Advanced Seminar on Relativity and Cosmology
–
Singularity Theorems

Sebastian Schuster

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1 Introduction

Singularity theorems play an important role in the suggestion that general relativity is imcomplete. The singularities seen in many cosmological models with expansion or in the Schwarzschild metric on their own could still be only pathological remainders of the high degree of symmetry seen in these solutions for Einstein's field equations. The singularities seen in these examples hint at possible contact points with a quantum theory of gravity. A warning is appropriate: Singularity theorems need a fair amount of mathematics. I will try to keep close to the fairly cursory statements and ideas from [3]. Still, I will give some hints at the hidden intricacies in 4. To mention them in the formal clarity they need would greatly overstretch this talk, so I will refrain from elaborating these hints. Rather, I want to give a feeling for different types of singularities and how easily they come up in the theory of general relativity.

First, I will give you some[1] definitions from the mathematical language as they appear in the context of singularity theorems. I will try to favor brevity. Then I will give some sample singularity theorems due to Hawking and Penrose. After this, I give a short introduction to the many mathematical details that might come up along the way and lead certainly to abundant complications with the definitions I will give here. Last, I will give you a guide to the literature and some linked topics in GR.

2 Preliminary Definitions

Definition 1 (Causal curve)**.** *A curve λ is called causal, if for every $t, \lambda(t)$ has either a future directed timelike or null tangent vector.*

Definition 2 (Chronological Future)**.** *The chronological future of a subset $S \subset M$ is given by:*

$$\begin{aligned} I^+(S) =&\{q \in M | \exists \text{ future directed, timelike curve } \lambda(t), \\ &s.t.\ \exists p \in S : \lambda(0) = p, \lambda(1) = q\} \end{aligned}$$

Definition 3 (Achronal Set)**.** *A subset $S \subset M$ is said to be achronal if there exist no p,$q \in S$ such that $q \in I^+(p)$.*

[1] maybe many

Obviously, this is equivalent to demanding that $I^+(S) \cap S = \emptyset$.

Definition 4 ($D(S)$)**.** *If $S \subset M$ is a part of space-time, we call $D^+(S)$ the set of all points q from which every past directed, timelike curve intersects with S. Similarily, $D^-(S)$ denots the set of points q such that all future directed curves emanating from q intersect S. Furthermore, $D(S) := D^+(S) \cup D^-(S)$.*

Definition 5 (Cauchy Surface)**.** *A closed achronal set Σ with $D(\Sigma) = M$ is called a Cauchy surface.*

Definition 6 (Global Hyperbolicity)**.** *A space-time which has a Cauchy surface Σ is globally hyperbolic.*

Definition 7 (Conjugate Points)**.** *A solution J^γ of the geodesic deviation equation*

$$v^\alpha \nabla_\alpha (v^\beta \nabla_\beta J^\gamma) = R^\gamma_{\alpha\beta\delta} J^\beta v^\alpha v^\delta$$

is called a Jacobi field. If their exists a nonzero Jacobi field that vanishes at two points p,q, these points are said to be conjugate to each other.

Roughly speaking the existence of conjugate points hints at timelike geodesics failing at being curves of maximal eigentime between two events.

2.1 Energy conditions

As I will explain in 3, energy conditions are a very important ingredient to singularity theorems. Physically, these conditions are fairly broad and include all known (classical) energy-momentum tensors, including a cosmological constant $\Lambda \leq 0$. As these energy conditions could easily fill a single talk (and therefore were even a possible topic in this seminar), I have to refrain from delving too deeply into this topic. So I will simply state the single one we will be concerned with, the generic energy condition as stated in [3]:

Definition 8. *For a timelike vector field v^μ we have:*

$$T_{\mu\nu} v^\mu v^\nu \geq \frac{1}{2} v^\mu v_\mu T$$

(This alone would be the so called strong energy condition.) Furthermore, we have: For every timelike or null geodesic exists a point such that

$$l_{[\alpha} R_{\beta]\gamma\delta[\epsilon} l_{\zeta]} l^\gamma l^\delta \neq 0$$

holds for the tangent vectors l^μ.

2.2 Singularity

There are many pathologies to be encountered when trying to give an all-encompassing definition of singularities. I will follow here the gist of [3] and [1]. This means, we will be considering singularities defined as:

Definition 9. *A space-time has a singularity, if timelike or lightlike geodesics are incomplete and the space-time cannot be embedded in a bigger space-time.*

The main point of this definition is to describe a singularity as something that either stops a particle's history abruptly in the past or future.

3 Two Sample Singularity Theorems

As described in [3], there are basically three key ingredients for a singularity theorem:

- An energy condition
- A condition on the global structure of space-time
- A trapped surface

Some of these properties can be hidden in more concrete conditions.

Many theorems use global hyperbolicity as global property, though there are more general ones. See [1] or [2].

The first theorem I would like to present is one related to cosmology.

Theorem 1. *Let $(M,g_{\mu\nu})$ be a globally hyperbolic space-time, which fulfills the strong energy condition. Let C be a negative constant. If there exists a Cauchy surface on which the trace of the external curvature K satisfies*

$$K \leq C < 0$$

then all past directed, timelike geodesics are incomplete.

Taking into account that all known (classical) matter satisfies the strong energy condition and globally hyperbolicity is a fairly common property on space-times, this tells that a singularity appears at the beginning of the universe in these cosmological models.

There are other theorems, which have similar relevance for cosmology but don't make use of global hyperbolicity and favor weaker conditions.

Another theorem, stated less technically in [3] would be:

Theorem 2. *A space-time fulfilling the generic energy condition, having no closed, timelike loops and having either trapped 2-surfaces or closed spacelike 3-surfaces, has a singularity*

This would, for example, tell that the Schwarzschild solution has a singularity. On the other hand, the Kerr metric would evade this theorem in certain cases as it could have the right values of charge and angular momentum to actually allow closed, timelike curves.

Proofs of these theorems can take quite a while as many concepts from topology would need to be introduced. These concepts play also a role in the statement of some singularity theorems.

As a last point, I'd like to give a very, very rough sketch of the proofs: The proofs given in [1] and [3] all come down to being proofs by contradiction. Due to the (different) assumptions and the Raychaudhuri equation telling us that gravity is a purely attractive force, we can deduce lemmata providing us with conjugate points in the given space time. At the same time we can prove, assuming geodesic completeness, the existence of a geodesic maximizing the proper time – with the conjugate points along its curve. This conjugate points would provide us with longer curves than the original curve with maximal proper time. Therefore, we arrive at a contradiction.

4 Here Be Dragons!

As already mentioned earlier, topology and its concepts play an increasing role when dealing with singularity theorems or the language of causality behind it. This easily gives rise to a lot of nitpicking. But there are also some problems from the physical point of view.

For example, there is no perfect definition of a singularity: already our definition of a singularity became rather cumbersome when demanding that the problem should not be related to an embedding. For example, cutting Minkowski space in half (by confining the variables) would easily result in incomplete geodesics, but regaining the original range of the variables would embed the new manifold in a way that extends its geodesics back to completeness. But even geodesic completeness runs into problems. Who is interested can read it up in [6] – there we have a curve (not a geodesic!) with finite proper time.

Also, the proofs of the singularity theorems often make use of theorems less easily interpreted in physical terms than the singularity theorem itself. Many are concerned with the properties of the space of curves or geodesics. In this respect the outline of the proofs above fairs worst.

5 Outlook and Links

Global hyperbolicity plays an important not only in the framework of singularity theorems, but also when dealing with the Hamiltonian formulation of GR as here the goal is to develop a initial value problem of GR. The Cauchy surface Σ can be seen as the initial values for a certain space-time. Therefore, this plays also a key role in canonical approaches to quantum gravity.

As already seen in the second theorem given here, or even earlier from the definitions, causal properties of the space-time also often come into play. In a way, already the global hyperbolicity can be linked to a causality as it is the background of an initial value formulation of GR.

The books found in the bibliography are all, with the exception of [3], rather mathematically involved. [1] would then come in second, but already being closer to the remaining three books than to [3].

References

[1] R. Wald – General Relativity, The University of Chicago Press, 1984

[2] S. Hawking, G. Ellis – Large Scale Structure of Space-Time, Cambridge University Press

[3] S. Hawking, R. Penrose – Raum und Zeit, Rowohlt, 1998

[4] B. O'Neill – Semi-Riemannian Geometry – with applications to relativity, Academic Press, 1983

[5] J. Natario – Relativity and Singularities – A Short Introduction for Mathematicians, arXiv:math/0603190v4, 7/2011

[6] R. Geroch, What is a singularity in general relativity?, Ann. Phys., Volume 48, Issue 3, July 1968, p526-540