Exercise 3 of Theoretische Physik II: Elektrodynamik
vector algebra, special functions

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Problem 1 (8 points): curvilinear coordinates
Consider spherical coordinates which are defined by the following equations
\( x^1 = r \cos \varphi \sin \theta \)
\( x^2 = r \sin \varphi \sin \theta \)
\( x^3 = r \cos \theta \)

1. Describe the hypersurfaces which are obtained by setting one of the new coordinates \( r, \varphi, \theta \) constant. Determine the new coordinate lines and the unit vectors \( e_r, e_{\varphi}, e_{\theta} \) which are tangent to them. Show their orthogonality. (4 points)

2. Derive \( \nabla \) in the new coordinates, i.e. determine the derivative operators \( D \) in \( \nabla = e_r + e_{\varphi} \partial_{\varphi} + e_{\theta} \partial_{\theta} \). What is the Laplacian \( \Delta = \nabla \cdot \nabla \) in the new coordinates? (4 points)

Problem 2 (8 points): vector algebra
1. Show that \( \Delta \phi \) is proportional to \( \delta^{(3)}(x) \), where \( \delta^{(3)}(x) = \delta(x)\delta(y)\delta(z) \) is the three-dimensional \( \delta \)-distribution, and calculate the value of the expression (cf. exercise 1, Problem 1.3). (2 points)

2. Let \( \phi, \psi \) be twice continuously differentiable functions. Derive the following identities:
\[ (i) \quad \int_V d^3x (\nabla \Psi \cdot \nabla \phi + \Psi \Delta \phi) = \int_{\partial V} dF \Psi (\mathbf{n} \cdot \nabla) \phi \] (Green's first identity)
\[ (ii) \quad \int_V d^3x (\phi \Delta \Psi - \Psi \Delta \phi) = \int_{\partial V} dF (\phi \nabla \Psi - \Psi \nabla \phi) \] (Green's second identity),
where \( \mathbf{n} \) is the unit normal vector to the area differential \( dF \). (2 points)
(These identities are important when dealing with boundary problems in electro- and magnetostatics.)

3. Show that every vector field \( \mathbf{A}(x) \in \mathcal{E} \) can be decomposed into a curlfree and a divergenceless part (decomposition theorem):
\[ \mathbf{A}(x) = \mathbf{A}_t(x) + \mathbf{A}_r(x), \]
where \( \text{curl} \mathbf{A}_t = 0 \) and \( \text{div} \mathbf{A}_r = 0 \).
Write the vector field in the form \( \mathbf{A}(x) = \int d^3x' \mathbf{A}(x') \delta(x-x') \) and use problem 2.1 as well as the expression for \( \Delta \mathbf{A}(x) \) (exercise 1, problem 1.1). (4 points)

Problem 3 (4 points): special functions
1. Consider the Legendre polynomials of the first kind
\[ P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \]
which are solutions of the Legendre equation
\[ (1 - x^2)P''_l - 2xP'_l + l(l + 1)P_l = 0. \]
Calculate the first five Legendre polynomials and make sketches of them. (2 points)

2. The spherical harmonics are defined by
\[ Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l + 1}{(2\pi)} \frac{(l - |m|)!}{(l + |m|)!}} P_l^m(\cos \theta) e^{im \varphi}. \]
Show their orthonormality, i.e. show that
\[ \int_0^{2\pi} d\varphi \int_0^\pi d\theta Y_{lm}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{ll'} \delta_{mm'}. \]
You may use the fact that the associated Legendre polynomials are orthogonal:
\[ \int_{-1}^1 dx P_l^m(x) P_l^{m'}(x) = \frac{2}{2l + 1} \frac{(l + |m|)!}{(l - |m|)!} \delta_{mm'}. \] (1 point)

3. Make sketches of the first five Bessel functions. (1 point)