of fermions

$$a < 0$$

attractive scattering length

$$|ak_\mathrm{F}| \sim |a/d| \ll 1$$

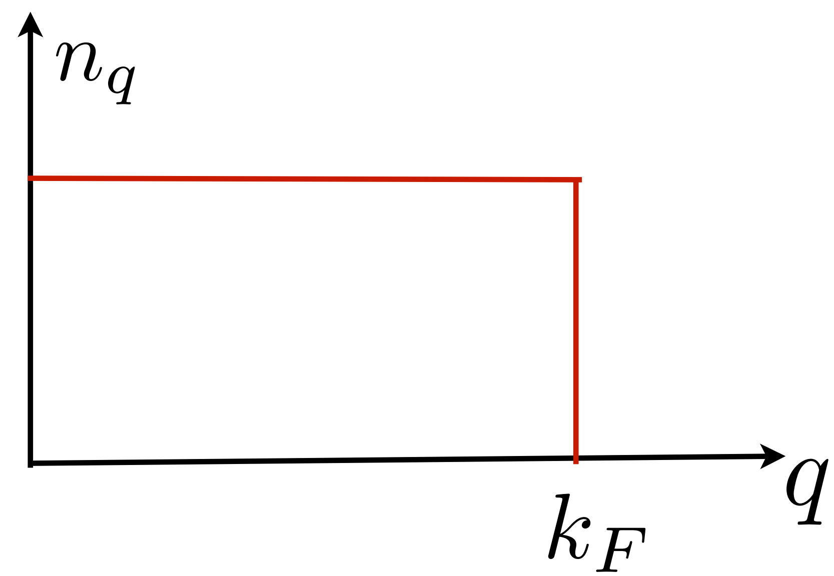
weakness/diluteness condition

- A small interaction scale will not be able to substantially modify the Fermi sphere. This is the key to BCS theory

Physical Picture for Weakly Attractive Fermions

$$|ak_{\mathrm{F}}| \sim |a/d| \ll 1$$

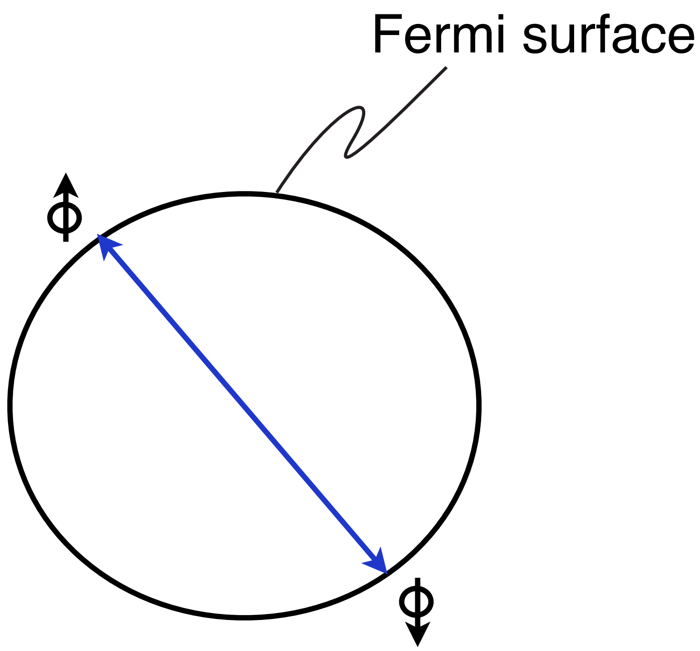
Fermi distribution at low T



- A small interaction scale will not be able to substantially modify the Fermi sphere
- However, pairing of fermions with momenta close to the Fermi surface is possible: "Cooper pairs":
  - These fermions attract each other with strength a
  - The total energy of the system is lowered when
    - bosonic pairs with zero cm energy (total momentum zero) form: local in momentum space
    - These pairs condense, i.e. occupy a single quantum state macroscopically:

$$\boxed{\langle \psi_\uparrow \psi_\downarrow \rangle \neq 0}$$

- Comments:
  - Distinguish pairing correlation from Bose condensation $\langle \varphi \rangle \neq 0$
  - But: in both cases, spontaneous breaking of U(1) symmetry



Cooper pairing: Local in momentum space

Action for weakly interacting fermions

- Two-component fermions are described by a spinor

$$\psi(\tau,\mathbf{x}) = \left(\begin{array}{c} \psi_\uparrow(\tau,\mathbf{x}) \\ \psi_\downarrow(\tau,\mathbf{x}) \end{array} \right), \quad \psi^\dagger(\tau,\mathbf{x}) = (\psi^*_\uparrow(\tau,\mathbf{x}), \psi^*_\uparrow(\tau,\mathbf{x}))$$

The components are Grassmann variables, i.e. they anticommute $\{\psi_\sigma(\tau.\mathbf{x}), \psi_{\sigma'}(\tau'.\mathbf{x}')\} = \{\psi^*_\sigma(\tau.\mathbf{x}), \psi^*_{\sigma'}(\tau'.\mathbf{x}')\}$ $\{\psi_\sigma(\tau.\mathbf{x}), \psi^*_{\sigma'}(\tau'.\mathbf{x}')\} = 0$ (reflecting anticommutation relations for fermionic operators)

- The kinetic (single particle) term describes nonrelativistic propagation:

$$S_{\mathrm{kin}} = \sum_{\sigma,\sigma'} \int d\tau dx [\psi_\sigma(\tau,x)(\partial_\tau - \frac{\triangle}{2M} - \mu)\delta_{\sigma\sigma'}\psi_{\sigma'}(\tau,x)] = \int_0^\beta d\tau dx [\psi^\dagger(\tau,x)(\partial_\tau - \frac{\triangle}{2M} - \mu)\psi(\tau,x)]$$

However, unlike bosons $\mu > 0$ to ensure a fixed particle number

- Local s-wave two-body density-density interactions between the two spin states are described by

$$S_{\mathrm{int}} = \frac{g}{2}\int_0^\beta d\tau dx [\psi^\dagger(\tau,x)\psi(\tau,x)]^2$$

The precise relation of the coupling constant $g < 0$ to the scattering length a is discussed below

- Validity: as for bosons, in particular, $a \ll d \sim k^{-1}{}_{\mathrm{F}}$
- Symmetries: as for the boson action, supplemented by global $SU(2)$ spin rotation invariance

Cooper Pairing and Hubbard-Stratonovich Transformation

- We perform a Hubbard Stratonovich transformation, which makes the expected pairing of the fermions ($\langle\psi_\uparrow\psi_\downarrow\rangle = \frac{1}{2}\langle\psi^T\epsilon\psi\rangle \neq 0$) explict. There are 3 steps:
  1. Rearrange the fermion fields into local singlets, to make the pairing explicit (Fierz transformation)

$$(\psi^\dagger\psi)^2 = (\psi^\dagger\delta\psi)^2 = -\tfrac{1}{2}(\psi^\dagger\epsilon\psi^*)(\psi^T\epsilon\psi), \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  2. Introduce a properly written factor of unity into the fermionic functional integral $Z = \int \mathcal{D}\psi \exp -(S_{\mathrm{kin}} + S_{\mathrm{int}})$, (note $\left(\psi^T\epsilon\psi\right)^\dagger = -\psi^\dagger\epsilon\psi^*$; we use $X = (\tau, \mathbf{x})$, $\int_X = \int d\tau\mathbf{x}$, $\int\mathcal{D}\psi = \int\mathcal{D}(\psi^*,\psi)$)

$$\mathbf{1} \quad = \mathcal{N}\int\mathcal{D}\varphi \exp -\int_X m^2(\varphi^* + \tfrac{h}{2m^2}\psi^\dagger\epsilon\psi^*)(\varphi - \tfrac{h}{2m^2}\psi^T\epsilon\psi)$$

  The parameters m^2, h are real and $m^2 > 0$ to make the Gaussian integral convergent but otherwise arbitrary. The resulting partition function is ($D = \partial_\tau - \frac{\triangle}{2M} - \mu$)

$$Z = \mathcal{N}\int\mathcal{D}\psi)D\varphi \exp -\int_X \left[\psi^\dagger D\psi + m^2\varphi^*\varphi + \tfrac{h}{2}(\varphi\psi^\dagger\epsilon\psi^* - \varphi^*\psi^T\epsilon\psi) - \tfrac{1}{4}(g + \tfrac{h^2}{m^2})(\psi^\dagger\epsilon\psi^*)(\psi^T\epsilon\psi)\right]$$

  3. For attractive interaction $g < 0$, we can now choose the parameters such that $g = -\frac{h^2}{m^2}$
     - Then we get rid of the fermion interaction term
     - Only this ratio is physical. We can redefine $\varphi \to h\varphi$ and get

$$Z = \mathcal{N}\int\mathcal{D}\psi\mathcal{D}\varphi \exp -S_{\mathrm{HS}}, \quad S_{\mathrm{HS}}[\psi,\varphi] = \int_X \left[\psi^\dagger D\psi + \tfrac{1}{|g|}\varphi^*\varphi + \tfrac{1}{2}(\varphi\psi^\dagger\epsilon\psi^* - \varphi^*\psi^T\epsilon\psi)\right]$$

Cooper Pairing and Hubbard-Stratonovich Transformation

- We discuss the Hubbard-Stratonovich action

$$S_{\mathrm{HS}}[\psi,\varphi] = \int_X \left[\psi^\dagger D\psi + \tfrac{1}{|g|}\varphi^*\varphi + \tfrac{1}{2}(\varphi\psi^\dagger\epsilon\psi^* - \varphi^*\psi^T\epsilon\psi)\right]$$

- The interacting fermion theory is mapped into a coupled fermion-boson theory.
- The boson field represents Cooper pairs, $\phi \sim \psi_\uparrow\psi_\downarrow$. A nonzero expectation value $\phi = \langle\varphi\rangle \neq 0$ breaks the $U(1)$ symmetry and describes Cooper pair condensation
- The theory is quadratic in the fermions. We introduce Nambu-Gorkov fields $\Psi^T = (\psi^T, \psi^\dagger)$ to make this explicit,

$$S_{\mathrm{HS}} = \frac{1}{2}\int_X \left[\Psi^T S^{(2)}[\varphi]\Psi + \tfrac{1}{|g|}\varphi^*\varphi\right], \quad S^{(2)}[\varphi] = \begin{pmatrix} -\varphi^*\epsilon & -D\delta \\ D\delta & \varphi\epsilon \end{pmatrix}$$

  They can be eliminated by Gaussian integration. However, the result is an interacting boson theory due to the dependence $S^{(2)}[\varphi]$
- The BCS approximation consists in neglecting the fluctuations of the bosonic Cooper pair field: Mean field approximation for the boson dofs

BCS Effective Action

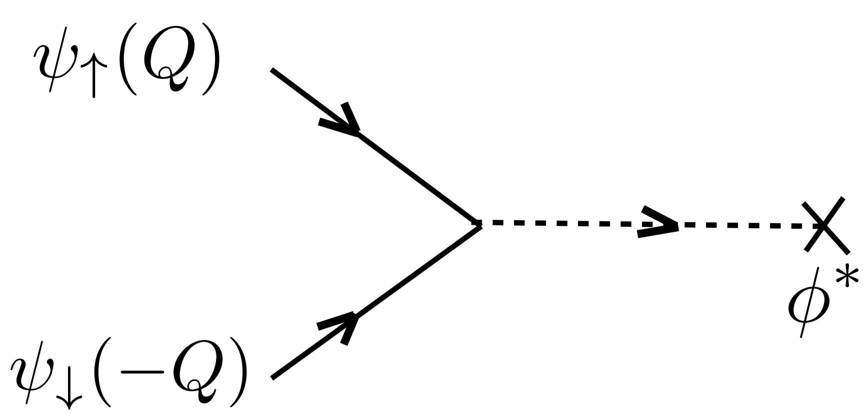
- We implement the above strategy on the effective action for homogeneneous classical field configurations, [1]

$$\begin{aligned}\Gamma[\psi_{\rm cl},\phi]/(V/T) &= -\log\int\mathcal{D}\delta\psi\mathcal{D}\delta\varphi\exp -S_{\rm HS}[\psi_{\rm cl}+\delta\psi,\phi+\delta\varphi]\\ &= \tfrac{1}{|g|}\phi^*\phi-\log\int\mathcal{D}\delta\psi\exp -S_{\rm HS}[\delta\psi,\phi]/(V/T)\end{aligned}$$

- Due to the approximation on φ, $S_{\rm HS}$ is now truly quadratic, in the background of a classical field ϕ. It is useful to switch to momentum space where

$$\begin{aligned}S_{\rm HS}[\delta\psi,\phi] &= \tfrac{1}{2}\int_Q \Psi^T(-Q)S^{(2)}[\phi]\Psi(Q),\\ S^{(2)}[\phi] &= \begin{pmatrix} -\phi^*\epsilon & -(-\mathrm{i}\omega_n+\epsilon_{\mathbf{q}}-\mu)\delta\\ (\mathrm{i}\omega_n+\epsilon_{\mathbf{q}}-\mu)\delta & \phi\epsilon\end{pmatrix}, \quad \Psi(Q)=\begin{pmatrix}\psi(Q)\\ \psi^*(-Q)\end{pmatrix}\end{aligned}$$

- NB: the classical field couples $\phi\psi^*_\uparrow(Q)\psi^*_\downarrow(-Q)$. Interpretation similar to Bogoliubov theory: creation of fermions with opposite momenta out of the Cooper pair condensate



annihilation process of two oppositely directed fermions into the pair condensate

1

- The fermion field expectation takes the physical value $\psi_{\rm cl}=0$ but serves to carry out successive derivatives to generate the 1PI correlation functions

BCS Effective Action

- Evaluation of the Grassmann Gaussian integral ($2\log\cosh x = \sum_n \log(1 + x^2/((n+1/2)\pi)^2)$)

$$\begin{aligned}
\Gamma[\psi_{\mathrm{cl}}, \phi]/(V/T) &= \frac{1}{|g|}\phi^*\phi - \frac{1}{2}\mathrm{tr}\log S^{(2)}[\phi]/(V/T) \\
&= \frac{1}{|g|}\phi^*\phi - 2T\int\frac{d^3\mathbf{q}}{(2\pi)^3}\log\cosh(E_{\mathbf{q}}/2T) + \mathrm{const.} \\
E_{\mathbf{q}} &= \sqrt{(\epsilon_{\mathbf{q}} - \mu)^2 + \phi^*\phi}
\end{aligned}$$

- NB: The homogeneous BCS effective action depends on the chemical potential μ and on the $U(1)$ invariant combination $\rho = \phi^*\phi$, $\Gamma = \Gamma[\mu, \rho]$

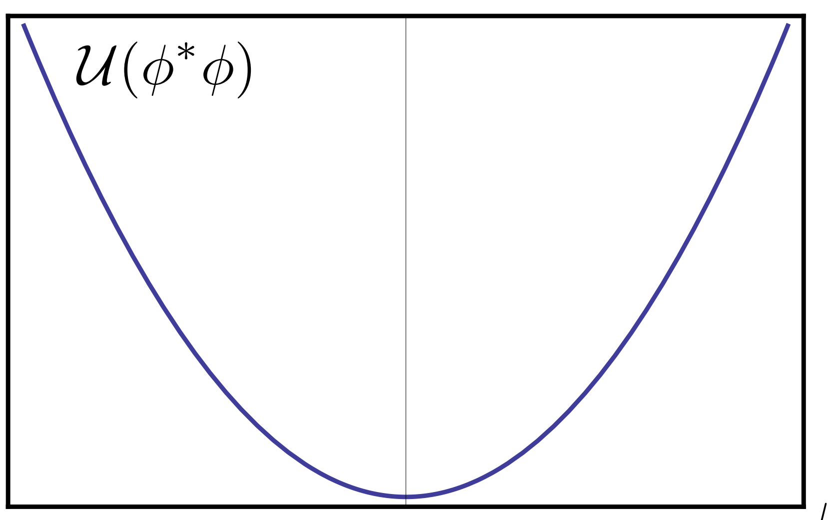
Spontaneous Symmetry Breaking

- The phase transition towards a state with macroscopic Cooper pairing at low T can be obtained from analyzing symmetry breaking patterns
- Study the field equation/ equilibrium condition for the effective potential $\mathcal{U}[\mu,\rho] = \Gamma[\mu,\rho]/(V/T)$,

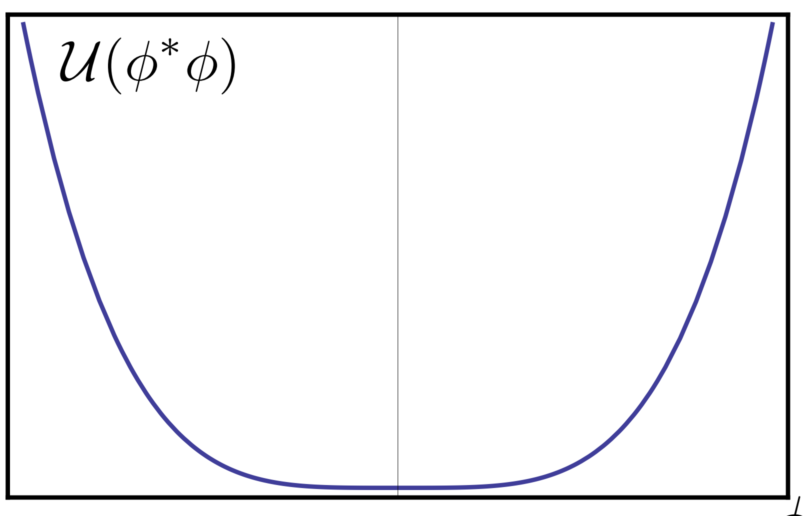
$$\frac{\partial \mathcal{U}}{\partial \phi^*} = \phi \cdot \frac{\partial \mathcal{U}}{\partial \rho} \stackrel{!}{=} 0$$

- There are three possibilities:
  1. Spontaneously symmetry broken phase: $\phi, \phi^* \neq 0, \quad \frac{\partial \mathcal{U}}{\partial \rho} = 0$
  2. "Symmetric" phase: $\phi = \phi^* = 0, \quad \frac{\partial \mathcal{U}}{\partial \rho} \neq 0$
  3. Critical point: $\phi = \phi^* = 0, \quad \frac{\partial \mathcal{U}}{\partial \rho} = 0$
- NB: in the symmetry broken phase, $\frac{\partial \mathcal{U}}{\partial \rho} = 0$ signals the vanishing mass of the Golstone mode

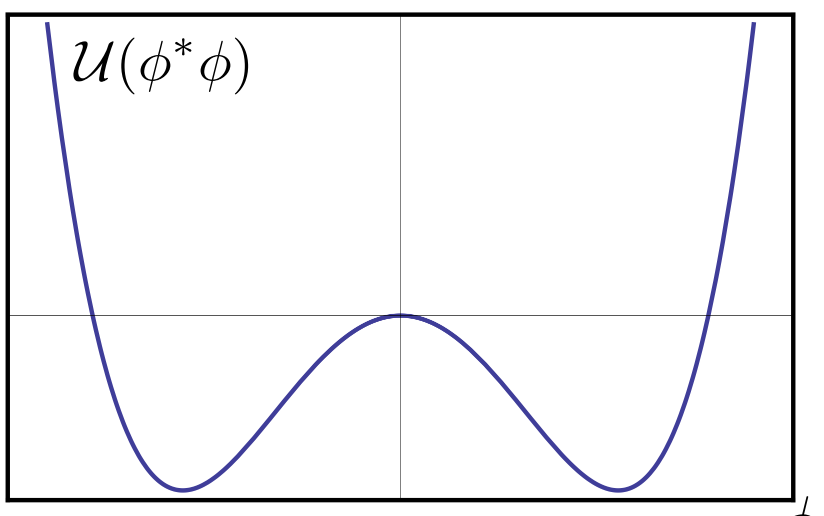
Cuts through potential landscape for U(1) symmetric theory



Symmetric phase



Critical point



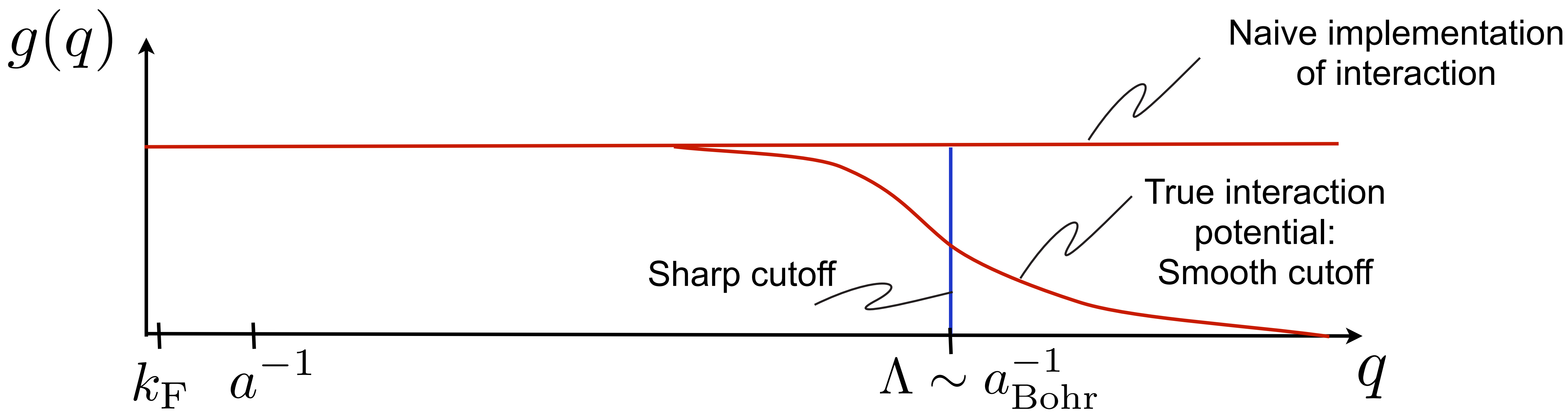
Symmetry broken phase

Critical Temperature and a Problem at Large Momenta

- We focus on the critical point for the Cooper instability first, i.e. we study $\frac{\partial \mathcal{U}}{\partial \rho}\Big|_{\rho=0} = 0$. Explicitly:

$$0 = -\frac{1}{g} - \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\epsilon_{\mathbf{q}} - \mu}\tanh\left(\frac{\epsilon_{\mathbf{q}} - \mu)}{2T}\right) = \frac{1}{|g|} - \frac{1}{4\pi^2}\int dq \frac{q^2}{q^2/(2M) - \mu}\tanh\left(\frac{q^2/(2M) - \mu)}{2T}\right)$$

- The integral is linearly divergent, as the integrand tends to 1 for large momenta $q \to \infty$: “Ultraviolet divergence”
- This is because we have assumed spatially local interactions, i.e. constant in momentum space up to arbitrarily large momenta
- In reality, this is not the case. There is a (smooth) momentum cutoff at large momenta. Details of the precise cutoff function should not matter (cf. discussion of effective theories above)



Ultraviolet Renormalization

- The problem is cured by a proper ultraviolet (UV) renormalization procedure:
  - Introduce a (sharp) UV cutoff Λ in the momentum space integral (UV regularization)
  - Interpret g as “bare” coupling with cutoff dependence, $g = g_\Lambda$
  - Trade the bare coupling for a physical observable: Require that $\partial\mathcal{U}/\partial\rho$ performed in vacuum ($\mu = 0\,(n=0),\, T=0, \rho=0$) produce the physical inverse scattering length (UV renormalization):

$$\frac{\partial\mathcal{U}}{\partial\rho}\Big|_{\text{vac}} = -\frac{1}{g_\Lambda} - \frac{1}{4\pi^2}\int^\Lambda dq\,\frac{q^2}{q^2/(2M)} = -\frac{1}{g_\Lambda} - \frac{M\Lambda}{2\pi^2} \overset{!}{=} -\frac{1}{g_p} = -\frac{M}{4\pi a}$$

NB: The relation between g_Λ and a is not perturbative; a more systematic treatment uses $g_\Lambda(q) = g_\Lambda\theta(\Lambda - q)$ and interprets the above equation as a resummed loop equation; the relation between a physical fermionic two-body coupling g_p and a scattering lenght is $a = Mg_p/(4\pi)$

- The renormalized equation for the critical temperature reads

$$0 = -\frac{1}{a} - \frac{2}{\pi}\int dq\left[\frac{q^2}{q^2 - 2M\mu}\tanh\left(\frac{q^2/(2M) - \mu)}{2T}\right) - 1\right]$$

bosons: 8pi;
bosons indistinguishable,
fermions distinguishable
(spin)

It will be analyzed below

The Equation of State

- The explicit expression for the density is obtained from

$$n_\Lambda = -\frac{\partial \mathcal{U}}{\partial \mu} = -\int \frac{d^3\mathbf{q}}{(2\pi)^3} \tanh(E_\mathbf{q}/2T)$$

  Again, the expression is UV divergent, here $\sim \Lambda^3$

- Using $\tanh(x/2) = 1 - 2(\exp x + 1)^{-1}$, we split the integral into a physical (depending on μ and ρ) and an unphysical contribution:

$$n_\Lambda = 2\left(\int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{e^{E_\mathbf{q}/T} + 1} - \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{2}\right)$$

  The overall factor of two counts the degenerate spin states

- Interpretation:
  - First term: involves the Fermi-Dirac distribution. The physical density is

$$n = 2\int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{e^{E_\mathbf{q}/T} + 1}$$

  - Second term: Unobservable zero-point energy shift due to the fact that the functional integral does not respect operator ordering: consider single fermion mode field ψ and associated operator c ($\{c, c^\dagger\} = 1$),

$$\langle \psi^* \psi \rangle = \tfrac{1}{2}\langle \psi^*\psi - \psi\psi^* \rangle = \tfrac{1}{2}\langle c^\dagger c - c c^\dagger \rangle = \langle c^\dagger c \rangle - \tfrac{1}{2}$$

BCS Equations for the Critical Temperature

- We have derived two equations governing the thermodynamics of weakly interacting fermions:
  - The equation of state

$$\boxed{n = 2\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{e^{(\epsilon_{\mathbf{q}}-\mu)/T}+1}}$$

  - The equation for the critical temperature

$$\boxed{0 = -\frac{1}{a} - \frac{2}{\pi}\int dq\left[\frac{q^2}{q^2-2M\mu}\tanh\left(\frac{q^2/(2M)-\mu}{2T}\right)-1\right]}$$

- Given μ we could solve the second equation. For $T/T_{\mathrm{F}} \ll 1$, the corrections to the chemical potential from $\epsilon_{\mathrm{F}} = \mu(T=a=0)$ are negligible. We therefore work with $\mu \approx \epsilon_{\mathrm{F}}$

The Critical Temperature

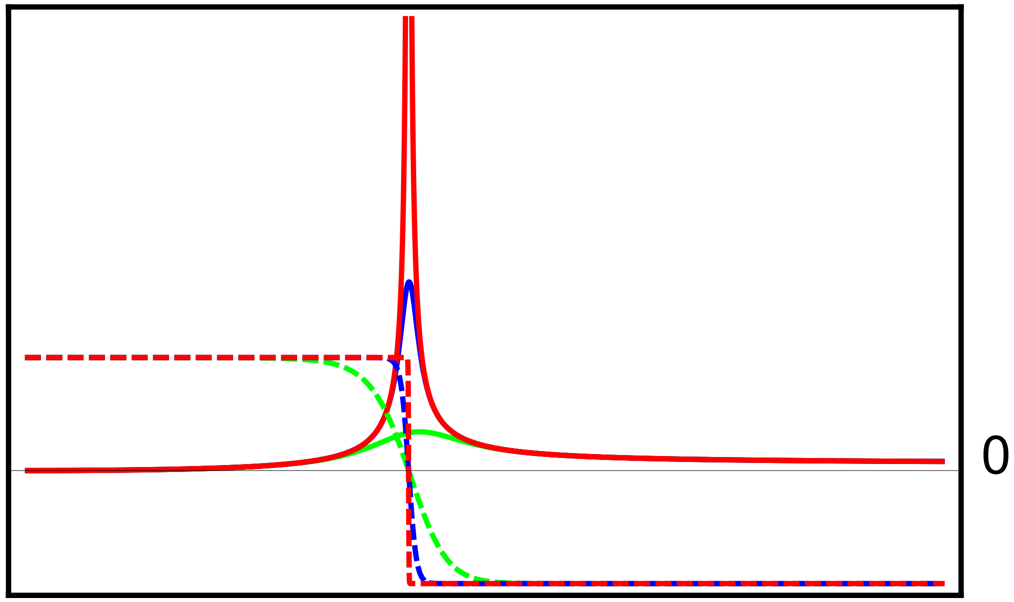
- We rescale momenta and energies as $\tilde{q} = q/k_\mathrm{F}, \tilde{E} = E/\epsilon_\mathrm{F}$. The dimensionless critical temperature is determined from

$$0 = \frac{1}{|ak_\mathrm{F}|} - \frac{2}{\pi} \int d\tilde{q} \left[\left(\frac{\tilde{q}^2}{\tilde{q}^2 - 1} \tanh \left(\frac{\tilde{q}^2 - 1)}{2\tilde{T}} \right) \right) - 1 \right]$$

- For $\tilde{T} \to 0$, the integral develops a logarithmic divergence at the Fermi surface where $\tilde{q} = 1$. Thus, the equation has a solution for arbitrarily weak interaction a if T is lowered sufficiently
- This establishes the BCS transition with critical temperature

$$\boxed{\frac{T_\mathrm{c}}{\epsilon_\mathrm{F}} = \frac{8\gamma}{\pi e^2} e^{-\frac{\pi}{2|ak_\mathrm{F}|}}}$$

  The exponential dependence reflects the log divergence. The prefactor is ≈ 0.61 with the Euler constant $\gamma \approx 1.78$
- For $T < T_\mathrm{c}$ a gap $\rho > 0$ develops to cure the log divergence



divergent part of the integrand, and distribution function (dashed)

green: $\tilde{T} = 0.1$

blue: $\tilde{T} = 0.01$

red: $\tilde{T} = 0.001$

BCS Equations for the Gap at Zero Temperature

- We have derived two equations governing the thermodynamics of weakly interacting fermions at $T = 0$:
  - The equation of state

$$\boxed{n = 2\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{e^{E_\mathbf{q}/T}+1}}$$

  - The dimensionless equation for the gap $\rho = \phi^*\phi$ (from $\partial\mathcal{U}/\partial\rho = 0$ in the presence of symmetry breaking $\rho \neq 0$ and $T = 0$)

$$\boxed{0 = \frac{1}{|ak_\mathrm{F}|} - \frac{2}{\pi}\int d\tilde{q}\left[\frac{\tilde{q}^2}{\tilde{E}_\mathbf{q}} - 1\right]}$$

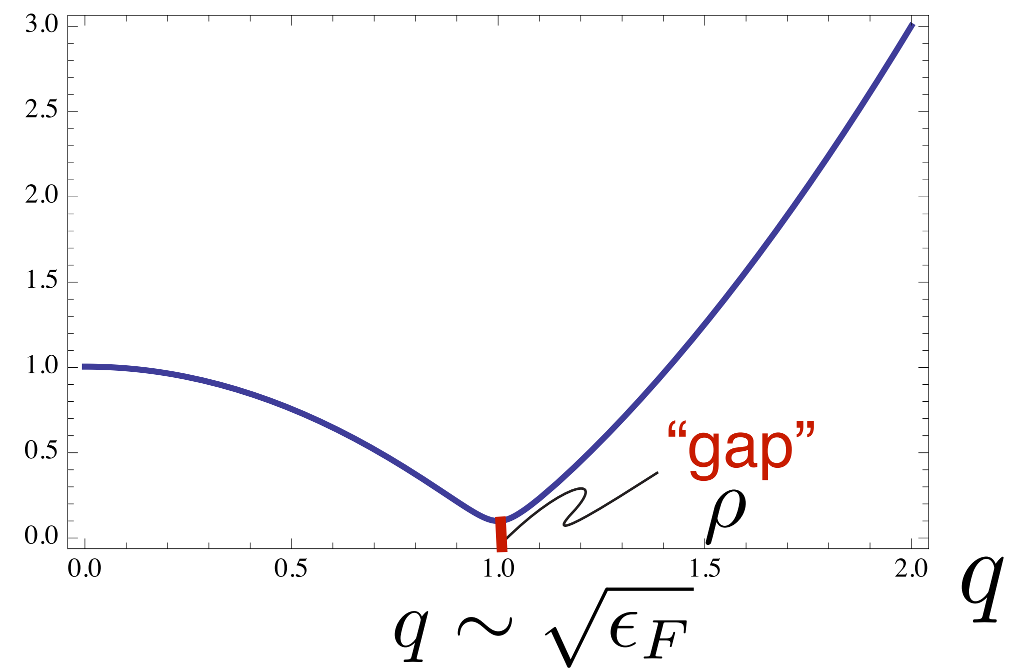
    with $E_\mathbf{q} = \sqrt{(\epsilon_\mathbf{q} - \mu)^2 + \rho}$
- The scale T is exchanged for the scale ρ, wich now regularizes the log divergence. Again, for $\rho/T_\mathrm{F} \ll 1$, the corrections to the chemical potential from $\epsilon_\mathrm{F} = \mu(T = a = 0)$ are negligible. We therefore work with $\mu \approx \epsilon_\mathrm{F}$
- This establishes the BCS gap at zero temperature ($\pi/\gamma \approx 1.76$)

$$\boxed{\frac{\rho(T=0)}{\epsilon_\mathrm{F}} = \frac{8}{e^2}\,e^{-\frac{\pi}{2|ak_\mathrm{F}|}} = \frac{\pi}{\gamma}\frac{T_\mathrm{c}}{\epsilon_\mathrm{F}}}$$

Fermion Gap and Superfluidity

- Below the critical temperature, a “gap” opens up
- This means that single fermion excitations are suppressed even very close to the Fermi surface (unlike the behavior above T_c). The spectrum is given by

$$E_{\mathbf{q}} = \left[\left(\frac{q^2}{2M} - \mu \right)^2 + \rho \right]^{1/2}$$



- The suppression of single fermion excitations is responsible for superfluidity:
  - The low lying modes are bosonic phonons with linear dispersion
  - This ensures superfluidity according to Landau’s criterion

Summary: Weakly Interacting Fermions

- We have derived BCS theory within the effective action formalism
  - Condensation of the fermion field is impossible due to Pauli's principle, but a second order pairing correlation can develop.
  - The BCS mechanism builds on two cornerstones:
    - the sharpness of the Fermi surface, guaranteed by $ak_{\mathrm{F}} \ll 1$
    - Cooper pairing of fermions close to the Fermi surface with opposite momenta
  - Pairing is introduced in the functional integral by a Hubbard-Stratonovich transformation, mapping the purely fermionic to a fermion-boson theory
  - BCS theory integrates out the quadratic fermions and treats the bosons classically
  - The exponential dependence of critical temperature and gap at zero temperature persists to arbitrarily small interactions. It can be traced to a logarithmic divergence at the Fermi surface.
  - Like the condensation amplitude for bosons, the pairing correlation breaks the U(1) symmetry. The finite gap prevents single fermion excitations and leads to superfluidity

Experimental (Ir)relevance of Weakly Interacting Atomic Fermions

- We compare the critical temperatures for a noninteracting BEC and weakly attractive fermions
  - Free bosons of mass M undergo condensation at $n\lambda_{dB} = \zeta(3/2), \quad \lambda_{dB} = (2\pi/(MT))^{1/2}$
  - Rewrite by using definitions from fermions $n = k_{\rm F}^3/(3\pi^2), \quad \epsilon_{\rm F} = k_{\rm F}^2/(2M)$

$$\frac{T_{\rm c}^{(\rm BEC)}}{\epsilon_{\rm F}} = 4\pi(3\pi^2\zeta(3/2))^{-2/3} \approx 0.69 = \mathcal{O}(1)$$

  - In contrast, the BCS critical temperature is exponentially small for $ak_{\rm F} \ll 1$

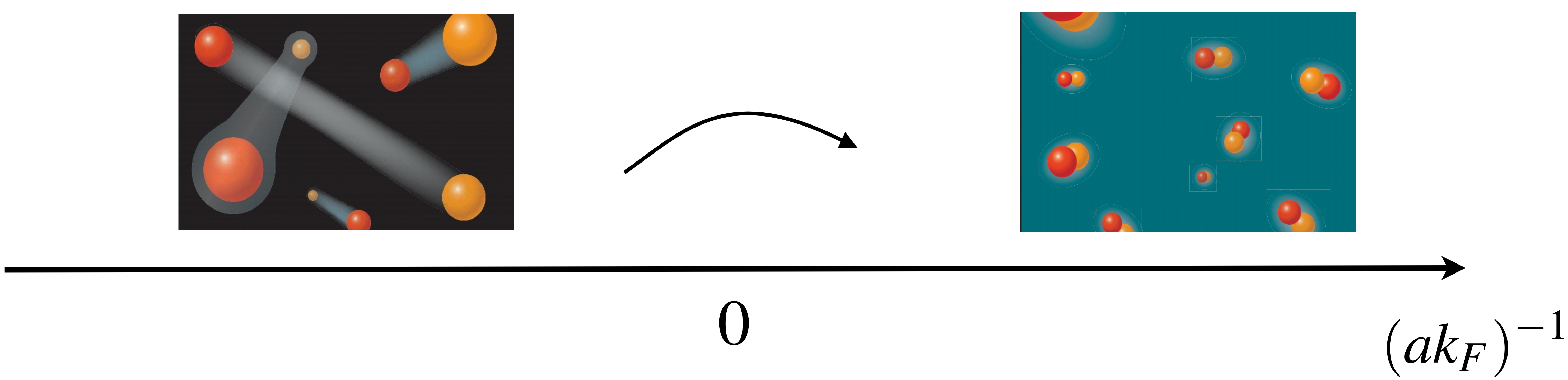
$$\frac{T_{\rm c}^{(\rm BCS)}}{\epsilon_{\rm F}} = \frac{8\gamma}{\pi e^2}\, e^{-\frac{\pi}{2|ak_{\rm F}|}} \approx 0.61 e^{-\frac{\pi}{2|ak_{\rm F}|}}$$

- additionally, cooling of degenerate fermions is experimentally more challenging due to Pauli blocking

---

- On the other hand, note a (formal) exponential increase of T_c for rising $ak_{\rm F}$ i.e. towards strong interactions
- Q: What is the fate of the exponential increase in T_c for rising $ak_{\rm F}$
- A: BCS-BEC crossover

Strong Interactions
and
the BCS-BEC Crossover



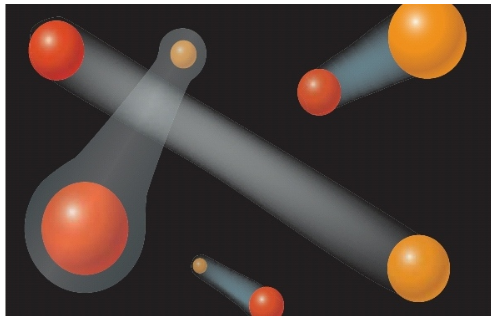
Physical picture: BCS-BEC Crossover

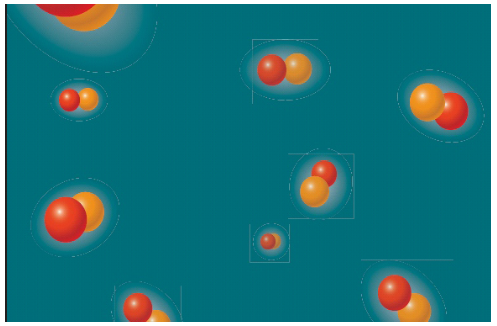
- We have discussed two cornerstones for quantum condensation phenomena:

  - fermions with attractive interactions
    ➡ BCS superfluidity at low T

  - weakly interacting bosons
    ➡ Bose-Einstein Condensate (BEC) at low T

bosons could be realized as
tightly bound molecules
("effective theory")





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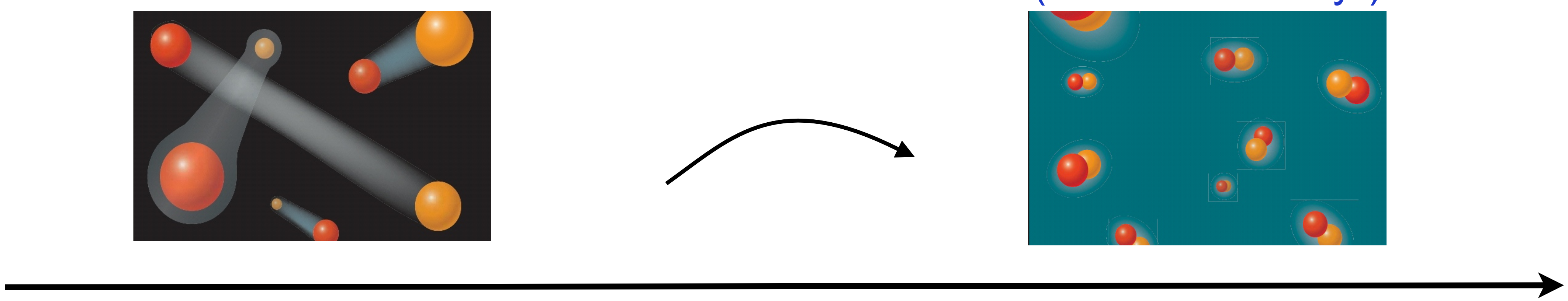


0 $\qquad$ $(ak_F)^{-1}$

- There is an experimental knob to connect these scenarios: Feshbach resonances
  - microscopically, the phenomenon is due to a bound state formation at the resonance $\frac{1}{ak_{\mathrm{F}}} = 0$
  - from a many-body perspective, the phenomenon is understood as
    - Localization in position space
    - Delocalization in momentum space

➡ In the strongly interacting regime, no simple ordering principle is known:
Challenge for Many-Body methods

Microscopic Origin: Feshbach Resonances

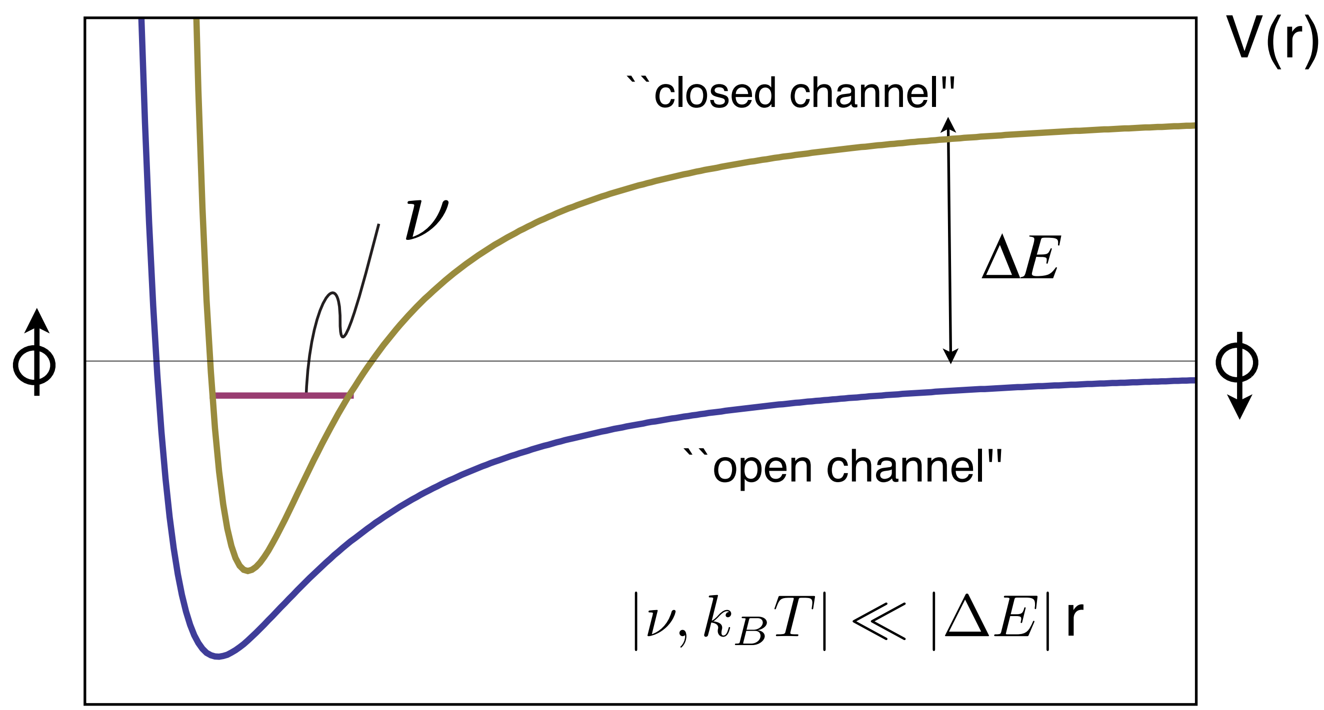
- Start from fermions: (Euclidean) Action

fermion field:
two hyperfine states

$$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$$

$$S_\psi[\psi] = \int d\tau d^3x \big(\psi^\dagger(\partial_\tau - \tfrac{\triangle}{2M})\psi + \tfrac{\lambda_\psi}{2}(\psi^\dagger\psi)^2\big)$$

- Consider a second interaction channel with bound state close to scattering threshold V=0, detuned by ν



examples: $^6\mathrm{Li}, {}^{40}\mathrm{K}$

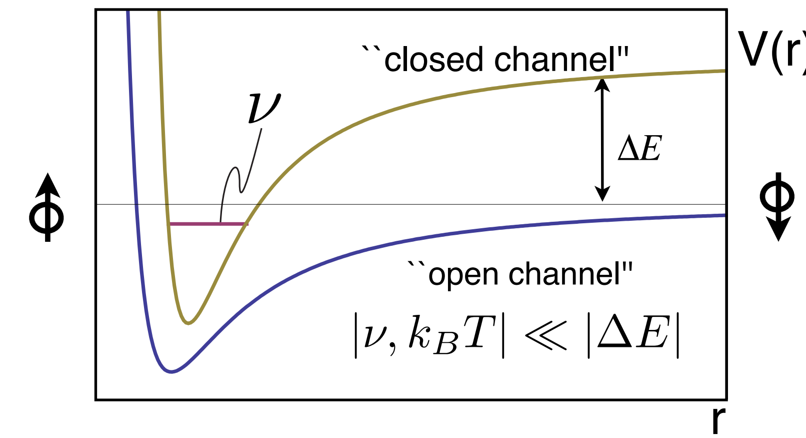
- Detuning ν can be controlled with magnetic field

$$\nu(B) = \mu_B(B - B_0)$$

magnetic moment

resonance position

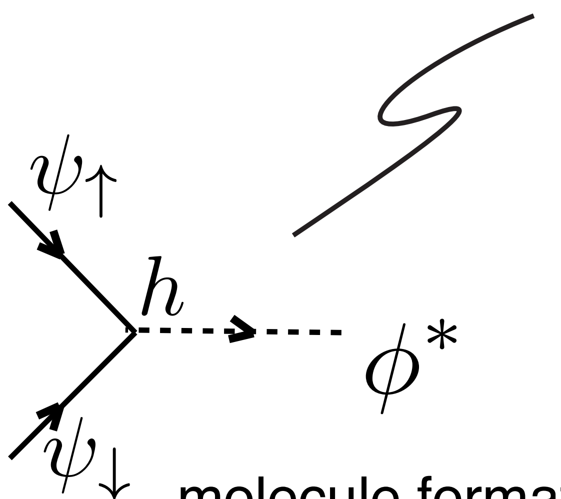
Microscopic Origin: Feshbach Resonances



- Effective Model to describe this situation: Interconversion of two fermions into a molecule

bosonic molecule field: $$S_\phi[\phi] = \int d\tau d^d x \phi^* (\partial_\tau - \frac{\triangle}{4M} + \nu)\phi$$

interconversion:
Feshbach, Yukawa term $$S_F[\psi, \phi] = -h \int d\tau d^d x (\phi^* \psi_\uparrow \psi_\downarrow - \phi \psi_\uparrow^* \psi_\downarrow^*)$$



molecule formation

- NB: cf. BCS Cooper pairing with condensate amplitude:

$$\phi_0^* = \text{const.}$$

- Now we allow for dynamic bosonic degrees of freedom

$$\phi^*(\tau, \mathbf{x}) \quad \text{or} \quad \phi^*(\omega, \mathbf{q})$$

- Parameters:
  - (background scattering in open channel) λ_ψ
  - Feshbach coupling: width of resonance h
  - detuning: distance from resonance ν

Relation to a strongly interacting theory

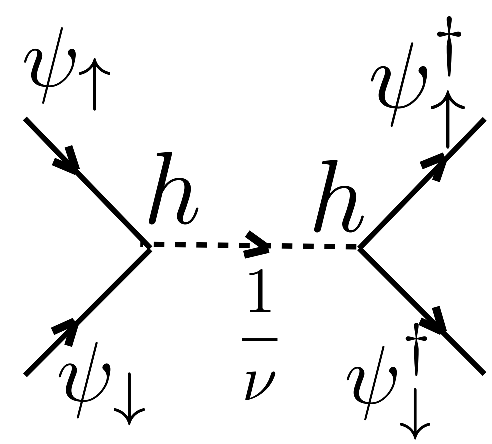
- Total action: $S = S_\psi + S_\phi + S_F[\psi, \phi]$
- Field equations:

$$\frac{\delta S}{\delta \phi^*} = 0 \qquad \Rightarrow (\partial_t - \tfrac{\triangle}{4M} + \nu)\phi = h\psi_\uparrow\psi_\downarrow$$

$$\Rightarrow \phi = \frac{h}{\partial_t - \frac{\triangle}{4M} + \nu}\psi_\uparrow\psi_\downarrow$$

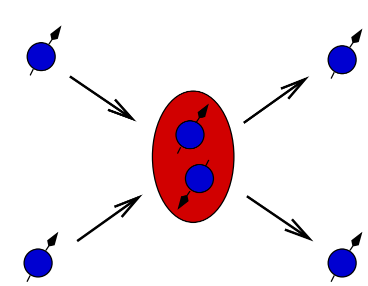
- Formally solve for ϕ, ϕ^*, and insert solution into the Feshbach term

$$S = S_\psi + \int d\tau d^3x\, \psi_\uparrow\psi_\downarrow \frac{h^2}{\cancel{\partial_t - \frac{\triangle}{2M}} + \nu} \psi_\uparrow^\dagger \psi_\downarrow^\dagger$$



- take constrained "broad resonance" limit:
  pointlike interactions

$$h \to \infty, \qquad \frac{h^2}{\nu} \to \text{const.}$$

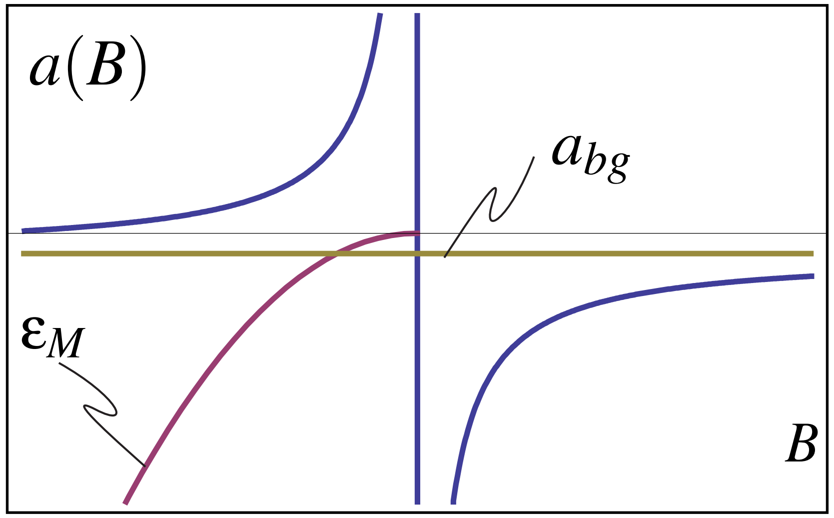


$$S \to S_\psi + \frac{h^2}{\nu} \int d\tau d^3x\, \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\uparrow \psi_\downarrow = S_\psi - \frac{1}{2}\frac{h^2}{\nu}\int d\tau d^3x\, (\psi^\dagger\psi)^2$$

Relation to a strongly interacting theory

- pointlike/broad resonance limit: The action takes the form

$$S[\psi] = \int d\tau d^d x \big(\psi^\dagger(\partial_t - \tfrac{\triangle}{2M})\psi + \tfrac{\lambda_\psi^{\rm eff}}{2}(\psi^\dagger\psi)^2\big)$$



scattering length a and binding energy

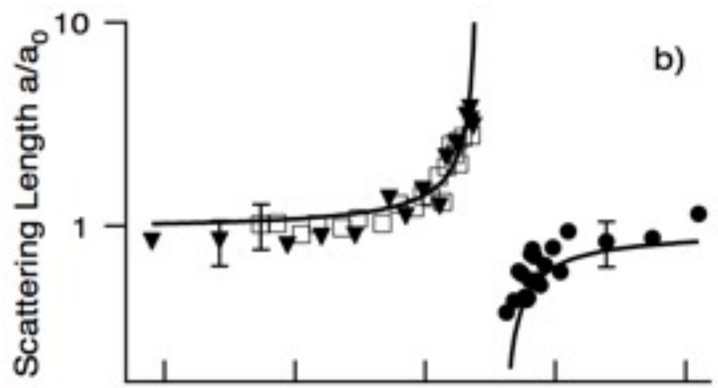
$$\lambda_\psi^{\rm eff} = \lambda_\psi - \frac{h^2}{\nu(B)}$$

effective fermionic interaction

$$a = \frac{4\pi\lambda_\psi^{\rm eff}}{M}$$

scattering length (nonidentical fermions)

observation of divergent scattering length
Ketterle Group, MIT (1999)
bosonic sodium



- remember $\nu(B) = \mu_B(B - B_0)$

  ➡ resonant (divergent) interaction at B_0

- in the following, we shall ignore the background scattering for simplicity

Regimes in the BCS-BEC Crossover

- Compare the scattering length to the mean interparticle spacing $d = (3\pi^2 n)^{-1/3}$

  ➡ three regimes

$a < 0,\ |a/d| \ll 1$ weakly interacting (dilute) fermions

$|a/d| \gtrsim 1$ strong interactions, dense

$a > 0,\ |a/d| \ll 1$ molecular bound states: dilute bosons

➡ see below!

- We identify the inverse scattering length as an adequate "crossover parameter"

$$a^{-1}(B) = -\frac{M\nu(B)}{4\pi h^2}$$

since the Feshbach resonance is located at the zero crossing of the detuning $\nu(B)$

- Cf. microscopic justification: a/d > 1 does not invalidate the microscopic Hamiltonian (as could be suspected from the discussion of weakly interacting gases). The relevant ratio for the validity is $r_{vdW}/d, r_{vdW}/\lambda_{dB} \ll 1$. Feshbach resonances violate the generic relation $r_{vdW}/a \approx 1$: "anomalously large scattering length"

Evaluation Strategy: Effective Action

- So far: microscopic physics of few scattering particles
- Quantize the many-body theory via functional integral for the effective action
  - ➡ Consider S, the classical action:

$$S[\chi] \qquad\qquad \frac{\delta S[\chi]}{\delta \chi} = 0$$

  - ➡ The effective action is given by

$$\Gamma[\chi] = -\log \int \mathcal{D}\delta\chi \exp -S[\chi + \delta\chi] \qquad\qquad \frac{\delta \Gamma[\chi]}{\delta \chi} = 0$$

- Result:
  - Effective action includes all quantum and statistical fluctuations by integrating over all possible paths
  - Leverages over the intuition from classical physics:
    - symmetries
    - action principle, field equation
  - Lends itself for powerful modern field theory techniques

Integrating out the fermions

- For practical purposes, the Feshbach type action is favorable:
  - the potential bosonic bound state is already explicit
  - NB: can be seen as result of Hubbard-Stratonovich transformation of fermionic theory
  - for simplicity restrict to broad resonance limit, no background coupling

- This theory is quadratic in the fermions

$$S[\psi] = \int d\tau d^3x \left(\psi^\dagger(\partial_t - \frac{\triangle}{2M} - \mu)\psi + (\nu - 2\mu)\phi^*\phi - h(\phi^*\psi_\uparrow\psi_\downarrow - \phi\psi_\uparrow^*\psi_\downarrow^*\right)$$

$$= \frac{1}{2}\int_Q (\Psi^T, \Phi^T) \begin{pmatrix} S_{FF}^{(2)} & 0 \\ 0 & S_{BB}^{(2)} \end{pmatrix} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}$$

Nambu-Gorkov fields

$$\Psi = \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \qquad \Phi = \begin{pmatrix} \phi \\ \phi^* \end{pmatrix}$$

Fourier transform

spin indices

$$S_{FF}^{(2)}[\phi] = \begin{pmatrix} -\epsilon_{\alpha\beta} h\phi^* & -P_F(-Q)\delta_{\alpha\beta} \\ P_F(Q)\delta_{\alpha\beta} & \epsilon_{\alpha\beta} h\phi \end{pmatrix} \qquad S_{BB}^{(2)} = \begin{pmatrix} 0 & \nu - 2\mu \\ \nu - 2\mu & 0 \end{pmatrix}$$

$$P_F(Q) = \mathrm{i}\,\omega + \frac{q^2}{2M} - \mu$$

- Inverse propagator matrix depends on the fluctuating bosonic field

boson carries two atoms

- fermions can be integrated out
- the result is a (nontrivial) purely bosonic theory, to be analyzed subsequently

Gaussian Integral for Fermions

- Effective Action:

$\Psi_0 = 0$ (Pauli principle)

$$\Gamma[\Psi_0, \Phi_0] = -\log \int \mathcal{D}\delta\Psi \mathcal{D}\delta\Phi \exp -S[\Psi_0 + \delta\Psi, \Phi_0 + \delta\Phi]$$

$$= -\log \int \mathcal{D}\delta\Phi \exp -S_{\rm int}[\Phi_0 + \delta\Phi]$$

composite field expectation value (condensation for $\Phi_0 \neq 0$)

with intermediate action

$$S_{\rm int}[\Phi] = S_\phi^{(cl)}[\Phi] - \frac{1}{2}\log\det S_{FF}^{(2)}[\Phi] = S_\phi^{(cl)}[\Phi] - \frac{1}{2}\mathrm{Tr}\log S_{FF}^{(2)}[\Phi].$$

$$\int (\nu - 2\mu)\phi^*\phi$$

- The Tr Log term features arbitrary powers of Φ (compatible with symmetries)

➡ This gives rise to an effective bosonic theory

Effective Quadratic Theory for Bosons

- Integrating out the fermions yields bosonic theory: Interpretation?
- Progress can be made by expansion in fluctuation field powers:

$$-\frac{1}{2}\mathrm{Tr}\log\left(\mathcal{P}_F+\mathcal{F}\right) = -\frac{1}{2}\mathrm{Tr}\log\mathcal{P}_F - \frac{1}{2}\mathrm{Tr}\mathcal{P}_F^{-1}\mathcal{F} + \frac{1}{4}\mathrm{Tr}(\mathcal{P}_F^{-1}\mathcal{F})^2 + \ldots$$

$$= S_{FF}^{(2)}[\Phi].$$

$\mathcal{O}(\delta\Phi)$ $\quad$ $\mathcal{O}(\delta\Phi^2)$

vanishes due to field equation

- $\mathcal{P}_F$ depends on the condensate $\quad \Phi_0$
- $\mathcal{F}$ is the “fluctuation matrix” $\quad \sim \delta\Phi$

- Good qualitative understanding at arbitrary temperature: Truncate after quadratic term
- Yields effective Bogoliubov theory for boson fluctuations

$$S_{\mathrm{Bog}} = \frac{1}{4}\mathrm{Tr}(\mathcal{P}_F^{-1}\mathcal{F})^2 = \frac{1}{2}\int_K \left(\delta\phi(-K),\delta\phi^*(K)\right)\begin{pmatrix}\lambda_\phi(K)\phi_0^*\phi_0^* & P_\phi(K)\\ P_\phi(-K) & \lambda_\phi(K)\phi_0\phi_0\end{pmatrix}\begin{pmatrix}\delta\phi(K)\\ \delta\phi^*(-K)\end{pmatrix}$$

$K = (\omega, \mathbf{k})$

frequency and momentum

- The coefficients are given by the classical + fluctuation contributions, e.g.

$$P_\phi(K) = \nu - h^2\int_Q \frac{P_F(-Q-K)P_F(Q)}{P_F^{|2|}(Q)P_F^{|2|}(Q+K)} = \mathrm{i}Z_\phi\omega + \frac{A_\phi}{4M}\mathbf{k}^2 + m_\phi^2 + \ldots$$

“mass term”

low energy/derivative expansion

Crossover Effective Potential I

- The steps in summary (and a last one...): Effective Potential

integration of fermion fluctuations

$$\Gamma[\Phi_0] = -\log\int\mathcal{D}\delta\Phi\,\exp - S_{\rm int}[\Phi_0+\delta\Phi]$$

quadratic expansion of boson fluctuations

$$= \int_Q \big((\nu-2\mu)\phi_0^*\phi_0 - \tfrac{1}{2}\mathrm{Tr}\log\mathcal{P}_F[\phi_0^*\phi_0]\big) - \log\int\mathcal{D}\delta\Phi\,\exp - S_{\rm Bog}[\delta\Phi]$$

integration of quadratic boson fluctuations

$$= V/T\big((-2\mu+\nu)\phi_0^*\phi_0 - \tfrac{1}{2}\mathrm{Tr}\log\mathcal{P}_F[\phi_0^*\phi_0] + \tfrac{1}{2}\mathrm{Tr}\log\mathcal{P}_B[\phi_0^*\phi_0]\big) \equiv V/T\;\mathcal{U}[\phi^*\phi]$$

"classical contribution": condensed bosons

fermionic fluctuations

bosonic fluctuations (approximation)

effective potential: homogeneous contribution to effective action

important in regime: BEC | BCS | BEC

Crossover Effective Potential II

- The Crossover Physics can now be extracted from the Effective Potential

$$\mathcal{U}[\rho;\mu] = (-2\mu+\nu)\rho - \tfrac{1}{2}\mathrm{Tr}\log\mathcal{P}_F[\rho;\mu] + \tfrac{1}{2}\mathrm{Tr}\log\mathcal{P}_B[\rho;\mu] \qquad \rho = \phi_0^*\phi_0$$

perform spin and frequency traces

$$= (\nu-2\mu)\rho - 2T\int\frac{d^3q}{(2\pi)^3}\log\cosh(E_{\mathbf{q}}^{(\mathrm{F})}/2T) + T\int\frac{d^3q}{(2\pi)^3}\log\sinh(E_{\mathbf{q}}^{(\mathrm{B})}/2T)$$

$$E_{\mathbf{q}}^{(\mathrm{F})} = [(\tfrac{q^2}{2M} - \mu)^2 + h^2\rho]^{1/2}$$

- single fermion excitation energies, coefficients from fermionic action

$$E_{\mathbf{q}}^{(\mathrm{B})} = [(A_\phi q^2 + m_\phi^2)^2 + 2\lambda_\phi\rho(A_\phi q^2 + m_\phi^2)]^{1/2}$$

- effective boson excitation energies
- coefficients: integrals from TrLog + derivative expansion
- mu-dependence: mainly $m_\phi^2(\mu)$

- The effective potential depends on μ and ρ
- These quantities will be determined by:
  - The gap equation
  - The equation of state

Self-consistency relations for the solution of the crossover thermodynamics

The Gap Equation and Spontaneous Symmetry Breaking

- field equation for the effective action: $$\frac{\delta\Gamma[\phi_0(x)^*,\phi_0(x)]}{\delta\phi_0(y)} = 0$$

- simplifications:
  - the effective potential depends only on homogeneous field configuration
  - due to U(1) (global phase) symmetry, it only depends on the invariant $\rho = \phi_0^*\phi_0$

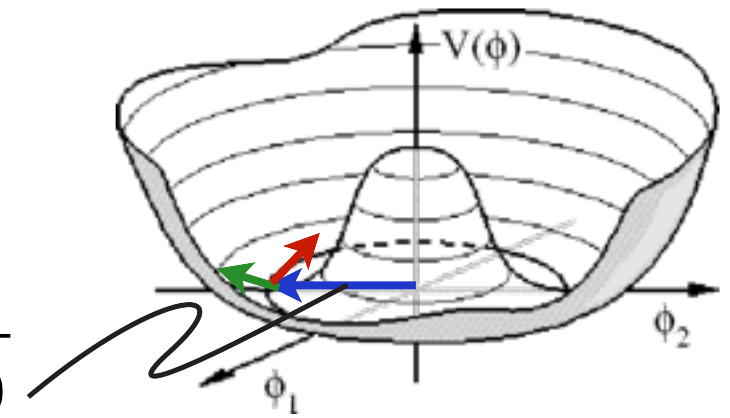
$$\frac{\partial\mathcal{U}[\phi_0^*\phi_0]}{\partial\phi_0} = \frac{\partial\mathcal{U}[\rho]}{\partial\rho}\phi_0^* = 0$$

- if the symmetry is broken spontaneously, $\phi_0^* \neq 0$, then we must have

$$\frac{\partial\mathcal{U}[\rho]}{\partial\rho} = 0$$

gap equation (cf. fully analogous treatment in BCS theory)

$\sqrt{\rho_0}$



- NB: This criterion based on symmetry breaking works throughout the whole crossover. At T=0, the symmetry will be broken for any value of scattering length. Therefore, there is no quantum phase transition, but only a crossover phenomenon

The Equation of State

- We would like to consider a situation with fixed density
- Our problem is formulated in the grand canonical ensemble, $S \ni \int dt d^3x (-\mu \psi^\dagger \psi)$

chemical potential

- The density can be obtained from the effective potential:

$$N/V = \lim_{\mathbf{x}\to 0} \langle \delta\psi^\dagger(\mathbf{x}) \delta\psi(0) \rangle = \lim_{\mathbf{x}\to 0} \int \mathcal{D}\delta\psi [\delta\psi^\dagger(\mathbf{x})\delta\psi(0)] \exp - S[\psi_0 + \delta\psi]/\mathcal{N}$$

$$= -\frac{\partial \mathcal{U}}{\partial \mu}$$

- In our approximation, there are three additive contributions to U. Thus

$$n = 2\phi_0^*\phi_0 + \tfrac{1}{2}\mathrm{Tr}\mathcal{P}_F^{-1} + \mathrm{Tr}\tfrac{1}{2}\mathcal{P}_B^{-1}$$

$$= 2\phi_0^*\phi_0 + n_F + n_B$$

effective bosonic propagator

classical bosonic condensate contribution

fermionic (quasiparticle) contribution

~ integral over Fermi distribution

contribution of fermions bound in bosons

~ integral over Bose distribution

in the presence of condensates

➡ definite interpretations only possible in the BCS and BEC limits

The Extended Mean Field Theory of the BCS-BEC Crossover

- The two self-consistency equations from the Effective Potential read
  - The Gap equation

$$0 = \frac{\partial \mathcal{U}}{\partial \rho} = \nu - 2\mu - \frac{h^2}{2} \int \frac{d^3q}{(2\pi)^3} \left[\frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}} \tanh \frac{E_{\mathbf{q}}^{(\mathrm{F})}}{2T} - \frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}} \Big|_{\phi=\mu=0} \right]$$

NB: Bosonic contribution neglected for simplicity

UV Renormalization: in complete analogy to BCS theory

  - The Equation of State

$$n = -\frac{\partial \mathcal{U}}{\partial \mu} = 2\phi_0^*\phi_0 + n_F + n_B$$

- Solve for μ and ρ

$$a^{-1} = -\frac{M}{4\pi}\frac{\nu}{h^2}$$

- Plot as a function of dimensionless crossover parameter

$$n = \frac{k_F^3}{3\pi^2}$$

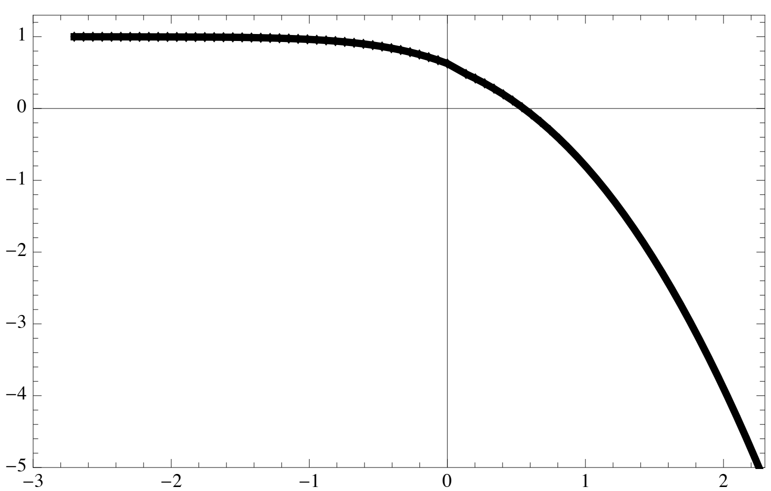
Fermi momentum

$$(a/d)^{-1} = (ak_F)^{-1}$$

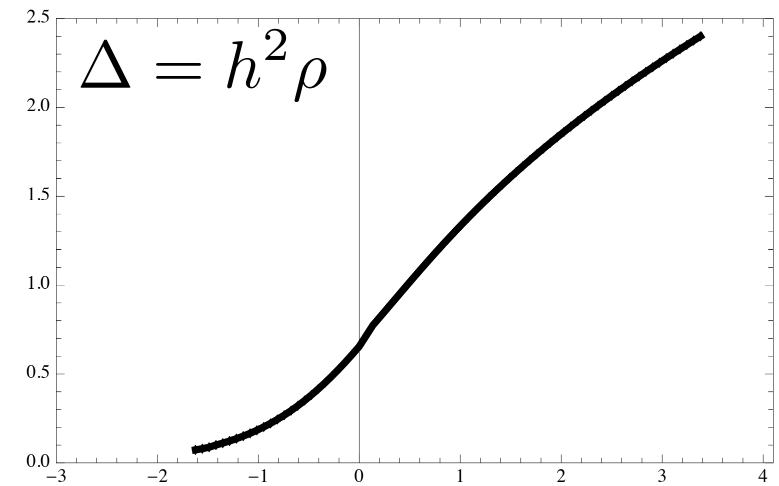
$$\frac{\mu}{\epsilon_F}$$

$$\epsilon_F = \frac{k_F^2}{2M}$$

Fermi energy



0 $(ak_F)^{-1}$



0 $(ak_F)^{-1}$

➡ What do these solutions tell us?

The Extended Mean Field Theory of the BCS-BEC Crossover

- The two self-consistency equations from the Effective Potential read
  - The Gap equation

$$0 = \frac{\partial \mathcal{U}}{\partial \rho} = \nu - 2\mu - \frac{h^2}{2} \int \frac{d^3 q}{(2\pi)^3} \Big[\frac{1}{E^{(\mathrm{F})}_{\mathbf{q}}} \tanh \frac{E^{(\mathrm{F})}_{\mathbf{q}}}{2T} - \frac{1}{E^{(\mathrm{F})}_{\mathbf{q}}} \Big|_{\phi=\mu=0} \Big]$$

NB: Bosonic contribution neglected for simplicity

  - The Equation of State

$$n = -\frac{\partial \mathcal{U}}{\partial \mu} = 2\phi_0^* \phi_0 + n_F + n_B$$

UV Renormalization: in complete analogy to BCS theory

- Solve for μ and ρ

$$a^{-1} = -\frac{M}{4\pi} \frac{\nu}{h^2}$$

- Plot as a function of dimensionless crossover parameter

$$(a/d)^{-1} = (a k_F)^{-1}$$

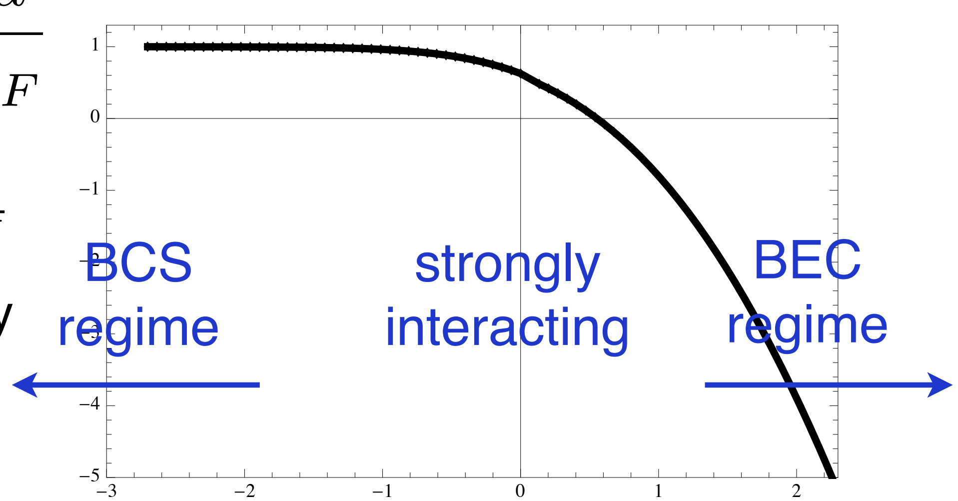
$$n = \frac{k_F^3}{3\pi^2}$$

Fermi momentum

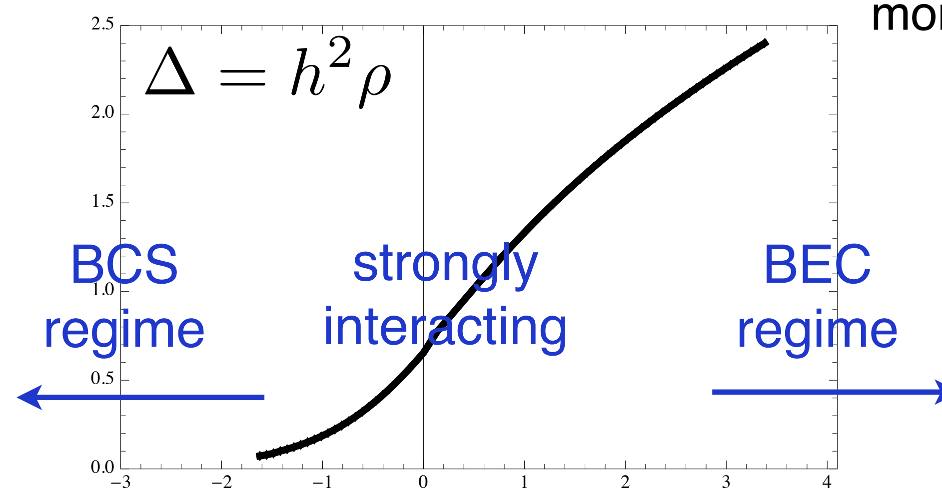
$$\frac{\mu}{\epsilon_F}$$

$$\epsilon_F = \frac{k_F^2}{2M}$$

Fermi energy



0 $(a k_F)^{-1}$



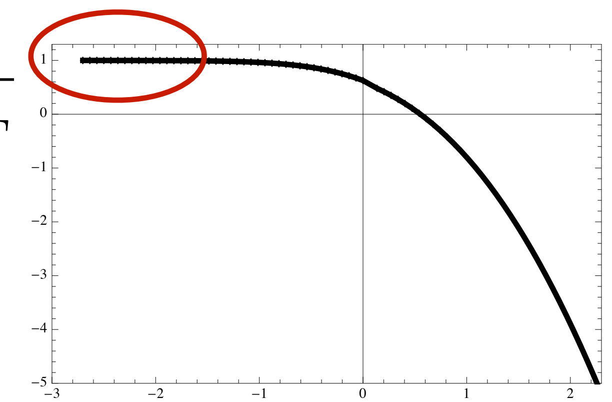
0 $(a k_F)^{-1}$

➡ Discuss the limiting cases!

The limiting cases: BCS limit

- Solution above: $\frac{\mu}{\epsilon_F} \to 1$
  - ➡ Expression of a Fermi surface, weakly interacting fermion gas is approached

BCS regime



- Simplifications
  - ➡ The EoS reduces to

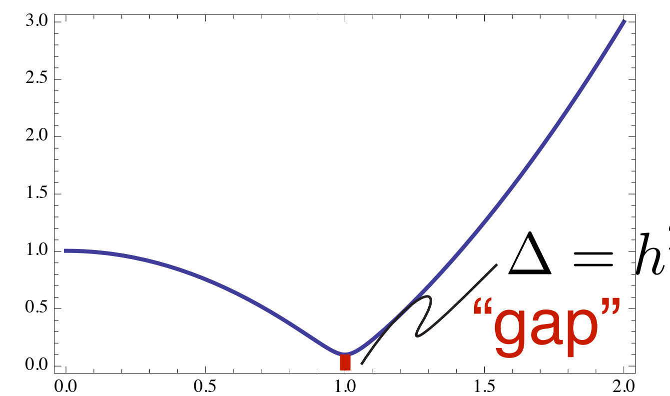
$$n = 2\phi_0^*\phi_0 + n_F + n_B \to n_F$$

  - ➡ The gap equation can be solved analytically

$$0 = \frac{\nu - 2\mu}{h^2} - \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\Big[\frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\tanh\frac{E_{\mathbf{q}}^{(\mathrm{F})}}{2T} - \frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\Big|_{\phi=\mu=0}\Big]$$

single fermion excitation spectrum

$$E_{\mathbf{q}}^{(\mathrm{F})} = \big[(\tfrac{\mathbf{q}^2}{2M} - \mu)^2 + h^2\rho\big]^{1/2}$$



  - ➡ NB: For broad resonances: $h \to \infty, \ \nu \propto h^2$

$$\frac{\nu - 2\mu}{h^2} \to \frac{\nu}{h^2} = -\frac{4\pi}{M}a^{-1}$$

$$0 = -\frac{1}{a} - \frac{M}{8\pi}\int \frac{d^3q}{(2\pi)^3}\Big[\frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\tanh\frac{E_{\mathbf{q}}^{(\mathrm{F})}}{2T} - \frac{1}{E_{\mathbf{q}}^{(\mathrm{F})}}\Big|_{\phi=\mu=0}\Big]$$

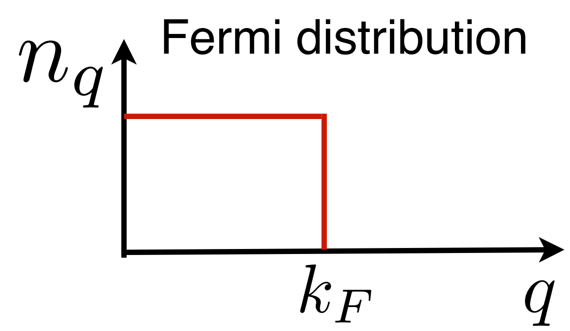
cf. above: BCS Gap equation reproduced

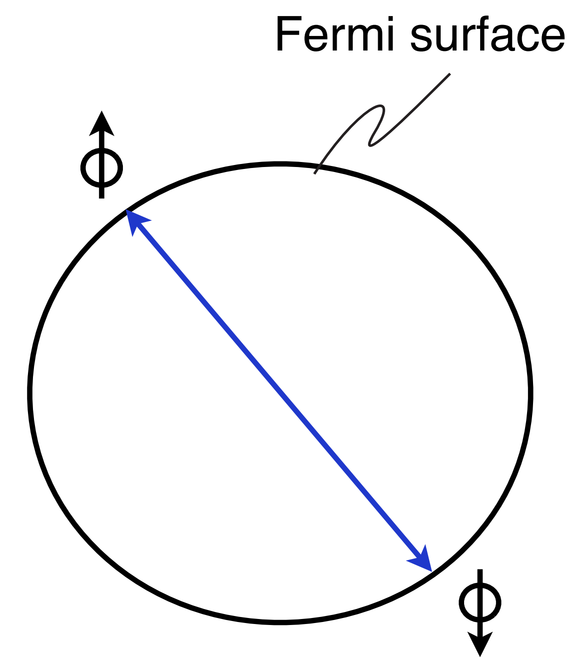
The limiting cases: BCS limit

- Interpretations:
  - Strongly expressed Fermi surface
    - ➡ Scattering/Pairing highly local in momentum space
  - The result for the gap:

$$\Delta = 0.61\epsilon_F \, e^{-\frac{\pi}{2ak_F}}$$

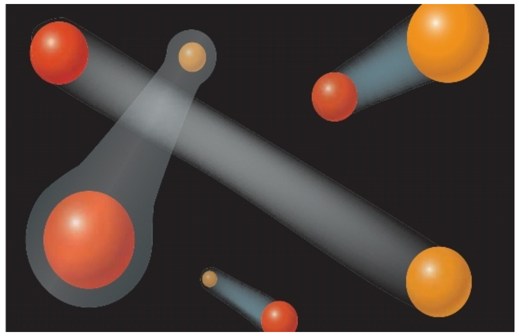
➡ Condensation is very weakly expressed: only Fermions close to Fermi surface contribute

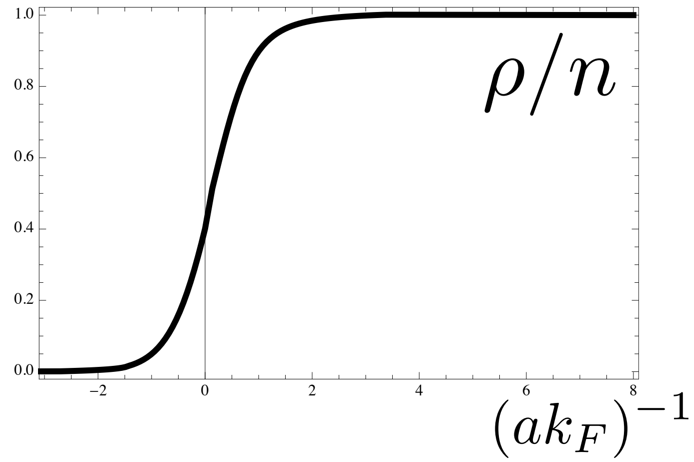




Locality in momentum space

Delocalization in position space

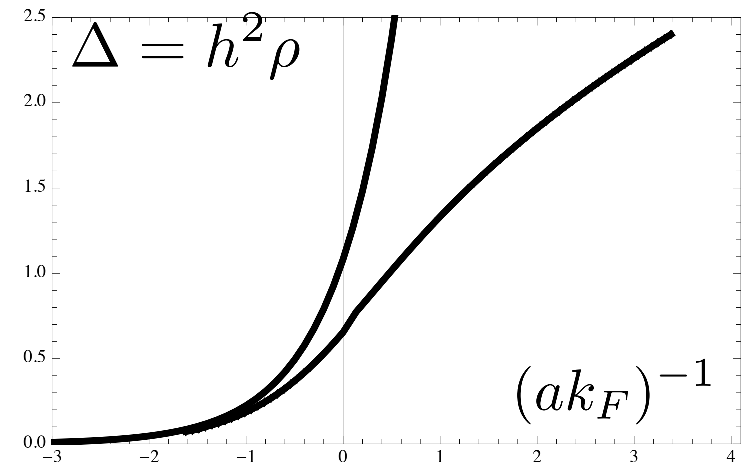




slight cheating here, this plots the renormalized condensate (below)

- Comparison to full result
  - ➡ Strong deviations from BCS result once

$$(ak_F)^{-1} \sim -1$$



The limiting cases: BEC limit

- Solution above: $\dfrac{\mu}{\epsilon_F} \to -\infty$

➡ Strong gap $-\mu$ develops on the normal $(\psi^\dagger\psi)$ sector of the inverse fermion propagator

➡ However, there is a piece from the anomalous part that is independent of $-\mu$

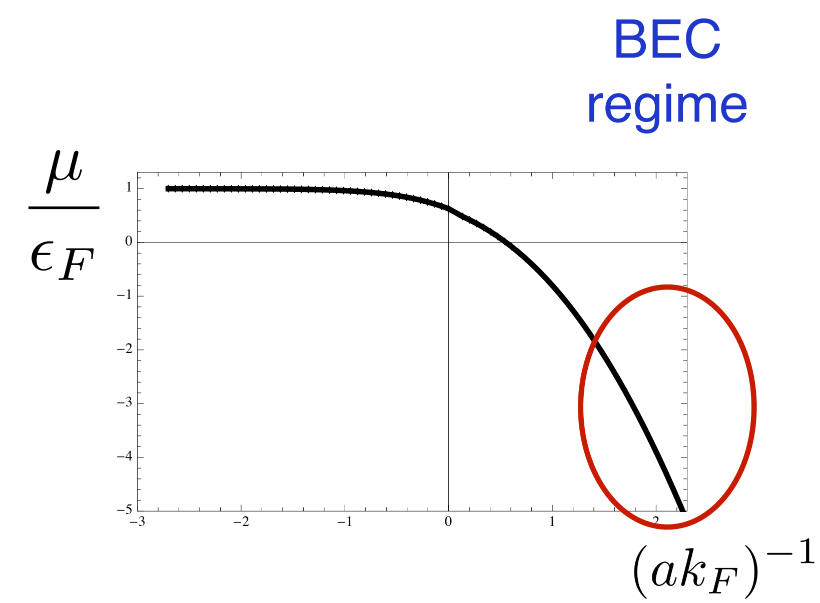
➡ The fermion density can be written

$\psi\psi$

$$n_F = \tfrac{1}{2}\mathrm{Tr}\,\mathcal{P}_F^{-1} \to -\frac{\partial \Delta m_\phi^2}{\partial \mu}\,\phi_0^*\phi_0$$

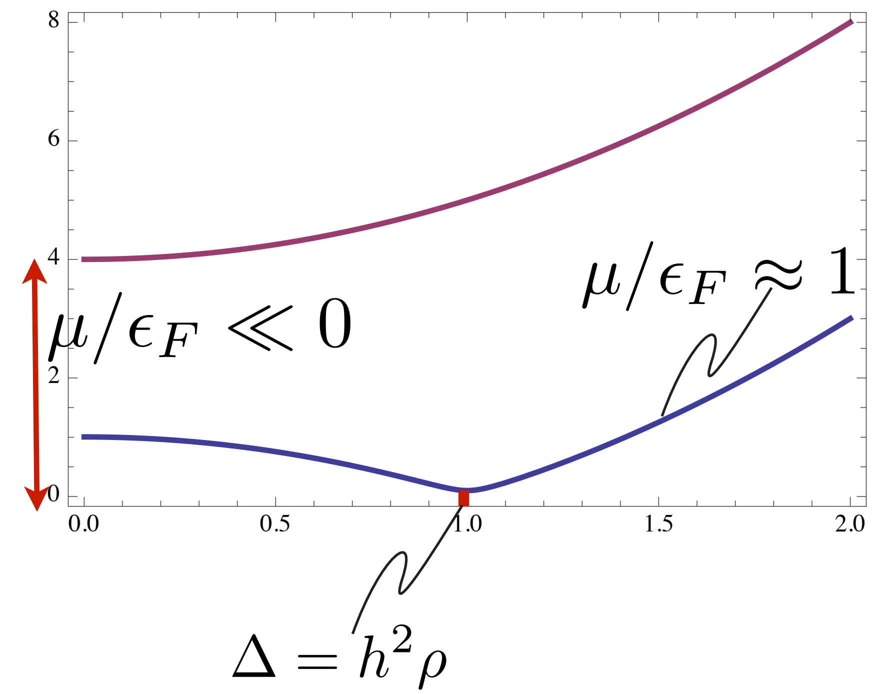
Δm_ϕ^2 is identified as the fluctuation correction to the bosonic mass term

➡ Interpretation?



single Fermion excitation spectrum

$$E_{\mathbf{q}}^{(\mathrm{F})} = \left[\left(\frac{\mathbf{q}^2}{2M} - \mu\right)^2 + h^2\rho\right]^{1/2}$$



Renormalized Condensate

- We consider the equation of state in the BEC regime:

total mass term:
classical + fluctuations

$$n = 2\phi_0^*\phi_0 + n_F + n_B \to (2 - \frac{\partial \Delta m_\phi^2}{\partial \mu})\phi_0^*\phi_0 + n_B = -\frac{\partial m_\phi^2}{\partial \mu}\phi_0^*\phi_0 + n_B$$

- We consider the bosonic density further (T=0):

Bosonic propagator

$$n_B = -\frac{\partial}{\partial\mu}\frac{1}{2}\mathrm{Tr}\log\mathcal{P}_B[\rho;\mu] \approx -\frac{\partial m_\phi^2}{\partial \mu}\mathrm{Tr}\log\mathcal{P}_B^{-1}[\rho;\mu]$$

- For $\mu \to -\infty$ the bosonic propagator matrix simplifies to

$$\mathcal{P}_B = \begin{pmatrix} \lambda_\phi(K)\phi_0^*\phi_0^* & P_\phi(K) \\ P_\phi(-K) & \lambda_\phi(K)\phi_0\phi_0 \end{pmatrix} \to Z_\phi \begin{pmatrix} 2aZ_\phi\phi_0^*\phi_0^* & \mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2aZ_\phi\phi_0^*\phi_0 \\ -\mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2aZ_\phi\phi_0^*\phi_0 & 2aZ_\phi\phi_0\phi_0 \end{pmatrix}$$

- We observe that the expression

$$Z_\phi := -\frac{1}{2}\frac{\partial m_\phi^2}{\partial \mu}$$

is ubiquitous to all the formulas here

- We introduce a renormalized condensate density as $\boxed{\tilde{\phi}^*\tilde{\phi} = Z_\phi\phi_0^*\phi_0^*}$

The limiting cases: BEC limit

- Expressing all quantities in terms of the renormalized condensate gives:
  - Equation of state

$$\boxed{n = 2\tilde{\phi}_0^* \tilde{\phi}_0 + 2\mathrm{Tr}\tilde{\mathcal{P}}_B^{-1} = 2\tilde{\phi}_0^* \tilde{\phi}_0 + 2\tilde{n}_B}$$

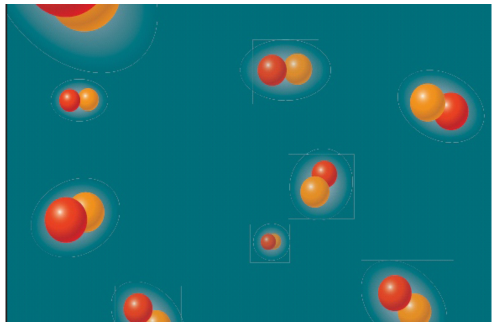
  - with renormalized inverse boson propagator

$$\tilde{\mathcal{P}}_B = \mathcal{P}_B / Z_\phi = \begin{pmatrix} 2a\tilde{\phi}_0^* \tilde{\phi}_0^* & \mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2a\tilde{\phi}_0^* \tilde{\phi}_0 \\ -\mathrm{i}\omega + \frac{\mathbf{q}^2}{4M} + 2a\tilde{\phi}_0^* \tilde{\phi}_0 & 2a\tilde{\phi}_0 \tilde{\phi}_0 \end{pmatrix}$$

➡ Reduction to an effective theory of “renormalized” bosonic bound states

- Mass 2M
- Interaction strength 2a
- Atom number 2

Local objects in position space



- All reference to the concrete value of Z is gone in the renormalized quantities
- Macroscopic measurements probe the renormalized quantities
- Microscopic probes can measure Z, but this is beyond the scope here
- NB: While boson mass and atom number follow from symmetry, the interaction strength 2a is an approximation. The exact answer is 0.6a (four-body problem)

Connection to Scattering Physics: Vacuum limit

- Heuristics: Study the gap equation (broad resonance) for $\frac{\mu}{\epsilon_F} \to -\infty$

$$\frac{1}{a} = \sqrt{-\mu \cdot 2M}$$

   - ➡ The density scale k_F (also: temperature) have disappeared from the gap equation:
   - ➡ only two-body physics left!
   - ➡ Very different from BCS limit, cf. $\frac{\Delta}{\epsilon_F} = 0.61\, e^{-\frac{\pi}{2ak_F}}$

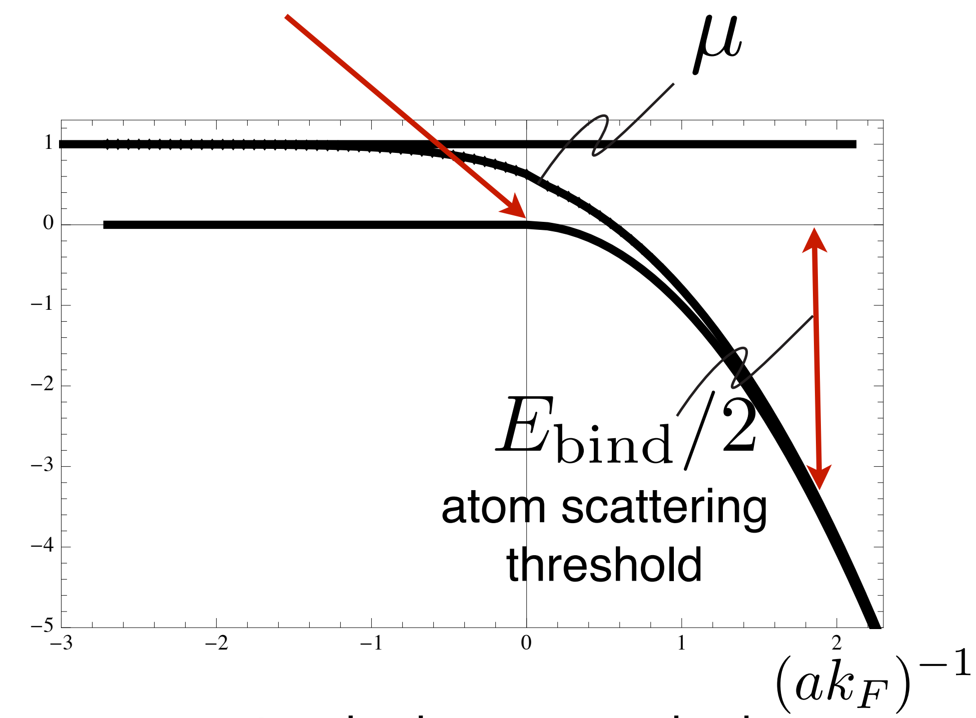
Connection to Scattering Physics: Vacuum limit

- More systematically: Project on physical vacuum by

$$\Gamma_{k\to 0}(vak) = \lim_{k_F\to 0} \Gamma_{k\to 0}\big|_{T/\varepsilon_F > T_c/\varepsilon_F = \text{const.}}$$

- Diluting procedure: $d \sim k_F^{-1} \to \infty$
- Getting cold: $T \sim \varepsilon_F$

$$n = \frac{k_F^3}{3\pi^2}$$



two-body vs. many-body

- The chemical potential plays the role of half the binding energy

$$\boxed{E_{\text{bind}} = -1/(Ma^2)}$$

- Smooth crossover terminates in sharp "second order phase transition" in vacuum

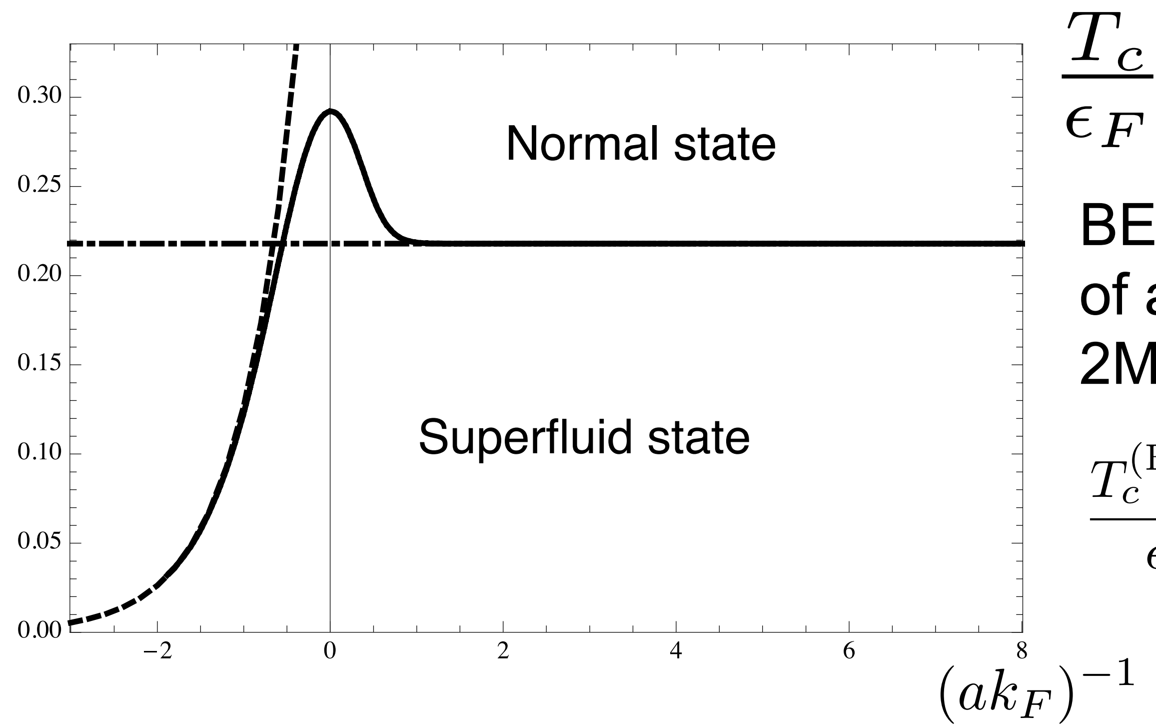
- NB: Using this vacuum procedure, the above functional integral treatment solves the two-body scattering problem -- as quantum mechanics would do

➡ microscopic origin of the crossover is bound state formation

Finite Temperatures

- So far: Crossover Physics at T=0
- The above formalism is readily applied to finite temperature: Frequency integration -> Matsubara sum
- Finite temperature phase diagram:

BCS limit:
BCS theory



BEC limit: Free bosons of atom number 2, mass 2M

$$\frac{T_c^{(\mathrm{BEC})}}{\epsilon_\mathrm{F}} = 2\pi(6\pi^2\zeta(3/2))^{-2/3} \approx 0.218$$

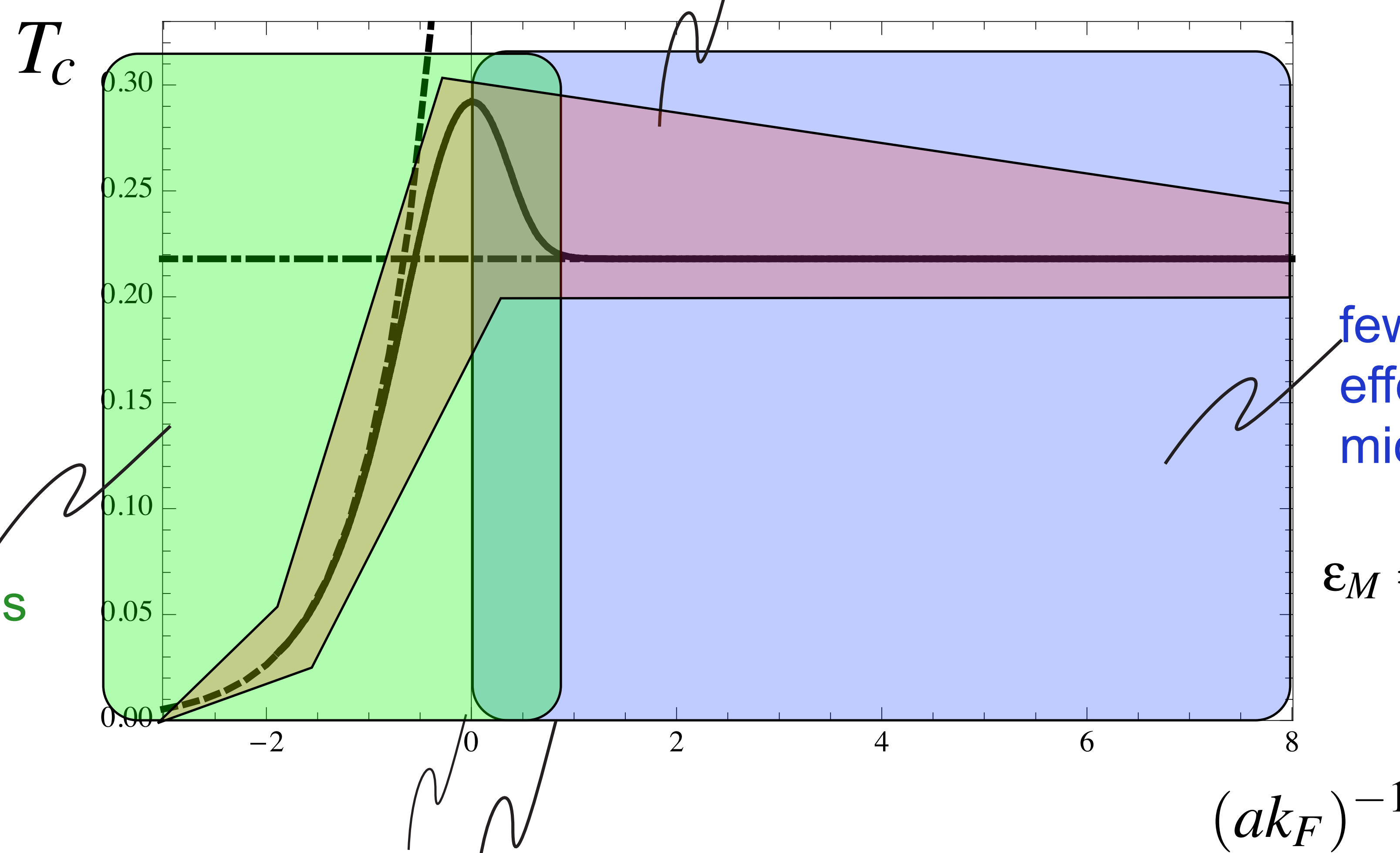
Challenges beyond Mean Field

Beyond mean field effects and challenges
at very different scales:

critical behavior:
long distance scales $k_{ld} \gg n^{1/3}, T^{1/2}, \epsilon_M^{1/2}$

Many-Body fermion physics:
Thermodynamic scales

$$n = \frac{k_F^3}{3\pi^2}, T$$



few-body physics of effective dimers:
microscopic scales

$$\epsilon_M = -\frac{1}{Ma^2} \gg T, \frac{k_F^2}{2M}$$

two-body bound state

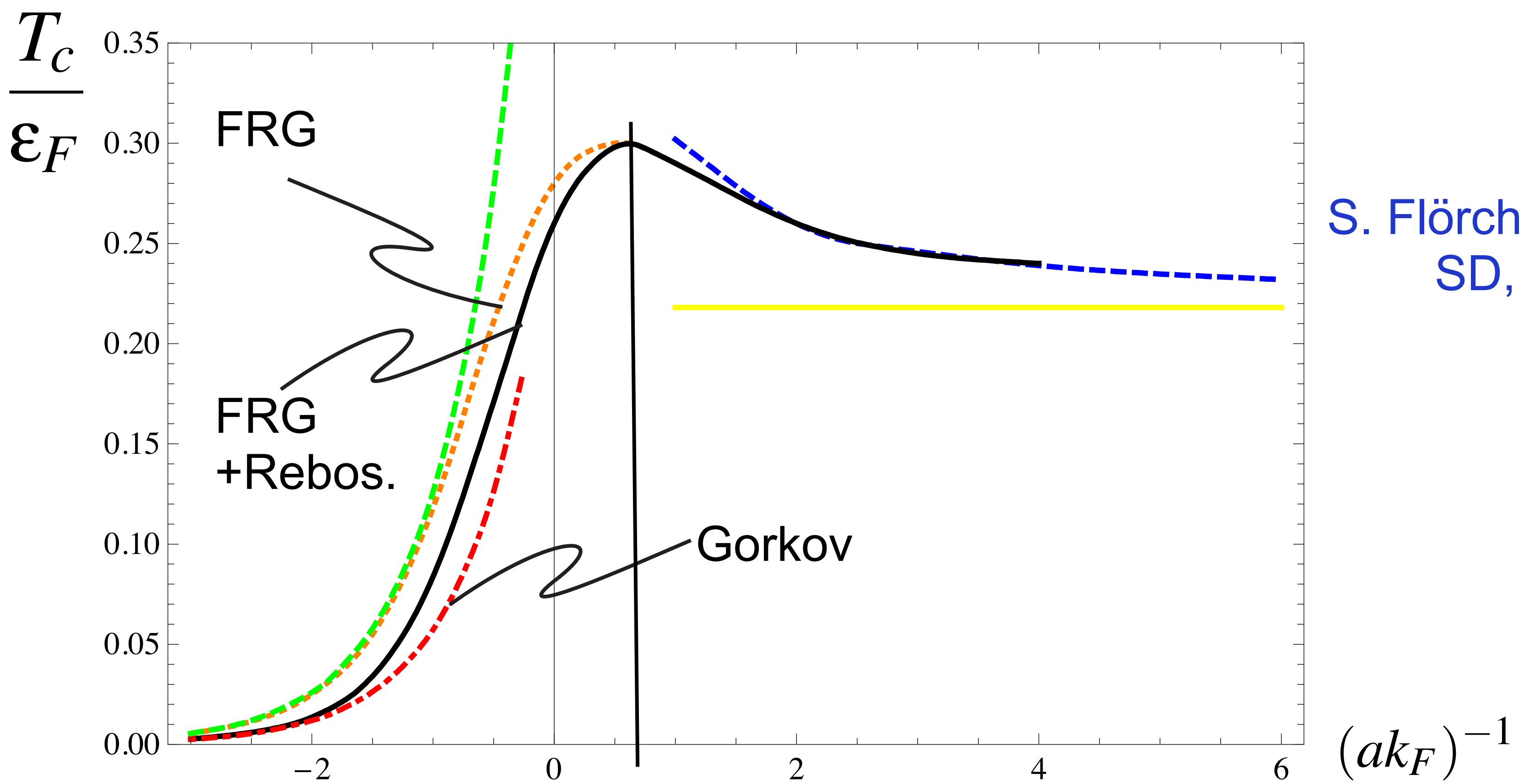
zero crossing of fermion chemical potential

Strategy: Find an interpolation scheme which incorporates known physical effects in the limiting cases
Methods: t-matrix approaches, 2PI Effective Action, Functional RG, ...

Results beyond Mean Field

- Functional RG treatment: connecting micro- and macrophysics, unified treatment of physics on various scales

microscopic	thermodynamic	long distance
$\varepsilon_M = -\frac{1}{Ma^2}$	$n = \frac{k_F^3}{3\pi^2}, T$	$k_{ld} \gg n^{1/3}, T^{1/2}, \varepsilon_M^{1/2}$

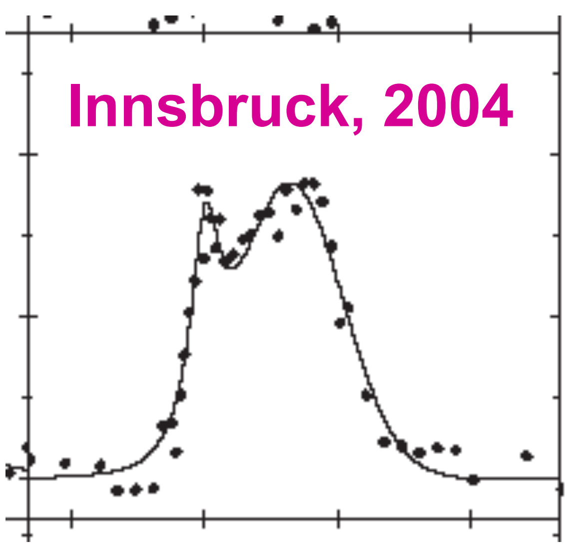


zero crossing of fermion chemical potential

S. Flörchinger, M. Scherer, SD, C. Wetterich

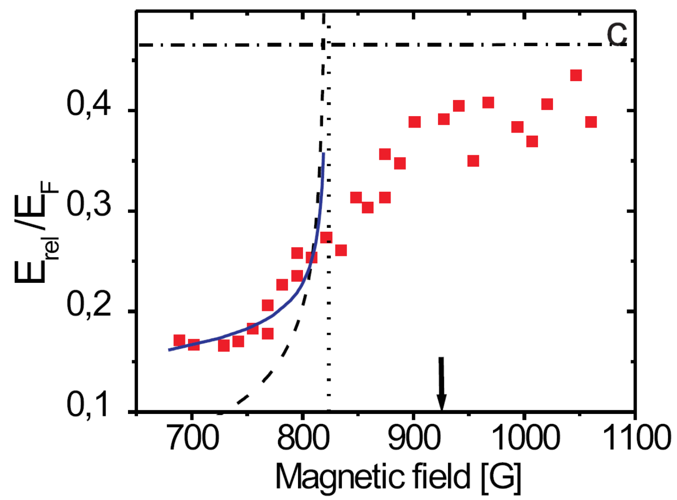
- Accurate treatment of molecular scattering physics in BEC regime
- Accurately reproduce Gorkov effect in the BCS regime from rebosonization procedure
- Long wavelengths: second order phase transition

Experiments in the BCS-BEC Crossover



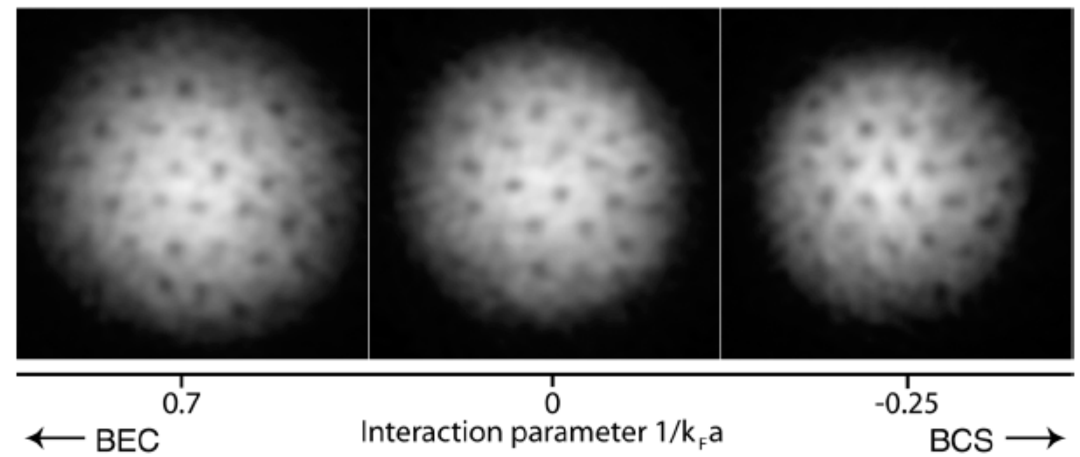
pairing gap

ENS, 2004

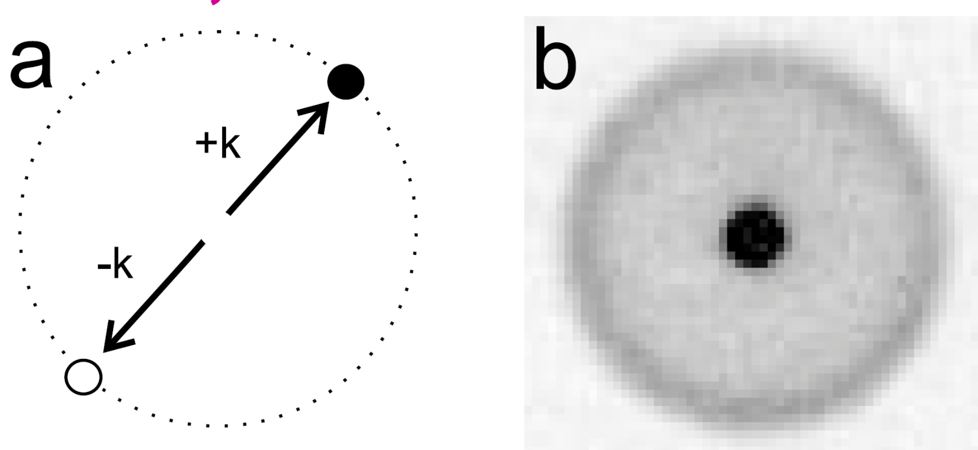


release energy

MIT, 2005 *vortices*

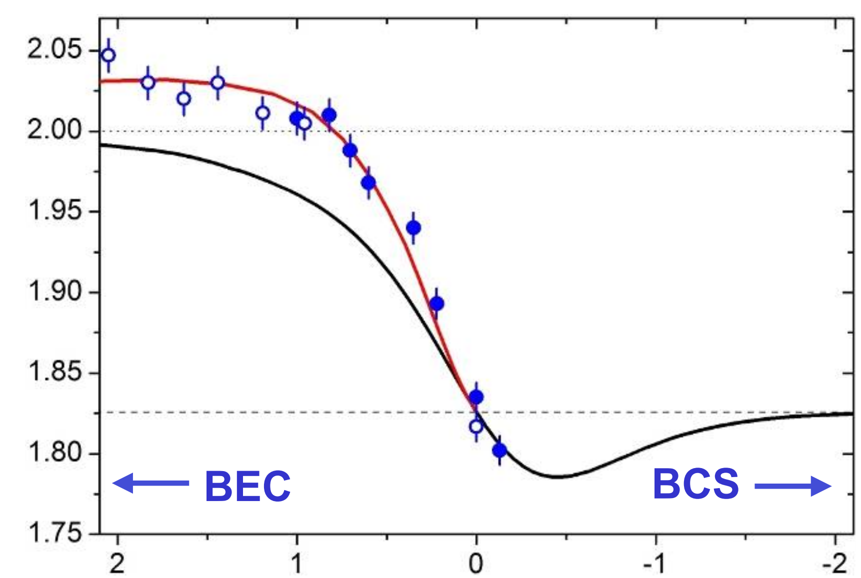


JILA, 2005



pair correlations

Duke ´04, Innsbruck 2004 & `06



collective modes

Summary: BCS-BEC Crossover

- We have discussed a simple approximation scheme based on the effective action capturing the qualitative features of the BCS-BEC crossover at zero and finite temperatures
  - The BCS-BEC crossover interpolates between the two cornerstones for quantum condensation phenomena, BCS and BEC type superfluidity
  - The microscopic origin is a Feshbach resonance. The divergence of the scattering length is accompanied by the formation of a bosonic bound state. There are three regimes for the many-body physics

$a < 0,\ \lvert a k_{\mathrm{F}}\rvert \ll 1$	weakly interacting (dilute) fermions
$\lvert a k_{\mathrm{F}}\rvert \gtrsim 1$	dense regime, strong interactions
$a > 0,\ \lvert a k_{\mathrm{F}}\rvert \ll 1$	molecular bound states: dilute bosons

  - The crossover from BCS to BEC regimes may be viewed as a localization process in real space, or a delocalization in momentum space
  - Though there are profound quantitiative changes in the thermodynamics, the ground state symmetry properties are unchanged: Crossover instead of quantum phase transition