Problem 1 (3 points): some more Fourier analysis
1. Prove the following properties of the Fourier transformation: \( \hat{f} = \mathcal{F}(f) \)
   
   (i) \( \hat{f}(t) = f(-t) \)
   
   (ii) \( \hat{f}(t) = \hat{g}(\hat{t}) \)
   
   (iii) \( \mathcal{F}(f \ast g) = \hat{f} \hat{g} \)

   (3 points)

Problem 2 (6 points): distributions
1. Show the following properties of the \( \delta \)-distribution (by means of test functions \( \varphi \in \mathcal{S} \)):
   
   (i) \( \delta(at) = \frac{1}{|a|}\delta(t) \)

   (ii) \( \delta(g(t)) = \sum_{i} \frac{1}{|g'(t_i)|} \delta(t - t_i) \), \( t_i \) zero of \( g(t) \)

   (2 points)

2. Consider the functions
   
   \( f_n(t) = \frac{1}{\pi} \frac{n}{t + n^2 \sigma^2} \).

   Show that in the limit of \( n \to \infty \), \( f_n(t) \) converges to \( \delta(t) \).

   (2 points)

3. Prove by letting the expressions act on test functions \( \varphi \in \mathcal{S} \):
   
   (i) \( f(t) \delta(t) = f(0) \delta(t) \)

   (ii) \( f(t) \delta'(t) = f(0) \delta'(t) - f'(t) \delta(t) \),

   where \( f \in \mathcal{S} \).

   (2 points)

Problem 3 (10 points): Green's functions
The goal of this problem is to derive the retarded Green's function \( G_n \) of the wave equation in \( n \) spatial dimensions (throughout this problem, we set \( c=1 \))

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta_n \right) G_n(t, x) = \delta(t) \delta(x) \tag{1}
\]

where \( G_n(t < 0, x) = 0 \) and the spatial derivatives vanish in spatial infinity. Let \( x = (x_1, ..., x_n) \) and

\[
\Delta_n = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}.
\]

Make the ansatz \( G_n(t, x) = g_n(t, r) \) where \( r = \sqrt{x_1^2 + ... + x_n^2} \).

1. Show that
   
   \[
   G_n(t, x) = \int_{-\infty}^{\infty} dx_{n+1} G_{n+1}(t, x, x_{n+1})
   \]

   is a Green's function according to (1) in \( n \) spatial dimensions, if \( G_{n+1} \) is also one in \( n + 1 \) spatial dimensions. (reduction method of Hadamard) (1 point)

2. Show by direct calculation that
   
   \[
   g_1(t, r) = \frac{1}{2} \Theta(t - r), \quad g_2(t, r) = \frac{1}{4\pi r} \delta(t - r)
   \]

   and, from this, calculate \( g_2(t, r) \).

   (4 points)

3. By applying the reduction method twice in a row, write \( g_n(t, r) \) as an area integral over an expression which includes \( g_{n+2} \). Use polar coordinates for the area integral to derive the recursion formula
   
   \[
   g_{n+2}(t, r) = -\frac{1}{2\pi} \frac{1}{r} \frac{\partial}{\partial r} g_n(t, r)
   \]

   (2 points)

4. Give the exact formulas for \( g_2 \) and \( g_{2k+1} \). The expressions can still contain derivative operators. From this, calculate \( g_4 \) and \( g_6 \).

   (3 points)

5. Physically, the Green's function here is the temporal and spatial evolution of a \( \delta \)-light pulse which is emitted at time \( t = 0 \) at the place \( x = 0 \). Which signal (qualitatively) does an observer see for \( t \geq r \) for even and odd spatial dimensions? (It can be shown that generally for even spatial dimensions, the observer would register a "reverberation"; this effect is not present for odd spatial dimensions.) (1 point)