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Exercise 3 of Theoretische Physik II: Elektrodynamik

vector algebra, special functions

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Problem 1 (8 points): *curvilinear coordinates*

Consider spherical coordinates which are defined by the following equations (x^1, x^2, x^3 are Cartesian coordinates):

$$x^1 = r\,cos\,\varphi\,sin\,\theta$$
$$x^2 = r\,sin\,\varphi\,sin\,\theta$$
$$x^3 = r\,cos\,\theta$$

1. Describe the hypersurfaces which are obtained by setting one of the new coordinates r, φ, θ constant. Determine the new coordinate lines and the unit vectors $\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_\theta$ which are tangent to them. Show their orthonormality. (4 points)

2. Derive ∇ in the new coordinates, i.e. determine the derivative operators D in $\nabla = \mathbf{e}_r D_r + \mathbf{e}_\varphi D_\varphi + \mathbf{e}_\theta D_\theta$. What is the Laplacian $\Delta = \nabla \cdot \nabla$ in the new coordinates? (4 points)

Problem 2 (8 points): *vector algebra*

1. Show that $\Delta \frac{1}{r}$ is proportional to $\delta^{(3)}(\mathbf{x})$, where $\delta^{(3)}(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$ is the three-dimensional δ-distribution, and calculate the value of the expression (cf. exercise 1, Problem 1.3). (2 points)

2. Let ϕ, ψ be twice continuously differentiable functions. Derive the following identities:

$$(i)\quad \int_V d^3x\,(\nabla\Psi \cdot \nabla\phi + \Psi\Delta\phi) = \int_{\partial V} dF\,\Psi\,(\hat{\mathbf{n}} \cdot \nabla)\phi \qquad \text{(Green's first identity)}$$

$$(ii)\quad \int_V d^3x\,(\phi\Delta\Psi - \Psi\Delta\phi) = \int_{\partial V} dF\,\hat{\mathbf{n}} \cdot (\phi\nabla\Psi - \Psi\nabla\phi) \qquad \text{(Green's second identity)},$$

where $\hat{\mathbf{n}}$ is the unit normal vector to the area differential $d\mathbf{F}$. (2 points)
(These identities are important when dealing with boundary problems in electro- and magnetostatics.)

3. Show that every vector field $\mathbf{A}(\mathbf{x}) \in \mathscr{S}$ can be decomposed into a curlfree and a divergenceless part (decomposition theorem):

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}_l(\mathbf{x}) + \mathbf{A}_t(\mathbf{x}),$$

where curl $\mathbf{A}_l = 0$ and div $\mathbf{A}_t = 0$.
Write the vector field in the form $\mathbf{A}(\mathbf{x}) = \int d^3x'\,\mathbf{A}(\mathbf{x}')\delta(\mathbf{x} - \mathbf{x}')$ and use problem 2.1 as well as the expression for $\Delta\mathbf{A}(\mathbf{x})$ (exercise 1, problem 1.1). (4 points)

Problem 3 (4 points): *special functions*

1. Consider the Legendre polynomials of the first kind

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l$$

which are solutions of the Legendre equation

$$(1 - x^2)P_l'' - 2xP_l' + l(l+1)P_l = 0.$$

Calculate the first five Legendre polynomials and make sketches of them. (2 points)

2. The spherical harmonics are defined by

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^m(cos\theta)e^{im\varphi}.$$

Show their orthonormality, i.e. show that

$$\int_0^\pi d\theta sin\theta \int_0^{2\pi} d\varphi Y_{lm}^*(\theta, \varphi)Y_{l'm'}(\theta, \varphi) = \delta_{ll'}\delta_{mm'}.$$

You may use the fact that the associated Legendre polynomials are orthogonal:

$$\int_{-1}^{1} dx\,P_l^m(x)P_{l'}^m(x) = \frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}\delta_{ll'}.$$

(1 point)

3. Make sketches of the first five Bessel functions. (1 point)