Exercise 4 (6 points): Motion in the gravitational field
The equation of motion for a test particle in the gravitational field is given by
\[ \ddot{x}^i + \Gamma^i_{kl} \dot{x}^k \dot{x}^l = 0, \]
where \( \dot{x}^i = \frac{dx^i}{ds} \) and \( \Gamma^i_{kl} = \frac{1}{2} g^{ij} (\partial_k g_{jl} + \partial_l g_{jk} - \partial_j g_{kl}) \).

1. Repeat briefly the derivation of (1) from the variational principle \( \delta \int ds = 0 \) as presented in the lecture course. Why can the derivation not be used for photons?
2. Derive (1) from the alternative variational principle
\[ \delta \int g_{ik} \dot{x}^i \dot{x}^k d\lambda \equiv \delta \int \mathcal{L} d\lambda = 0, \]
where \( \lambda \) is an affine parameter, and \( \dot{x}^i = \frac{dx^i}{d\lambda} \). Show that the derivation holds also for photons and determine \( \mathcal{L} \) for the solution of (1).

Exercise 5 (6 points): Christoffel symbols
Derive the transformation property of the Christoffel symbols
\[ \Gamma^i_{kl} = \frac{1}{2} (g_{ik,l} + g_{li,k} - g_{kl,i}) \]
under a coordinate transformation \( x'^i(x^a) \). The result shows that they do not form a tensor.

Exercise 6 (4 points): Rotating reference frame
Calculate in the Newtonian approximation the Christoffel symbols for a system which rotates with constant angular velocity \( \omega \) around the \( z \)-axis and formulate the geodesic equation (1) for this case. Identify the centrifugal and the Coriolis force in the resulting equation of motion.

Exercise 7 (4 points): Freely falling observer
The equation of motion of a mass point in a (flat) 1+1-dimensional Minkowski space be given by the example \( m \ddot{x} - mg = 0 \). In analogy to the equation of motion (1) we set \( \Gamma^1_{00} = -g \) and \( \Gamma^i_{kl} = 0 \) otherwise. On physical grounds it is obvious that there should exist a reference frame in which the Christoffel symbols vanish and the equation of motion for a free mass point therefore reads \( m \ddot{x} = 0 \). Find such a coordinate system by "integrating" the Christoffel symbol.