The conserved energy-momentum current of matter as the basis for the gauge theory of gravitation

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Hundred Years of Gauge Theory, 678th Wilhelm and Else Heraeus Seminar, 30 July to 3 August 2018

I'd like to thank Claus Kiefer and Silvia De Bianchi for the invitation.

This repres. joint work with Yuri Obukhov, Yakov Itin, and Jens Boos.

• Blagojević & Hehl (eds.) Gauge Theories of Gravitation, "A Reader", Lond./Singap. (2013), for updates: Itin et al., PRD 95, 084020 (2017)

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1. Yang-Mills theory, gauge theory

- 100 years of gauge theory, a modern definition of gauge theory:

  A gauge theory is a heuristic scheme within the Minkowski space of special relativity (SR) for the purpose of deriving from a conserved current and the attached rigid ('global') symmetry group a new interaction. This new interaction is induced by demanding that the rigid symmetry should be extended to a locally valid symmetry.

- Yang-Mills (1954) and $SU(2)$ as example: *"Conservation of isotopic spin and isotopic gauge invariance."* Experimentally established is the charge independence of the nuclear interaction. This implies the conserved isotopic spin current:

- conserved isospin current $dI = 0 \stackrel{\text{recipr. Noether}}{\longrightarrow}$ rigid $SU(2)$ invariance
- rigid $SU(2)$ inv. $\stackrel{\text{heur. princ.}}{\longrightarrow}$ local $SU(2)$ inv. $\longrightarrow$ comp. (or gauge) pot. A

- Compensating field (or rather the potential) A. Matter fields Ψ, invariant under an internal symmetry described by a semi-simple Lie group with generators T_a, implying a conserved current, the symmetry is localized by introducing the compensating field $A := A_\mu{}^a T_a dx^\mu$ by means of the minimal coupling prescription of the matter field to the new gauge interaction:

$$d\Psi \longrightarrow \overset{A}{D}\Psi := (d + A)\Psi\,, \quad L_{\rm mat}(\Psi, d\Psi) \longrightarrow L_{\rm mat}(\Psi, \overset{A}{D}\Psi)\,.$$

- The 1-form A is a Lie-algebra valued *connection*. It acts on the components of the fields, Ψ, with respect to some reference frame, indicating that it can be properly represented as the connection of a frame bundle, which is associated to the symmetry group.
- A, becomes a true dynamical variable by adding a suitable kinetic term, V, to the minimally coupled matter Lagrangian. This supplementary term has to be gauge invariant, such that the gauge invariance of the action is kept. Gauge invariance of V is obtained by constructing V in terms of the field strength.

- Field strength $F = \overset{A}{D}A = dA + A \wedge A$. That is, $V = V(F)$. Hence, the gauge Lagrangian V, as in Maxwell's theory, is assumed to depend only on F, not, however, on its derivative.
- Therefore, the (inhomogeneous) Yang–Mills field equation will be of second order in the gauge potential, A, and its *general form* is (H is the excitation, see electrodynamics)

$$\boxed{\overset{A}{D}H = dH + A \wedge H = I\,,} \quad \text{with} \quad H = -\partial V/\partial F\,, \quad I = \partial L_{\mathrm{mat}}/\partial A\,.$$

- The homogeneous Yang–Mills equation is

$$\overset{A}{D}\,F = 0\,,$$

  a Bianchi type identity following from the definition of the field strength, $F = \overset{A}{D}A$.
- The Yang–Mills field equations are analogs of the inhomogeneous and the homogeneous Maxwell equations $dH = J$ and $dF = 0$, respectively (also in electrodynamics the experimentally established conservation law of the electric charge is the basis of the whole theory)..

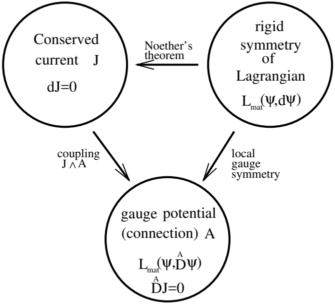
- The original conservation law, after gauging, gets modified and is only gauge covariantly conserved, $dI = 0 \;\longrightarrow\; \overset{A}{D}I = 0$.
- The gauge field is charged. Its *isospin* current is

$$\overset{A}{I} := -A \wedge H\,, \qquad \text{with} \qquad \check{I} := I + \overset{A}{I} \qquad \text{and} \qquad d\check{I} \cong 0\,.$$

  The non gauge-covariant *isospin complex*, $\check{I}$, is “weakly” conserved, provided the inhomog. field equation is fulfilled.
- The (inhomogeneous) Yang–Mills equation becomes *quasi-linear,* as long as the gauge Lagrangian, V, depends on F no more than *quadratically.* Accordingly, H must be linear in F, namely $H = \alpha\, {}^\star F$, with α as a coupling constant. The Hodge star, *, is required in order to make H a differential form with twist. Then the Yang–Mills equation finally reads

$$\overset{A}{D}\,{}^\star F = d\,{}^\star F - \alpha^{-1}\,\overset{A}{I} = \alpha^{-1}\,I \qquad \text{with} \qquad \overset{A}{I} := -\alpha\, A \wedge {}^\star F\,.$$

- Later, Mills (1979) also discussed a *non*linear, Born–Infeld type “constitutive” relation between H and F. But this didn’t prove to be useful.



The structure of a gauge theory à la Yang–Mills is depicted in this diagram, which is adapted from Mills.

$\star$ A gauge theory is based on a CONSERVED CURRENT and the SYMMETRY connected with it. The symmetry is first rigid—and there is no interaction—then, subsequently, made local, and the gauge potential A and the gauge field strength $F \sim \overset{A}{D}A$ emerge in this procedure.

$\star$ A caveat: In this lecture, classical field theory is used. We have *not* investigated quantum field theoretical consequences.

2. Newton-Einstein gravity

- In Newtonian gravity, mass is the source of gravity and—in its quasi field-theoretical formulation—the mass density: $\Delta\Phi(\vec{r},t) \sim G\rho(\vec{r},t)$, ρ mass density is source of gravity. Mass is conserved, continuity equation for ρ.
- In special relativity (SR), the starting point for a gravitational theory [see Einstein, *Meaning of Relativity* (1922)] is Minkowski space: supported today by all high-energy physics experiments.
- Wigner's 1939 classification of elementary quantum mechanical object: mass-spin classification; for massless particles, the helicity classification.
- In special relativity, the mass density is superseded by the *energy-momentum* current of matter and the conservation of mass by the conservation of energy-momentum:$d\,\Sigma_\alpha = 0$:

conserved energy-mom. current $\longrightarrow$ rigid transl. inv. in (1+3)d spacetime

- Thus, the conserved energy-momentum current of matter in special relativity is doubtlessly the starting point for a gauge theory of gravity.

Let me quote three different people who came to similar conclusions:

1. Sakurai (1960): "...there exists a deep connection between energy conservation and the very existence of the gravitational coupling. The gravitational field, being the dynamical manifestation of energy, is to be coupled to energy-momentum density....hence the gravitational field can interact with itself in the same way as the $T = 1$ Yang-Mills $B_\mu^{(T)}$ field (which is the dynamical manifestation of isospin) can interact with itself."
2. Glashow & Gell-Mann (1961): "...if we set up the Einstein theory by gauge methods then the conclusions are slightly different. Instead of an isotopic rotation, we perform a 4-dimensional translation at each point of space..."
3. Feynman (1962): "The equations of physics are invariant when we make coordinate displacements [by] any constant amount a^μ...it is possible to investigate how we might make the equations of physics invariant when we allow space dependent *variable* displacements..."

Technical remark re different formalisms: The canonical

energy-momentum tensor (tensor calculus) $T_\alpha{}^\beta$ and the canonical energy-momentum current 3-form (exterior calculus) Σ_α are equivalent:

$$\Sigma_\alpha = T_\alpha{}^\beta \epsilon_\beta = \frac{1}{3!} T_\alpha{}^\beta \epsilon_{\beta\mu\nu\rho} \vartheta^\mu \wedge \vartheta^\nu \wedge \vartheta^\rho \,, \qquad T_\alpha{}^\beta = {}^\diamond(\vartheta^\beta \wedge \Sigma_\alpha)\,.$$

A priori, they have 16 independent components. If a metric $g_{\alpha\beta}$ is used, we can define a symmetric tensor $T_{\alpha\beta} = g_{\beta\gamma}\, T_\alpha{}^\gamma$, with 10 independent components, which is the case for fluid and electromagnetic matter in classical physics.

3. Translational gauge theory (TG)

- For gravity, the mass density (Newton-Poisson) and, in special relativity, the conserved ENERGY-MOMENTUM CURRENT is the source of gravity.

  Thus, TRANSLATION symmetry (in Minkowski space) emerges.

  $\longrightarrow$ translational gauge invariance. If localized, the premetric teleparallelsim scheme, see the Tonti-diagram in our joint paper. Hence the source is the canonical 16 component energy-momentum tensor.

- Rigid translational invariance is made *local* at the price of introducing 4 translational gauge potentials (the coframe ϑ) which compensate the violation of the rigid invariance:

rigid transl. inv. $\overset{\text{heur. princ.}}{\longrightarrow}$ local transl. inv. $\longrightarrow$ coframe ϑ^α compensates

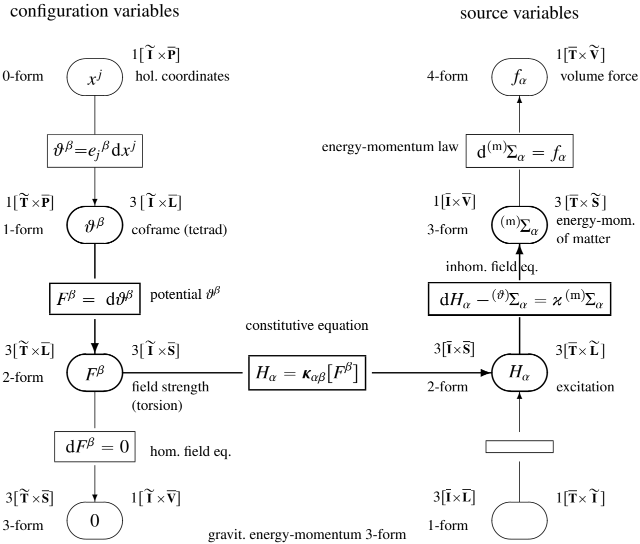
- The curl of ϑ^α, the torsion, corresp. to the grav. field strength:

$$F^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$$

- The corresponding curvature vanishes: $R_\alpha{}^\beta := d\Gamma - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = 0$. Teleparallelism takes place in a so-called Weitzenböck spacetime. Analogously as above, $F \neq 0$ is the criterion for the emerging of a new non-trivial gravitational/translational gauge field. It can be shown that this so-called telparallelism theory, for a suitable Lagrangian, is equivalent to general relativity of 1916, provided a symmetric energy-momentum tensor is chosen.
- Teleparallelism spacetime is an AFFINE SPACETIME without metric. This teleparallelism scheme cannot be directly compared with nature.

  For defining a symmetric energy-momentum tensor we need a metric. Hence the premetric teleparallelism scheme does not qualify as a bona fide physical theory. However, since we started from SR, we have a metric available and we can use it for formulating the constitutive law of a teleparallelism theory. Then TG becomes $\mathrm{GR}_{||}$, see the Tonti diagram of our new paper: Itin, Obukhov, Boos, Hehl, to be published. We find that for scalar and electromagentic matter TG is equivalent to GR.

Tonti diagram of the premetric teleparallel theory of gravity⋆ (TG)



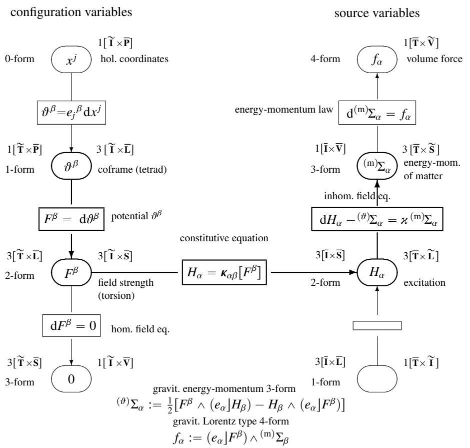
gravit. energy-momentum 3-form

$${}^{(\vartheta)}\Sigma_\alpha := \tfrac{1}{2}\left[F^\beta \wedge (e_\alpha \rfloor H_\beta) - H_\beta \wedge (e_\alpha \rfloor F^\beta)\right]$$

gravit. Lorentz type 4-form

$$f_\alpha := (e_\alpha \rfloor F^\beta) \wedge {}^{(\mathrm{m})}\Sigma_\beta$$

Tonti diagram of the premetric teleparallel theory of gravity* (TG)



Patterned after E. Tonti: *The Mathematical Structure of Classical and Relativistic Physics,* A general classification diagram (Birkhäuser-Springer, New York, 2013) pages 402 and 315.

Notation: Y. Itin, F. W. Hehl, Yu. N. Obukhov, Phys. Rev. D **95**, 084020 (2017), arXiv:1611.05759. We denoted here the torsion 2-form with F^β in order to underline its function as a field strength. Usually, however, we use for torsion T^β.

We chose everywhere the 'teleparallel gauge' such that the connection 1-form vanishes globally: $\Gamma^{\alpha\beta}(x) \stackrel{*}{=} 0$. For the reversible case, we have the gravitational Lagrangian as ${}^{(\vartheta)}\Lambda = -\frac{1}{2}F^\alpha \wedge H_\alpha$ and $\Lambda = {}^{(\vartheta)}\Lambda + {}^{(\mathrm{m})}\Lambda$. The gravitational constant is denoted by $\varkappa$.

*) Also known as translation gauge theory of gravity.

4. Poincaré gauge theory (PG)

- Since the translation group $T(4)$ in special relativity is a subgroup of the Poincaré group $T(4) \rtimes SO(1,3)$, the symmetry group of special relativity, one has to straightforwardly extend the gauging of the translations to the gauging of full Poincaré transformations. Thereby also the conservation law of the *angular momentum* current is included. The emerging Poincaré gauge theory of gravity will take place in a Riemann-Cartan space with torsion and curvature.

- In this last step, one has to switch from the Einstein laboratory to the Kibble laboratory, since fermions have specific spin properties not known from ordinary macroscopic matter, which is conventionally considered in GR, like Euler fluids and electromagnetic fields. Accordingly, we arrive at PG and, more realistically, at the quadratic PG with even and odd pieces. In the recent paper of M. Blagojević and B. Cvetković, “General Poincaré gauge theory: Hamiltonian structure and particle spectrum,” Phys. Rev. D 98, 024014 (2018), one can find a coherent discussion of the emerging structures.

- Equipment and constructs in “Einstein’s laboratory”:

(i) A neutral point particle with mass m;
(ii) an inertial frame K;
(iii) an accelerated (i.e., non-inertial) frame K';
(iv) a homogeneous gravitational field; and
(v) light rays.

- Equipment and constructs in “Kibble’s laboratory”:

(i) An unquantized Dirac spinor (fermionic field with mass m and spin $s = \hbar/2$);
(ii) an inertial frame, $\vartheta^{\alpha} = \delta_i^{\alpha} dx^i$;
(iii) a translational and rotational accelerated frame, $\vartheta^{\alpha'}$;
(iv) homogeneous gravitational fields; and
(v) light rays.

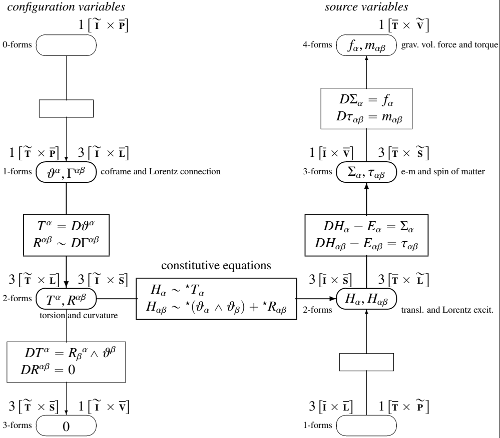
- The objects considered are different, see (i).

The notions of inertial systems are different, see (ii).
The rotational acceleration plays an additional role, see (iii).

The rest is similar. Because we consider spinors, we get, by using Einstein’s original ideas, a modified outcome.

	Einstein's laboratory	Kibble's laboratory
Elementary object in SR	Mass point m with velocity u^i	Dirac spinor $\Psi(x)$ of mass m (with 4 components)
Inertial frame (IF)	Cart. coo. system x^i $ds^2 \stackrel{*}{=} o_{ij}dx^i dx^j$	Holonomic orthon. frame $e_\alpha \stackrel{*}{=} \delta^i_\alpha \partial_i, \quad e_\alpha \cdot e_\beta = o_{\alpha\beta}$
Force-free motion in IF	$\dot{u}^i \stackrel{*}{=} 0$	$(i\gamma^i\partial_i - m)\Psi \stackrel{*}{=} 0$
Non-inertial frame (NIF)	Curvilinear coord. system x^i	Anholon. orthon. frame $e_\alpha = e^i{}_\alpha \partial_i$ or coframe $\vartheta^\alpha = e_i{}^\alpha dx^i$
Force-free motion in NIF	$\dot{u}^i + u^j u^k \widetilde{\Gamma}_{jk}{}^i = 0$	$[i\gamma^\alpha e^i{}_\alpha(\partial_i + \Gamma_i) - m]\Psi = 0$ $\Gamma_i := \frac{i}{4}\Gamma_i{}^{\beta\gamma}\rho_{\beta\gamma}$ Lorentz
Non-inertial geom. objects	$\widetilde{\Gamma}_{ij}{}^k$ 40	e_α or ϑ^α, $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ 16 + 24
Constraints in SR	$\widetilde{Riem}(\partial\widetilde{\Gamma}, \widetilde{\Gamma}) = 0$ 20	$Tor(\partial e, e, \Gamma) = 0, Riem(\partial\Gamma, \Gamma) = 0$ 24 + 36
Global IF	$g_{ij} \stackrel{*}{=} o_{ij}\,,\ \widetilde{\Gamma}_{ij}{}^k \stackrel{*}{=} 0$	$(e_i{}^\alpha,\ \Gamma_i{}^{\alpha\beta}) \stackrel{*}{=} (\delta_i^\alpha, 0)$
Archetypal experiment	Apple in grav. field (Newton)	Neutron in grav. field (COW)
Switch on gravity	$\widetilde{Riem} \neq 0$ Riemann spacetime	$Tor \neq 0, \quad Riem \neq 0$ Riemann-Cartan spacetime
Local IF "Einstein elev."	$g_{ij}\vert_P \stackrel{*}{=} o_{ij}\,,\ \widetilde{\Gamma}_{ij}{}^k\vert_P \stackrel{*}{=} 0$	$(e_i{}^\alpha,\ \Gamma_i^{\alpha\beta})\vert_P \stackrel{*}{=} (\delta_i^\alpha, 0)$
Field equations	$\widetilde{Ric} - \frac{1}{2}tr(\widetilde{Ric}) \sim mass$ **GR**	$Ric - \frac{1}{2}tr(Ric) \sim mass$ $Tor + 2\ tr(Tor) \sim spin$ **EC**

Poincaré gauge theory of gravity (PG)



grav. volume force $\quad f_\alpha := (e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge \tau_{\beta\gamma}$

grav. volume torque $\quad m_{\alpha\beta} := -\vartheta_{[\alpha} \wedge \Sigma_{\beta]}$

grav. energy-momentum $\quad E_\alpha := e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R^{\beta\gamma}) \wedge H_{\beta\gamma}$

grav. spin ang. momentum $\quad E_{\alpha\beta} := -\vartheta_{[\alpha} \wedge H_{\beta]}$

Einstein-Cartan theory (EC) emerges from the constitutive equations

$H_\alpha = 0\,,\ H_{\alpha\beta} = \frac{1}{2\kappa}{}^\star(\vartheta_\alpha \wedge \vartheta_\beta)$; for $\tau_{\alpha\beta} = 0$, one recovers general relativity (GR)

Teleparallelism (TG) emerges from $H_\alpha \sim {}^\star T^\alpha$, $H_{\alpha\beta} = 0$, and $\tau_{\alpha\beta} = 0$

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