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06/29/2004

Exercise 10 of Theoretische Physik II: Elektrodynamik
method of images in dielectrics, Maxwell's equations

Submission date: 07/07/2004

Problem 1 (8 points): *method of images in dielectrics*

The complete space is filled with a dielectric, which in the upper half-space ($z > 0$) has dielectric constant ϵ_1 and in the lower half-space ($z < 0$) ϵ_2. A charge q is located on the positive z-axis at $\mathbf{x}_0 = d\mathbf{e}_z$.

1. Determine the potential in the whole space. To do this, consider the upper and lower half-spaces separately.

   (i) $z > 0$: Consider the whole space as a dielectric with ϵ_1. The influence of the dielectric ϵ_2 can be taken into account by a charge q' at $\mathbf{x}' = -\mathbf{x}_0$.

   (ii) $z < 0$: Consider the whole space as a dielectric with ϵ_2. The potential which is caused by q is modified by the changed dielectric constant; this can be taken into account by a modified charge q'' at $\mathbf{x}'' = \mathbf{x}_0$.

Use the boundary conditions of the fields. (3 points)

2. Calculate the surface charge density σ_P as well as the total charge Q_P of the x-y-plane. (2 points)

3. What is the potential in the limit $\epsilon_1 = \epsilon_2$? For which value of ϵ_2 does one obtain the potential of a point charge in front of a conducting half-space? (1 point)

4. Make sketches of the field lines for the cases $\epsilon_1 = \epsilon_2$, $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$. (2 points)

Problem 2 (5 points): *wave equation in a conductor (Telegraphengleichung)*

1. Consider a charge free, homogeneous isotropic electric conductor with conductivity σ, which obeys Ohm's law. Starting from Maxwell's equations, derive the wave equations for $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$, and show that they are given by

$$\left[\left(\Delta - \frac{1}{v_P^2}\frac{\partial^2}{\partial t^2}\right) - \mu\mu_0\sigma\frac{\partial}{\partial t}\right]\mathbf{V}(\mathbf{x},t) = 0,$$

where $\mathbf{V}(\mathbf{x},t)$ stands for the $\mathbf{E}$- and $\mathbf{B}$-fields. Show that, for $\sigma = 0$, this describes an electromagnetic wave with propagation speed $v_P = \frac{c}{n}$ with the index of refraction $n = \sqrt{\mu\epsilon}$. (2 points)

2. Show that, using the above assumptions, a vanishing charge density $\rho(\mathbf{x},t)$ at time $t = 0$ leads to the condition $\rho(\mathbf{x},t) \equiv 0$ for all times t. (1 point)

3. With the ansatz $\mathbf{V}(\mathbf{x},t) = \mathbf{V}_0 e^{i(\mathbf{kx}-\omega t)}$, calculate the (generally complex) wave number $k = \frac{\omega}{c}(\bar{n} + i\kappa)$ for $\sigma = 0$ and $\sigma \neq 0$, and describe what this means physically for a wave impinging on a conductor with $\sigma \neq 0$. Let $n = \sqrt{\mu\epsilon(\omega)} \in \mathbf{R}$. (2 points)

Problem 3 (7 points): *reflection/transmission*

A plane, linearly polarized electromagnetic wave of the frequency ω impinges perpendicularly on a metal surface with conductivity σ and permeability μ. Let the following condition hold for the generalized index of refraction $\bar{n}$ and the attenuation coefficient (Extinktionskoeffizient) κ of the metal, defined by $k = \frac{\omega}{c}(\bar{n}+i\kappa)$, where k is the complex wave number:

$$\bar{n} = \kappa \approx \sqrt{\frac{\sigma\mu}{2\omega\epsilon_0}} \gg 1.$$

This condition is satisfied for a good conductor (cf. Problem 2, part 3).

1. What are the amplitudes of the reflected and the transmitted waves? Calculate both the $\mathbf{E}$- and $\mathbf{H}$-fields. Let the oscillation direction of the $\mathbf{E}$-field be parallel to the x-axis and the metal surface be the x-y-plane. Use the boundary conditions for the fields. (4 points)

2. The transmitted wave acts on the metal with the force

$$\mathbf{K} = \mu_0\mu\int d^3x\, \mathbf{j}\times\mathbf{H}.$$

What temporally averaged pressure is exerted on the metal by the wave? (3 points)