An Introduction to Symplectic Geometry

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These notes are a short sum up about two talks that I gave in August and September 2015 an the University of Cologne in my workgroup seminar by Prof. Dr. Claus Kiefer. The intension of those talks were to give a tiny introduction into fascination and often inaccessible topic of symplectic geometry, which is a generalization of the famous theory Hamilton mechanics, which has a big big success in physics. Without focusing too much on being mathematical rigorously I tried to be as precise as possible. Of course I also could not cover all introduced concepts in great details. For example, the part about vector bundles is rather short and maybe unsatisfactory. For further reading I refer the following references:

- **Foundations of mechanics**, R. Abraham and J.E. Mardsen, 1978.
- **An Introduction to Symplectic Geometry**, R. Berndt, 2000.
- **Lecture notes: Symplectic Geometry**, S. Sabatini, Sommersemester 2015, University of Cologne.
- **Mathematical methods in classical mechanics**, V.I. Arnold, 1989.
- **Embedding problems in Symplectic Geometry** - Felix Schlenk, 2005.

Every kind of comment or improvement proposal is highly welcome. Just contact me under my email: fasse@thp.uni-koeln.de.

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1 Motivation

Consider a particle moving in $\mathbb{R}^3$. Then the phase space M consists of all positions $\vec{q} \in \mathbb{R}^3$ and all momenta $\vec{p} = m\dot{\vec{q}} \in \mathbb{R}^3$.

Exemplum 1.1(a) For a free particle: $M \simeq \mathbb{R}^6$.

(b) For n free particles: $M \simeq \mathbb{R}^{6n}$.

(c) If the particle is forced to move on a surface $S \subset \mathbb{R}^3$ with projection $\pi : M \to S$, then $\pi^{-1}(q) \subset \mathbb{R}^2$ with $q \in S$. But $M \not\simeq S \otimes \mathbb{R}^2$, e.g. let S be the MÖBIUS strip embedded in $\mathbb{R}^3$.

Consider a non-free particle in $\mathbb{R}^3$ with $M = \mathbb{R}^6$ with conservative force field $\vec{F}$ and corresponding potential V. Then

$$H(\vec{q}, \vec{p}) = E + V \tag{1}$$

is the **Hamiltonian** of the system, where E denotes the Kinetic energy. H gives the total energy of the system and should therefore be conserved along the motion of the particle, i.e. let $\gamma(t) = (\vec{q}(t), \vec{p}(t))$ be the trajectory of a. Then $H(\gamma(t)) = \text{const}$ for all t. The equations of motion are

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i. \tag{2}$$

How can we generalize this concept and understand more about the geometry of phase space M?

2 Mathematical preliminaries

Before we can start with symplectic geometry we have to set up some definitions and have to fix notation.

2.1 Tensor algebra

Definitio 2.1
Let V be a vector space. A (covariant) **k-tensor** $A : V^k \to \mathbb{R}$ is a function that is linear in every argument. The space of k-tensors over V is a vector space denoted by $T_k^0(V)$.

Definitio 2.2
A k-form $\mu \in T_k^0(V)$ is called **alternating** if for every $s \in S_k$, where S_k denotes the permutation group of k elements, we have

$$\mu(v_1, \ldots, v_k) = \text{sgn}(s)\mu(v_{s(1)}, \ldots, v_{s(k)}). \tag{3}$$

The set of all alternating k-forms over V will be denoted by $\Lambda^k(V)$.

Definitio 2.3
The direct sum $\bigoplus_{i=1}^{\infty} \Lambda^k(V)$ together with its structure as a $\mathbb{R}$-vector space and the multiplication by the wedge product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is called the **Grassmann algebra** of V.

Notatio 2.1
Let V be a n-dimensional vector space. Then $\dim \Lambda^k = \binom{n}{k}$.

Definitio 2.4
The non-zero elements of $\Lambda^n(V)$ are called **volume elements**. If $\omega_1, \omega_2 \in \Lambda^n(V)$ we say that $\omega_1 \sim \omega_2$ if there exists a $c \in \mathbb{R}^+$ such that $\omega_1 = c\,\omega_2$. An element of ${}^{V}/_{\sim}$ is called an **orientation** on V.

Definitio 2.5
A 2-form $\mu \in \Lambda^2(V)$ is called **non-degenerate** if

$$\mu(v, w) = 0 \quad \forall v \in V \quad \Rightarrow w = 0 \tag{4}$$

and

$$\mu(v, w) = 0 \quad \forall w \in V \quad \Rightarrow v = 0. \tag{5}$$

2.2 Vector bundles

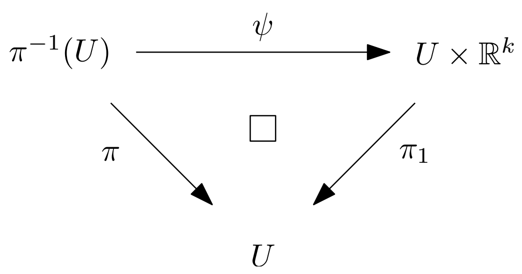
Definitio 2.6
A (real) **vector bundle** over $\mathbb{R}$ of rank k is a triple $\pi : E \to M$ with

(a) E and M are C^∞-manifolds.

(b) The map $\pi : E \to M$ is a surjeciton.

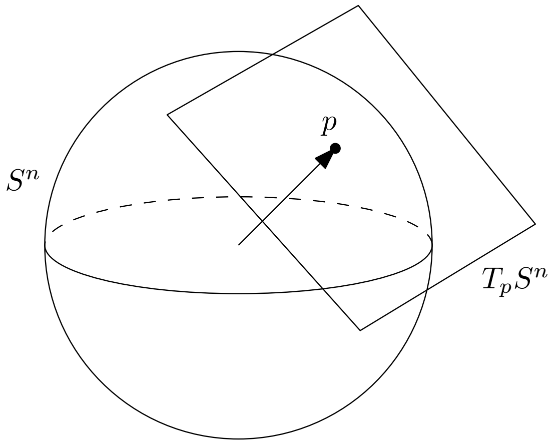
(c) For every $p \in M$ the preimage $\pi^{-1}(p)$ is a $\mathbb{R}$-vector space of dimension k.

(d) For every points $p \in M$ theres exists a neighborhood $U \subset M$ of p and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^K$ such that

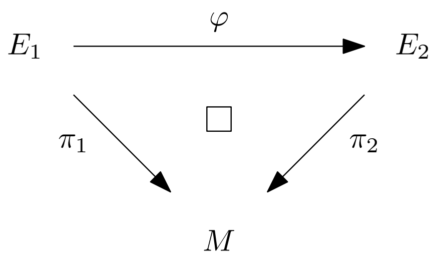


(A diagram with a square $\square$ in the center of a closed path means that it commutes, i.e. is does not matter which path I chose I will always get the same element, e.g. $\psi = \pi \circ \pi_1^{-1}$ and so forth).

Exemplum 2.1
Consider $M = S^2$ and $TS^2 = \bigsqcup_{p \in S^2} T_pS^2$, where T_pS^2 is the tangent space at $p \in S^2$. Then $\pi : TS^2 \to S^2$ is a vector bundle over $\mathbb{R}$ of rank 2.



Definitio 2.7
Two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ are called **isomorphic** to each other if there exists a diffeomorphism $\varphi : E_1 \to E_2$ such that



Definitio 2.8
A vector bundle $\pi : E \to M$ of rank k is called **trivial** if it is isomorphic to the trivial bundle $M \times \mathbb{R}^k$.

Definitio 2.9
Given a vector bundle $\pi : E \to M$, a **section** $s : M \to E$ is a smooth map such that $\pi \circ s = \mathrm{Id}_M$.

Notatio 2.2
This definition reflects our intuition that a tangent vector at p should be an element of T_pM. So the condition of $\pi \circ s = \mathrm{Id}_M$ says that what every I associated to $p \in M$ should be an element of the fiber $\pi^{-1}(p)$ of p.

Exemplum 2.2
Let M be a manifold and consider the vector bundle $\pi : TM \to M$. The sections of this vector bundle are the ordinary vector fields $\Gamma(M)$ of M.

Exemplum 2.3
Let M be a manifold. Define the **cotangent bundle** T^*M as

$$T^*M = \bigsqcup_{p \in M} T_p^*M = \bigsqcup_{p \in M} (T_pM)^*. \tag{6}$$

Given a function $f \in C^\infty(M)$. Then $\mathrm{d}f$ is a section of T^*M for all $p \in M$, i.e.

$$\mathrm{d}f\big|_p : T_pM \longrightarrow \mathbb{R}. \tag{7}$$

Notatio 2.3
In Hamiltonian mechanics a manifold $M \subset \mathbb{R}^n$ for some n will be the **configuration space** and T^*M will be the corresponding **phase space**.

3 Symplectic manifolds

3.1 Symplectic vector spaces

Before we can go to symplectic manifolds we will consider the case of a symplectic vector space. After this is done we can raise our considerations fo the case of a manifold M since T_pM has the structure of an $\mathbb{R}$-vector space for every $p \in M$.

Definitio 3.1
Let V be a $\mathbb{R}$-vector space. A non-degenrate and alternating two-form $\omega \in \Lambda^2(V)$ is called a **symplectic structure** on V and the pair (V, ω) is called a **symplectic vector space**.

Exemplum 3.1
Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ and $\omega_0 := \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i$. Then ω_0 is due to the properties of $\wedge$ alternating and linear since every $\mathrm{d}x_i$ and $\mathrm{d}y_i$ are linear.

Propositio 3.1
Every symplectic vector space is even dimensional.

Propositio 3.2
For every symplectic vector space (V, ω) of dimension $2n$ there exists a basis $\{e_i, f_i\}$ with $i \in \{1, \dots, n\}$ such that

$$\omega_0 = \sum_{i=1}^n f_i^* \wedge e_i^*. \tag{8}$$

*This form ω_0 is called the **canonical form** or **standard form** on V.*

Definitio 3.2
Let (V, ω) and (V', ω') be two symplectic vector spaces. Then (V, ω) and (V', ω') are called **symplectomorphic** to each other if there exists a linear isomorphism $\varphi : V \to V'$ such that $\varphi^*\omega' = \omega$. Then φ is called a **symplectic map**.

Exemplum 3.2
Consider $(\mathbb{R}^{2n}, \omega_0)$ and write $v, w \in V$ with respect to the standard basis as $v = (v_1, \dots, v_{2n})$ and $w = (w_1, \dots, w_{2n})$. Then

$$\omega_0(v, w) = v^T J_0 w, \tag{9}$$

where

$$J_0 = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \in \mathrm{SL}(2n, \mathbb{R}). \tag{10}$$

Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear isomorphism, written as a matrix $A \in \mathrm{GL}(2n, \mathbb{R})$. We want to know which A maps ω_0 to ω_0. Then we can insert:

$$A^*\omega_0(v, w) = \omega_0(Av, Aw) \overset{!}{=} \omega_0(v, w) \quad \forall v, w \in \mathbb{R}^{2n}. \tag{11}$$

Writing this as in (9) we get

$$v^T A^T J_0 A w \overset{!}{=} v^T J_0 w \quad \forall v, w \in \mathbb{R}^{2n}. \tag{12}$$

Therefore the following definition makes sense:

Definitio 3.3
A matrix $A \in \mathrm{Mat}(\mathbb{R}, 2n \times 2n)$ is called **symplectic** if

$$A^T J_0 A = J_0. \tag{13}$$

Lemma 3.3
*The set of matrices $A \in \mathrm{M}(\mathbb{R}, 2n \times 2n)$ satisfying (13) forms with respect to ordinary matrix multiplication a group, called the **symplectic group** and is denoted by $\mathrm{Sp}(2n)$. Moreover $A \in \mathrm{Sp}(2n)$ implies that $A^T \in \mathrm{Sp}(2n)$.*

Propositio 3.4
For $A \in \mathrm{Sp}(2n)$ we have $\det A = 1$ for all $n \in \mathbb{N}$.

3.2 Symplectic manifolds

Definitio 3.4
Let $\Omega^k(M)$ denote the space of k-forms over a manifold M, i.e. $\omega \in \Omega^k(M)$ means that $\omega\big|_p =: \omega_p \in \Lambda^2(T_pM)$ varies smoothly with $p \in M$.

Definitio 3.5
Let M be a manifold. Then $\omega \in \Omega^2(M)$ is called **non-degenerate** if ω_p is non-degenerate for all $p \in M$. In other words: (T_pM, ω_p) is a symplectic vector space for all $p \in M$.

Definitio 3.6
Let M be a manifold and $\omega \in \Omega^2(M)$ be a non-degenerate and closed, i.e. $\mathrm{d}\omega = 0$, 2-form on M. Then ω is called a **symplectic form** on M and the pair (M, ω) is called a **symplectic manifold**.

Notatio 3.1 (i) M has to be even dimensional since $\dim T_pM = \dim M$ and T_pM is by Propositio 3.1 even dimensional. Physically one could translate this to the property that to every coordinate q there has to exists an associated momenta p.

(ii) M has to be orientable since $\omega^n \neq 0$ defines a volume form on M.

Theorema 3.5
(Darboux). Let (M, ω) be a symplectic manifold of dimension $2n$. Then for every $p \in M$ there exists a neighborhood $U \subset M$ of p and a symplectomorphism $\varphi : U \to V$, where $V \subset \mathbb{R}^{2n}$ is a neighborhood of $\varphi(p)$, such that $\varphi^*\omega = \omega_0$.

3.3 Back to classical mechanics

In classical mechanics we normally consider the positions of a particle $(q_1, \ldots, q_n)$ to together with their momenta $(p^1, \ldots, p_n)$. This done by letting $q \in M$, where M is a manifold and denotes the configuration space and $p \in T^*M$ since p depends in general on q and computes n different real numbers. Therefore T^*M is our phase space of the system.

Of central importance in physics is the law of **conservation of energy**, i.e. there exists a function $H \in C^\infty(T^*M)$ such that H is conserved along the motion of a particle in phase space. Let $\gamma : \mathbb{R} \in T^*M$ with $\gamma(t) = (q(t), p(t))$ be such a solution, then $H(\gamma(t)) = \mathrm{const}$ for all $t \in \mathbb{R}$. Let us generalize this concept to symplectic manifolds:

Definitio 3.7
Let (M, ω) be a symplectic manifold and $\iota_X : \Gamma(TM) \times \Omega^k(M) \to \Omega^{k-1}(M$ be the insertion map that plugs in a vector field X into the first component of a k-form. A vector field $X \in \Gamma(M)$ is called

(a) **symplectic** if $\iota_X \omega$ is closed. The set (M, ω, X_H) is called a **symplectic system**.

(b) **Hamiltonian** if $\iota_X \omega$ is exact, i.e. there exists a function $H \in C^\infty(M)$ such that

$$\mathrm{d}H = \iota_{X_H} \omega. \tag{14}$$

Then H is called the **energy function** of X. The set (M, ω, X_H) is called a **Hamiltonian system**.

Propositio 3.6
Let $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ be canonical coordinates for ω, i.e. $\omega = \omega_0 = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$. Let X_H be a Hamiltonian vector field with energy function H. Then in these coordinates

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}. \tag{15}$$

Thus $(q(t), p(t))$ is an integral curve of X_H if and only if Hamilton's equations of motion hold:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \forall i \in \{1, \ldots, n\}. \tag{16}$$

Proof. Let X_H be defined by the formula (15). We then have to verify (14). By construction we have

$$\iota_{X_H} \mathrm{d}q^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \iota_{X_H} \mathrm{d}p_i = -\frac{\partial H}{\partial q^i} \tag{17}$$

and therefore

$$\iota_{X_H} \omega = \sum_i \left(\iota_{X_H} \mathrm{d}q^i \right) \wedge \mathrm{d}p_i - \mathrm{d}q^i \wedge \iota_{X_H} \mathrm{d}p_i = \sum_i \frac{\partial H}{\partial p_i} \mathrm{d}p_i + \frac{\partial H}{\partial q^i} \mathrm{d}q^i = \mathrm{d}H. \tag{18}$$

□

Propositio 3.7
Let (M, ω, X_H) be a Hamiltonian system and let $\gamma(t)$ be an integral curve for X_H. Then $H(\gamma(t)) = \mathrm{const}$ for all $t \in \mathbb{R}$.

Proof. By the chain rule and (14) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\gamma(t)) = \mathrm{d}H(\gamma(t)) \cdot \gamma'(t) = \mathrm{d}H(\gamma(t)) \cdot X_H(\gamma(t)) \tag{19}$$

$$= \omega \left(X_H(\gamma(t)), X_H(\gamma(t)) \right) = 0 \tag{20}$$

since ω is alternating. □

Notatio 3.2
Not every symplectic system is globally Hamiltonian. Think for example of a particle with friction. But every symplectic system is locally Hamiltonian by the Poincaré lemma.

Since ω is non-degenerate we can use to map vector fields to forms and vice versa. Let $f \in C^\infty$. Then X_f is defined via the equation

$$\mathrm{d}f(Y) = \iota_{X_f}\omega(Y) = \omega(X_f, Y) \quad \forall Y \in \Gamma(M). \tag{21}$$

Definitio 3.8
Let (M, ω) be a symplectic manifold and $f, g \in C^\infty(M)$. The **Poisson bracket** of f and g is the function

$$\{f, g\} := -\iota_{X_f}\iota_{X_g}\omega = \omega(X_f, X_g). \tag{22}$$

Propositio 3.8
In canonical coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ *we have*

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right). \tag{23}$$

Proof. We can compute

$$\{f, g\} = \mathrm{d}f(X_g) = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\mathrm{d}q^i + \frac{\partial f}{\partial p_i}\mathrm{d}p_i \right) \left(\sum_{j=1}^{n} \frac{\partial g}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial g}{\partial q^j}\frac{\partial}{\partial p_j} \right) \tag{24}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right). \tag{25}$$

□

4 GROMOV's theorem

In this part we want to explore to so-called Gromov's theorem from 1985 and its connections to the uncertainty principle. To do so we need some definitions:

Definitio 4.1
Let (U, ω) and $(\mathbb{R}^{2n}, \omega')$ be two symplectic manifolds. A symplectic map $\varphi : U \to \mathbb{R}^{2n}$ is called a **symplectic embedding** if it is injective.

Definitio 4.2
Let $(\mathbb{R}^{2n}, \omega)$ be a symplectic manifold. Then $B^{2n}(r) \subset \mathbb{R}^{2n}$ defined as

$$B^{2n}(r) = \left\{ (x, y) \in \mathbb{R}^{2n} \mid \sum_{i=1}^{n} x_i^2 + y_i^2 < r^2 \right\} \tag{26}$$

is called a **symplectic ball** of dimension $2n$ with radius $r \in \mathbb{R}^+$ and

$$Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n} \tag{27}$$

is called **sympletic cylinder** of dimension $2n$ with radius $r \in \mathbb{R}^+$.

Theorema 4.1
*(*GROMOV*'s non-sequeezing theorem). If there exists a symplectic embedding*

$$B^{2n}(r) \hookrightarrow Z^{2n}(R), \tag{28}$$

then

$$r \leq R. \tag{29}$$

Notatio 4.1
LALOND and MCDUFF improved this theorem to arbitrary sympletic manifolds (M, ω) of dimension $2n$. If there exists a symplectic embedding

$$B^{2n+2}(r) \hookrightarrow B^2(R) \times M \tag{30}$$

then $r \leq R$.

Proof. Even if the statements sound rather simple the proof of this is quite complicated and uses the techniques of so-called J-holomorphic curves. □

Notatio 4.2
This is an analog of the uncertainty principle. Let $a = (q^1, \ldots, q^n, p_1, \ldots, p_n) \in M$ be a point in phase space with radial uncertainty $\varepsilon \in \mathbb{R}^+$, denoted by U, i.e. U is a symplectic ball around a with radius ε. Then there is no dynamical evolution or choice of coordinates, i.e. a symplectomorphism, such that the uncertainty of two conjugate variables (q^i, p_i) can be reduced lower then ε. Note that this is so far only a classical statement.

GROMOV's theorem gave rise to the following definition:

Definitio 4.3
A **symplectic capacity** is a map that assigns to every symplectic manifold (M, ω) a non-negative but possibly infinite number $c(M, \omega)$ such that

(a) If there exists a symplectic embedding $(M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$ and $\dim M_1 = \dim M_2$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$.

(b) For every $\lambda \in \mathbb{R}$ one has $c(M, \lambda\omega) = |\lambda| c(M, \omega)$.

(c) $c(B^{2n}(1), \omega_0) > 0$ and $c(Z^{2n}(1), \omega_0) < \infty$.

Notatio 4.3
Condition (c) of the definition excludes volume to be a capacity. Moreover the requirement that $c(Z^{2n}(1), \omega_0)$ to be finite means that capacities are 2-dimensional invariants.

Notatio 4.4
The existence of a capacity c with

$$c(B^{2n}, \omega_0) = c(Z^{2n}(1), \omega_0) = \pi \tag{31}$$

is equivalent to Gromov's theorem (4.1).

5 Quantization

Roughly speaking quantization is a map that takes classical observables to symmetric operators over a certain HILBERT space. Of course this has to be made more precise. The notion of the corresponding Hilbert space can be obtained by equivalence classes over square integrable functions of a configuration space manifold Q:

Definitio 5.1
Let Q be a manifold and consider the set of all pair (f, μ), where μ is natural measure on Q and $f \in C^\infty(Q, \mathbb{C})$ is a complex measurable function such that $\int_Q |f|^2 \mathrm{d}\mu < \infty$, i.e. $f \in L^2(Q, \mu)$. Two pairs (f, μ) and (g, η) will be called **equivalent** provided that $f\sqrt{\mathrm{d}\mu/\mathrm{d}\eta} = g$. We denote the equivalence class of (f, μ) by $f\sqrt{\mathrm{d}\mu}$. Let $\mathcal{H}(Q)$ be the set of all such equivalence classes. Pick a natural measure μ. Then the map

$$\begin{array}{rccc} U_\mu : & L^2(Q, \mu) & \longrightarrow & \mathcal{H}(Q) \\ & f & \longmapsto & f\sqrt{\mathrm{d}\mu} \end{array}$$

is a bijection.

Definitio 5.2
Let Q be a manifold. A function $f \in C^\infty(Q)$ is called a **classical configuration observable** and the corresponding **quantum position observable** is a operator O_f on $\mathcal{H}(Q)$ such that

$$O_f(g\sqrt{\mathrm{d}\mu}) = f \cdot g\sqrt{\mathrm{d}\mu}, \tag{32}$$

i.e. O_f is the multiplication by f. Define the **classical momentum observable** $P(X)$ on T^*Q associated to vector field X on Q as

$$\begin{array}{rccc} P(X) : & T^*Q & \longrightarrow & \mathbb{R} \\ & \alpha & \longmapsto & \alpha(X) \end{array}$$

We shall call the operator $\widetilde{X}$ the corresponding **quantum momentum observable**.

Propositio 5.1
Let Q be a finite-dimensional manifold with HILBERT *space $\mathcal{H}(Q)$. Let $X, Y \in \Gamma(Q)$ be vector fields on Q and let $f, g \in C^\infty(Q)$. Then we have*

(i) $[\widetilde{X}, \widetilde{Y}] = -\mathrm{i}\widetilde{[X,Y]}$.

(ii) $[O_f, O_g] = 0$.

(iii) $[O_f, \tilde{X}] = \mathrm{i}O_{X(f)}$.

Definitio 5.3
A **full quantization** of a manifold Q is a map, denoted by a hat $\widehat{}$, taking classical observables f to self-adjoint operators $\widehat{f}$ on Hilbert space $\mathcal{H}(Q)$ such that:

(i) $\widehat{(f+g)} = \widehat{f} + \widehat{g}$.

(ii) $\widehat{\lambda f} = \lambda \widehat{f}$ for all $\lambda \in \mathbb{R}$.

(iii) $\widehat{\{f,g\}} = -\mathrm{i}[\widehat{f}, \widehat{g}]$.

(iv) $\widehat{1} = \mathrm{Id}$, where 1 is the function that is constantly 1 on Q and Id is the identity on $\mathcal{H}(Q)$.

(v) $\widehat{q}^i$ and $\widehat{p}_j$ act irreducibly on $\mathcal{H}(Q)$.

Notatio 5.1
By a theorem from Stone-von Neumann the condition (v) really means that we can take $\mathcal{H} = L^2(\mathbb{R}^n)$, where n is the dimension of Q, and that $\widehat{q}^i$ and $\widehat{p}_j$ are given by $\widehat{q}^i = O_{q^i}$ and $\widehat{p}_j = -\mathrm{i}\frac{\partial}{\partial q^j}$; that is, the SCHRÖDINGER representation.

In order to allow spin one has to relax the last condition to:

(v') The position and momentum operators are represented by a direct sum of **finitely many** copies of the Schrödinger representation. More precisely, we are asking that $\mathcal{H}(Q)$ can be realized as the space of L^2 functions from $\mathbb{R}^n$ to a d-dimensional Hilbert space $\mathfrak{H}_d$ with $d < \infty$, so that

$$\widehat{q}^i \phi(x) = q^i \phi(x) \quad \text{and} \quad \widehat{p}_j \phi(x) = -\mathrm{i}\frac{\partial \phi}{\partial x^j}(x) \tag{33}$$

Does such a full quantization exists? The answer is no! To see this we follow A. Joseph from 1970:

Theorema 5.2
Let U be the Lie algebra of real-valued polynomials of $\mathbb{R}^{2n}$, where the bracket is given by the Poisson bracket. Let $H = L^2(\mathbb{R}^n, \mathfrak{H}_d)$. Then there is ***no*** *map $f \mapsto \widehat{f}$ from U to the self-adjoint operators on H that has the following properties:*

(i) For each finite subset $S \subset U$ there is dense subspace $D_S \subset H$ such that for all $f \in S$: $D_S \subset D_{\widehat{f}}$ and $\widehat{f} D_S \subset D_S$.

(ii) $\widehat{(f+g)} = \widehat{f} + \widehat{g}$ *pointwise on* D_S *if* $f, g \in S$.

(iii) $\widehat{(\lambda f)} = \lambda \widehat{f}$ *for* $\lambda \in \mathbb{R}$.

(iv) $\widehat{\{f,g\}} = -\mathrm{i}[\widehat{f}, \widehat{g}]$ *on* D_S.

(v) $\widehat{1} = \mathrm{Id}$.

(vi) $\widehat{q}^i$ *is multiplication by* q^i *and* $\widehat{p}_j = -\mathrm{i}\frac{\partial}{\partial q^j}$.

Notatio 5.2
Since we have seen that quantization with finite representations are not working there is another procedure, the so-called **pre-quantization**, that is a quantization satisfying (i)-(iv) but not (v) or (v') of definition 5.3.

Definitio 5.4
Let M be a manifold. A vector bundle $\pi : Q \to M$ is called a **principal circle bundle** if every fiber $\pi^{-1}(p)$ for $p \in M$ is a circle S^1 and there is a consistent action $S^1 \times Q \to Q$, which is just multiplication on each fiber, i.e. $M = {Q}/{S^1}$.

Definitio 5.5
Let (P, ω) be a symplectic manifold. We say that (P, ω) is **quantizable** if and only if there is a principle circle bundle $\pi : Q \to P$ over P and a 1-form α on Q such that

(a) α is invariant under the action of S^1.

(b) $\pi^*\omega = \mathrm{d}\alpha$.

Q is then called the **quantizing manifold** of P.

Exemplum 5.1(a) If ω is exact, then P is quantizable, e.g. $P = T^*M$. If P is simply connected, then Q is unique, i.e. if $\omega = \mathrm{d}\theta$ we can let $Q = P \times S^1$ and $\alpha = \theta + \hbar\, \mathrm{d}s$ for some suitable $s \in C^\infty(Q)$ and $\hbar$ is a constant, ultimately to be defined with **Plank's constant**.

(b) For more details view Souriau 1970: Consider the two sphere S^2 in $\mathbb{R}^3$ with radius mK with the symplectic form

$$\omega = \frac{-\mathrm{i}}{\sqrt{8me}} \cdot \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_3}, \tag{34}$$

where m is the mass, K is the attractive constant and e is the energy. Then S^2 is quantizable if and only if $e = -2\pi^2 mK^2/N^2$ for an integer $n \in \mathbb{N}$.