These notes are a short sum up about two talks that I gave in August and September 2015 at the University of Cologne in my workgroup seminar by Prof. Dr. Claus Kiefer. The intention of those talks were to give a tiny introduction into fascination and often inaccessible topic of symplectic geometry, which is a generalization of the famous theory Hamilton mechanics, which has a big big success in physics. Without focusing too much on being mathematical rigorously I tried to be as precise as possible. Of course I also could not cover all introduced concepts in great details. For example, the part about vector bundles is rather short and maybe unsatisfactory. For further reading I refer the following references:


Every kind of comment or improvement proposal is highly welcome. Just contact me under my email: fasse@thp.uni-koeln.de.

Current version: September 1, 2015.
1 Motivation

Consider a particle moving in $\mathbb{R}^3$. Then the phase space $M$ consists of all positions $\vec{q} \in \mathbb{R}^3$ and all momenta $\vec{p} = m\dot{\vec{q}} \in \mathbb{R}^3$.

Exemplum 1.1 (a) For a free particle: $M \simeq \mathbb{R}^6$.
(b) For $n$ free particles: $M \simeq \mathbb{R}^{6n}$.
(c) If the particle is forced to move on a surface $S \subset \mathbb{R}^3$ with projection $\pi : M \to S$, then $\pi^{-1}(q) \subset \mathbb{R}^2$ with $q \in S$. But $M \ncong S \otimes \mathbb{R}^2$, e.g. let $S$ be the Möbius strip embedded in $\mathbb{R}^3$.

Consider a non-free particle in $\mathbb{R}^3$ with $M = \mathbb{R}^6$ with conservative force field $\vec{F}$ and corresponding potential $V$. Then

$$H(\vec{q}, \vec{p}) = E + V$$

is the Hamiltonian of the system, where $E$ denotes the Kinetic energy. $H$ gives the total energy of the system and should therefore be conserved along the motion of the particle, i.e. let $\gamma(t) = (\vec{q}(t), \vec{p}(t))$ be the trajectory of $a$. Then $H(\gamma(t)) = \text{const}$ for all $t$. The equations of motion are

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \text{and} \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i.$$  

(2)

How can we generalize this concept and understand more about the geometry of phase space $M$?

2 Mathematical preliminaries

Before we can start with symplectic geometry we have to set up some definitions and have to fix notation.

2.1 Tensor algebra

Definition 2.1
Let $V$ be a vector space. A (covariant) $k$-tensor $A : V^k \to \mathbb{R}$ is a function that is linear in every argument. The space of $k$-tensors over $V$ is a vector space denoted by $T^0_k(V)$.

Definition 2.2
A $k$-form $\mu \in T^0_k(V)$ is called alternating if for every $s \in S_k$, where $S_k$ denotes the permutation group of $k$ elements, we have

$$\mu(v_1, \ldots, v_k) = \text{sgn}(s)\mu(v_{s(1)}, \ldots, v_{s(k)}).$$

(3)

The set of all alternating $k$-forms over $V$ will be denoted by $\Lambda^k(V)$. 

2
Definizione 2.3
The direct sum $\bigoplus_{i=1}^{\infty} \Lambda^i(V)$ together with its structure as a $\mathbb{R}$-vector space and the multiplication by the wedge product $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is called the Grassmann algebra of $V$.

Notazione 2.1
Let $V$ be a $n$-dimensional vector space. Then $\dim \Lambda^k = \binom{n}{k}$.

Definizione 2.4
The non-zero elements of $\Lambda^n(V)$ are called volume elements. If $\omega_1, \omega_2 \in \Lambda^n(V)$ we say that $\omega_1 \sim \omega_2$ if there exists a $c \in \mathbb{R}^+$ such that $\omega_1 = c \omega_2$. An element of $V/\sim$ is called an orientation on $V$.

Definizione 2.5
A 2-form $\mu \in \Lambda^2(V)$ is called non-degenerate if
\[ \mu(v, w) = 0 \quad \forall v \in V \quad \Rightarrow \quad w = 0 \] \quad (4)
and
\[ \mu(v, w) = 0 \quad \forall w \in V \quad \Rightarrow \quad v = 0. \] \quad (5)

2.2 Vector bundles

Definizione 2.6
A (real) vector bundle over $\mathbb{R}$ of rank $k$ is a triple $\pi : E \to M$ with
(a) $E$ and $M$ are $C^\infty$-manifolds.
(b) The map $\pi : E \to M$ is a surjection.
(c) For every $p \in M$ the preimage $\pi^{-1}(p)$ is a $\mathbb{R}$-vector space of dimension $k$.
(d) For every points $p \in M$ there exists a neighborhood $U \subset M$ of $p$ and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi} & U \times \mathbb{R}^k \\
\downarrow \pi & & \downarrow \pi_1 \\
U & & \\
\end{array}
\]

(A diagram with a square $\square$ in the center of a closed path means that it commutes, i.e. it does not matter which path I chose I will always get the same element, e.g. $\psi = \pi \circ \pi_1^{-1}$ and so forth).
Exemplum 2.1
Consider \( M = S^2 \) and \( TS^2 = \bigsqcup_{p \in S^2} T_pS^2 \), where \( T_pS^2 \) is the tangent space at \( p \in S^2 \). Then \( \pi : TS^2 \to S^2 \) is a vector bundle over \( \mathbb{R} \) of rank 2.

\[ \begin{array}{c}
S^n \\
\hline
\hline
T_pS^n
\end{array} \]

Definitio 2.7
Two vector bundles \( \pi_1 : E_1 \to M \) and \( \pi_2 : E_2 \to M \) are called isomorphic to each other if there exists a diffeomorphism \( \varphi : E_1 \to E_2 \) such that

\[ E_1 \xrightarrow{\varphi} E_2 \]

\[ \pi_1 \quad \varnothing \quad \pi_2 \]

\[ M \]

Definitio 2.8
A vector bundle \( \pi : E \to M \) of rank \( k \) is called trivial if it is isomorphic to the trivial bundle \( M \times \mathbb{R}^k \).

Definitio 2.9
Given a vector bundle \( \pi : E \to M \), a section \( s : M \to E \) is a smooth map such that \( \pi \circ s = \text{Id}_M \).

Notaio 2.2
This definition reflects our intuition that a tangent vector at \( p \) should be an element of \( T_pM \). So the condition of \( \pi \circ s = \text{Id}_M \) says that what every I associated to \( p \in M \) should be an element of the fiber \( \pi^{-1}(p) \) of \( p \).

Exemplum 2.2
Let \( M \) be a manifold and consider the vector bundle \( \pi : TM \to M \). The sections of this vector bundle are the ordinary vector fields \( \Gamma(M) \) of \( M \).

Exemplum 2.3
Let \( M \) be a manifold. Define the cotangent bundle \( T^*M \) as

\[ T^*M = \bigsqcup_{p \in M} T^*_pM = \bigsqcup_{p \in M} (T_pM)^*. \] (6)
Given a function \( f \in C^\infty(M) \). Then \( df \) is a section of \( T^*M \) for all \( p \in M \), i.e.
\[
df_p : T_pM \to \mathbb{R}.
\] (7)

**Notatio 2.3**
In Hamiltonian mechanics a manifold \( M \subset \mathbb{R}^n \) for some \( n \) will be the **configuration space** and \( T^*M \) will be the corresponding **phase space**.

## 3 Symplectic manifolds

### 3.1 Symplectic vector spaces

Before we can go to symplectic manifolds we will consider the case of a symplectic vector space. After this is done we can raise our considerations for the case of a manifold \( M \) since \( T_pM \) has the structure of an \( \mathbb{R} \)-vector space for every \( p \in M \).

**Definition 3.1**
Let \( V \) be a \( \mathbb{R} \)-vector space. A non-degenerate and alternating two-form \( \omega \in \Lambda^2(V) \) is called a **symplectic structure** on \( V \) and the pair \( (V, \omega) \) is called a **symplectic vector space**.

**Example 3.1**
Consider \( \mathbb{R}^{2n} \) with coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) and \( \omega_0 := \sum_{i=1}^n dx_i \wedge dy_i \). Then \( \omega_0 \) is due to the properties of \( \wedge \) alternating and linear since every \( dx_i \) and \( dy_i \) are linear.

**Proposition 3.1**
Every symplectic vector space is even dimensional.

**Proposition 3.2**
For every symplectic vector space \( (V, \omega) \) of dimension \( 2n \) there exists a basis \( \{e_i, f_i\} \) with \( i \in \{1, \ldots, n\} \) such that
\[
\omega_0 = \sum_{i=1}^n f_i^* \wedge e_i^*.
\] (8)

This form \( \omega_0 \) is called the **canonical form** or **standard form** on \( V \).

**Definition 3.2**
Let \( (V, \omega) \) and \( (V', \omega') \) be two symplectic vector spaces. Then \( (V, \omega) \) and \( (V', \omega') \) are called **symplectomorphic** to each other if there exists a linear isomorphism \( \varphi : V \to V' \) such that \( \varphi^* \omega' = \omega \). Then \( \varphi \) is called a **symplectic map**.

**Example 3.2**
Consider \( \mathbb{R}^{2n}, \omega_0 \) and write \( v, w \in V \) with respect to the standard basis as \( v = (v_1, \ldots, v_{2n}) \) and \( w = (w_1, \ldots, w_{2n}) \). Then
\[
\omega_0(v, w) = v^T J_0 w,
\] (9)

where
\[
J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \in \text{SL}(2n, \mathbb{R}).
\] (10)

Let \( A : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) be a linear isomorphism, written as a matrix \( A \in \text{GL}(2n, \mathbb{R}) \). We want to know which \( A \) maps \( \omega_0 \) to \( \omega_0 \). Then we can insert:
\[
A^* \omega_0(v, w) = \omega_0(Av, Aw) \equiv \omega_0(v, w) \quad \forall v, w \in \mathbb{R}^{2n}.
\] (11)
Writing this as in (9) we get
\[ v^T A^T J_0 A w = \frac{1}{\lambda} v^T J_0 w \quad \forall v, w \in \mathbb{R}^{2n}. \] (12)

Therefore the following definition makes sense:

**Definition 3.3**
A matrix \( A \in \text{Mat}(\mathbb{R}, 2n \times 2n) \) is called symplectic if
\[ A^T J_0 A = J_0. \] (13)

**Lemma 3.3**
The set of matrices \( A \in \text{M}(\mathbb{R}, 2n \times 2n) \) satisfying (13) forms with respect to ordinary matrix multiplication a group, called the symplectic group and is denoted by \( \text{Sp}(2n) \). Moreover \( A \in \text{Sp}(2n) \) implies that \( A^T \in \text{Sp}(2n) \).

**Proposition 3.4**
For \( A \in \text{Sp}(2n) \) we have \( \det A = 1 \) for all \( n \in \mathbb{N} \).

### 3.2 Symplectic manifolds

**Definition 3.4**
Let \( \Omega^k(M) \) denote the space of \( k \)-forms over a manifold \( M \), i.e. \( \omega \in \Omega^k(M) \) means that \( \omega|_p = \omega_p \in \Lambda^k(T_p M) \) varies smoothly with \( p \in M \).

**Definition 3.5**
Let \( M \) be a manifold. Then \( \omega \in \Omega^2(M) \) is called non-degenerate if \( \omega_p \) is non-degenerate for all \( p \in M \). In other words: \( (T_p M, \omega_p) \) is a symplectic vector space for all \( p \in M \).

**Definition 3.6**
Let \( M \) be a manifold and \( \omega \in \Omega^2(M) \) be a non-degenerate and closed, i.e. \( d \omega = 0 \), 2-form on \( M \). Then \( \omega \) is called a symplectic form on \( M \) and the pair \( (M, \omega) \) is called a symplectic manifold.

**Notation 3.1**
(i) \( M \) has to be even dimensional since \( \dim T_p M = \dim M \) and \( T_p M \) is by Proposition 3.1 even dimensional. Physically one could translate this to the property that to every coordinate \( q \) there has to exist an associated momenta \( p \).

(ii) \( M \) has to be orientable since \( \omega^n \neq 0 \) defines a volume form on \( M \).

**Theorem 3.5**
(Darboux). Let \( (M, \omega) \) be a symplectic manifold of dimension \( 2n \). Then for every \( p \in M \) there exists a neighborhood \( U \subset M \) of \( p \) and a symplectomorphism \( \varphi : U \to V \), where \( V \subset \mathbb{R}^{2n} \) is a neighborhood of \( \varphi(p) \), such that \( \varphi^* \omega = \omega_0 \).

### 3.3 Back to classical mechanics

In classical mechanics we normally consider the positions of a particle \((q_1, \ldots, q_n)\) together with their momenta \((p^1, \ldots, p_n)\). This done by letting \( q \in M \), where \( M \) is a manifold and denotes the configuration space and \( p \in T^* M \) since \( p \) depends in general on \( q \) and computes \( n \) different real numbers. Therefore \( T^* M \) is our phase space of the system.
Of central importance in physics is the law of conservation of energy, i.e., there exists a function \( H \in C^\infty(T^*M) \) such that \( H \) is conserved along the motion of a particle in phase space. Let \( \gamma : \mathbb{R} \to T^*M \) with \( \gamma(t) = (q(t), p(t)) \) be such a solution, then \( H(\gamma(t)) = \text{const} \) for all \( t \in \mathbb{R} \). Let us generalize this concept to symplectic manifolds:

**Definition 3.7**

Let \((M, \omega)\) be a symplectic manifold and \( \iota_X : \Gamma(TM) \times \Omega^k(M) \to \Omega^{k-1}(M) \) be the insertion map that plugs in a vector field \( X \) into the first component of a \( k \)-form. A vector field \( X \in \Gamma(M) \) is called

(a) **symplectic** if \( \iota_X \omega \) is closed. The set \((M, \omega, X_H)\) is called a symplectic system.

(b) **Hamiltonian** if \( \iota_X \omega \) is exact, i.e., there exists a function \( H \in C^\infty(M) \) such that

\[
\frac{d}{dt} H = \iota_{X_H} \omega.
\]

Then \( H \) is called the energy function of \( X \). The set \((M, \omega, X_H)\) is called a Hamiltonian system.

**Proposition 3.6**

Let \((q^1, \ldots, q^n, p_1, \ldots, p_n)\) be canonical coordinates for \( \omega \), i.e., \( \omega = \omega_0 = \sum_i dq^i \wedge dp_i \). Let \( X_H \) be a Hamiltonian vector field with energy function \( H \). Then in these coordinates

\[
X_H = \frac{\partial H}{\partial p_i} \partial_{q^i} - \frac{\partial H}{\partial q_i} \partial_{p_i}.
\]

Thus \((q(t), p(t))\) is an integral curve of \( X_H \) if and only if Hamilton’s equations of motion hold:

\[
\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \forall i \in \{1, \ldots, n\}.
\]

**Proof.** Let \( X_H \) be defined by the formula (15). We then have to verify (14). By construction we have

\[
\iota_{X_H} dq^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \iota_{X_H} dp_i = -\frac{\partial H}{\partial q^i}
\]

and therefore

\[
\iota_{X_H} \omega = \sum_i \left( \iota_{X_H} dq^i \right) \wedge dp_i - dq^i \wedge \iota_{X_H} dp_i = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i = dH.
\]

**Proposition 3.7**

Let \((M, \omega, X_H)\) be a Hamiltonian system and let \( \gamma(t) \) be an integral curve for \( X_H \). Then \( H(\gamma(t)) = \text{const} \) for all \( t \in \mathbb{R} \).

**Proof.** By the chain rule and (14) we get

\[
\frac{d}{dt} H(\gamma(t)) = dH(\gamma(t)) \cdot \gamma'(t) = dH(\gamma(t)) \cdot X_H(\gamma(t))
\]

\[
= \omega(X_H(\gamma(t)), X_H(\gamma(t))) = 0
\]

since \( \omega \) is alternating.
Notatio 3.2
Not every symplectic system is globally Hamiltonian. Think for example of a particle with friction. But every symplectic system is locally Hamiltonian by the Poincaré lemma.

Since $\omega$ is non-degenerate we can use to map vector fields to forms and vice versa. Let $f \in C^\infty$. Then $X_f$ is defined via the equation

$$df(Y) = \iota_{X_f} \omega(Y) = \omega(X_f, Y) \quad \forall Y \in \Gamma(M).$$

(21)

Definitio 3.8
Let $(M, \omega)$ be a symplectic manifold and $f, g \in C^\infty(M)$. The Poisson bracket of $f$ and $g$ is the function

$$\{f, g\} := -\iota_{X_f} \iota_{X_g} \omega = \omega(X_f, X_g).$$

(22)

Propositio 3.8
In canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ we have

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

(23)

Proof. We can compute

$$\{f, g\} = df(X_g) = \left( \sum_{i=1}^n \frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i \right) \left( \sum_{j=1}^n \frac{\partial g}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial g}{\partial q^j} \frac{\partial}{\partial p_j} \right)$$

(24)

$$= \sum_{i=1}^n \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right).$$

(25)
4 GROMOV’S THEOREM

In this part we want to explore so-called Gromov’s theorem from 1985 and its connections to the uncertainty principle. To do so we need some definitions:

**Definition 4.1**
Let $(U, \omega)$ and $(\mathbb{R}^{2n}, \omega')$ be two symplectic manifolds. A symplectic map $\varphi : U \to \mathbb{R}^{2n}$ is called a symplectic embedding if it is injective.

**Definition 4.2**
Let $(\mathbb{R}^{2n}, \omega)$ be a symplectic manifold. Then $B^{2n}(r) \subset \mathbb{R}^{2n}$ defined as

\[
B^{2n}(r) = \left\{ (x, y) \in \mathbb{R}^{2n} \mid \sum_{i=1}^{n} x_i^2 + y_i^2 < r^2 \right\}
\]

is called a symplectic ball of dimension $2n$ with radius $r \in \mathbb{R}^+$ and $Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ is called symplectic cylinder of dimension $2n$ with radius $r \in \mathbb{R}^+$.

**Theorem 4.1**
(GROMOV’s non-sequeezing theorem). If there exists a symplectic embedding

\[
B^{2n}(r) \hookrightarrow Z^{2n}(R),
\]

then

\[
r \leq R.
\]

**Notation 4.1**
LALOND and McDUFF improved this theorem to arbitrary symplectic manifolds $(M, \omega)$ of dimension $2n$. If there exists a symplectic embedding

\[
B^{2n+2}(r) \hookrightarrow B^2(R) \times M
\]

then $r \leq R$.

**Proof.** Even if the statements sound rather simple the proof of this is quite complicated and uses the techniques of so-called $J$-holomorphic curves. \qed

**Notation 4.2**
This is an analog of the uncertainty principle. Let $a = (q^1, \ldots, q^n, p_1, \ldots, p_n) \in M$ be a point in phase space with radial uncertainty $\varepsilon \in \mathbb{R}^+$, denoted by $U$, i.e. $U$ is a symplectic ball around $a$ with radius $\varepsilon$. Then there is no dynamical evolution or choice of coordinates, i.e. a symplectomorphism, such that the uncertainty of two conjugate variables $(q^i, p_i)$ can be reduced lower than $\varepsilon$. Note that this is so far only a classical statement.

GROMOV’s theorem gave rise to the following definition:
Definitio 4.3
A symplectic capacity is a map that assigns to every symplectic manifold \((M, \omega)\) a non-negative but possibly infinite number \(c(M, \omega)\) such that

(a) If there exists a symplectic embedding \((M_1, \omega_1) \hookrightarrow (M_2, \omega_2)\) and \(\dim M_1 = \dim M_2\) then \(c(M_1, \omega_1) \leq c(M_2, \omega_2)\).

(b) For every \(\lambda \in \mathbb{R}\) one has \(c(M, \lambda \omega) = |\lambda| c(M, \omega)\).

(c) \(c(B^{2n}(1), \omega_0) > 0\) and \(c(Z^{2n}(1), \omega_0) < \infty\).

Notatio 4.3
Condition (c) of the definition excludes volume to be a capacity. Moreover the requirement that \(c(Z^{2n}(1), \omega_0)\) to be finite means that capacities are 2-dimensional invariants.

Notatio 4.4
The existence of a capacity \(c\) with
\[
c(B^{2n}, \omega_0) = c(Z^{2n}(1), \omega_0) = \pi
\] (31)
is equivalent to Gromov’s theorem \([4.1]\).

5 Quantization

Roughly speaking quantization is a map that takes classical observables to symmetric operators over a certain Hilbert space. Of course this has to be made more precise. The notion of the corresponding Hilbert space can be obtained by equivalence classes over square integrable functions of a configuration space manifold \(Q\):

Definitio 5.1
Let \(Q\) be a manifold and consider the set of all pair \((f, \mu)\), where \(\mu\) is natural measure on \(Q\) and \(f \in C^\infty(Q, \mathbb{C})\) is a complex measurable function such that \(\int_Q |f|^2 d\mu < \infty\), i.e. \(f \in L^2(Q, \mu)\). Two pairs \((f, \mu)\) and \((g, \eta)\) will be called equivalent provided that \(f \sqrt{d\mu/d\eta} = g\). We denote the equivalence class of \((f, \mu)\) by \(f \sqrt{d\mu}\). Let \(\mathcal{H}(Q)\) be the set of all such equivalence classes. Pick a natural measure \(\mu\). Then the map
\[
U_\mu : L^2(Q, \mu) \longrightarrow \mathcal{H}(Q)
\]
\[
f \longmapsto f \sqrt{d\mu}
\]
is a bijection.

Definitio 5.2
Let \(Q\) be a manifold. A function \(f \in C^\infty(Q)\) is called a classical configuration observable and the corresponding quantum position observable is an operator \(O_f\) on \(\mathcal{H}(Q)\) such that
\[
O_f(g \sqrt{d\mu}) = f \cdot g \sqrt{d\mu},
\] (32)
i.e. \(O_f\) is the multiplication by \(f\). Define the classical momentum observable \(P(X)\) on \(T^*Q\) associated to vector field \(X\) on \(Q\) as
\[
P(X) : T^*Q \longrightarrow \mathbb{R}
\]
\[
\alpha \longmapsto \alpha(X)
\]
We shall call the operator $\hat{X}$ the corresponding quantum momentum observable.

**Proposition 5.1**
Let $Q$ be a finite-dimensional manifold with Hilbert space $\mathcal{H}(Q)$. Let $X, Y \in \Gamma(Q)$ be vector fields on $Q$ and let $f, g \in C^\infty(Q)$. Then we have

(i) $[\hat{X}, \hat{Y}] = -i[\hat{X}, \hat{Y}]$.

(ii) $[O_f, O_g] = 0$.

(iii) $[O_f, \hat{X}] = iO_X(f)$.

**Definition 5.3**
A full quantization of a manifold $Q$ is a map, denoted by a hat $\hat{\cdot}$, taking classical observables $f$ to self-adjoint operators $\hat{f}$ on Hilbert space $\mathcal{H}(Q)$ such that:

(i) $\hat{(f + g)} = \hat{f} + \hat{g}$.

(ii) $\hat{\lambda f} = \lambda \hat{f}$ for all $\lambda \in \mathbb{R}$.

(iii) $\{\hat{f}, \hat{g}\} = -i[\hat{f}, \hat{g}]$.

(iv) $\hat{1} = \text{Id}$, where 1 is the function that is constantly 1 on $Q$ and $\text{Id}$ is the identity on $\mathcal{H}(Q)$.

(v) $\hat{q}^i$ and $\hat{p}_j$ act irreducibly on $\mathcal{H}(Q)$.

**Notation 5.1**
By a theorem from Stone-von Neumann the condition (v) really means that we can take $\mathcal{H} = L^2(\mathbb{R}^n)$, where $n$ is the dimension of $Q$, and that $\hat{q}^i$ and $\hat{p}_j$ are given by $\hat{q}^i = O_q^i$ and $\hat{p}_j = -i\frac{\partial}{\partial q^j}$; that is, the Schrödinger representation.

In order to allow spin one has to relax the last condition to:

(v') The position and momentum operators are represented by a direct sum of finitely many copies of the Schrödinger representation. More precisely, we are asking that $\mathcal{H}(Q)$ can be realized as the space of $L^2$ functions from $\mathbb{R}^n$ to a $d$-dimensional Hilbert space $\mathcal{H}_d$ with $d < \infty$, so that

$$\hat{q}^i \phi(x) = q^i \phi(x) \quad \text{and} \quad \hat{p}_j \phi(x) = -i\frac{\partial \phi}{\partial x^j}(x) \quad (33)$$

Does such a full quantization exist? The answer is no! To see this we follow A. Joseph from 1970:

**Theorem 5.2**
Let $U$ be the Lie algebra of real-valued polynomials of $\mathbb{R}^{2n}$, where the bracket is given by the Poisson bracket. Let $H = L^2(\mathbb{R}^n, \mathcal{H}_d)$. Then there is no map $f \mapsto \hat{f}$ from $U$ to the self-adjoint operators on $H$ that has the following properties:

(i) For each finite subset $S \subset U$ there is a dense subspace $D_S \subset H$ such that for all $f \in S$: $D_S \subset D_{\hat{f}}$ and $\hat{f} D_S \subset D_S$. 

11
\( (\hat{f} + \hat{g}) = \hat{f} + \hat{g} \) pointwise on \( D_S \) if \( f, g \in S \).

\( \hat{\lambda f} = \lambda \hat{f} \) for \( \lambda \in \mathbb{R} \).

\( \{ \hat{f}, \hat{g} \} = -i[\hat{f}, \hat{g}] \) on \( D_S \).

\( \hat{1} = \text{Id} \).

\( \hat{q}^i \) is multiplication by \( q^i \) and \( \hat{p}_j = -i \frac{\partial}{\partial q^j} \).

**Notation 5.2**
Since we have seen that quantization with finite representations are not working there is another procedure, the so-called **pre-quantization**, that is a quantization satisfying (i)-(iv) but not (v) or (v') of definition 5.3.

**Definition 5.4**
Let \( M \) be a manifold. A vector bundle \( \pi : Q \to M \) is called a **principal circle bundle** if every fiber \( \pi^{-1}(p) \) for \( p \in M \) is a circle \( S^1 \) and there is a consistent action \( S^1 \times Q \to Q \), which is just multiplication on each fiber, i.e. \( M = Q / S^1 \).

**Definition 5.5**
Let \( (P, \omega) \) be a symplectic manifold. We say that \( (P, \omega) \) is **quantizable** if and only if there is a principle circle bundle \( \pi : Q \to P \) over \( P \) and a 1-form \( \alpha \) on \( Q \) such that

(a) \( \alpha \) is invariant under the action of \( S^1 \).
(b) \( \pi^* \omega = d\alpha \).

\( Q \) is then called the **quantizing manifold** of \( P \).

**Exemplum 5.1**
(a) If \( \omega \) is exact, then \( P \) is quantizable, e.g. \( P = T^*M \). If \( P \) is simply connected, then \( Q \) is unique, i.e. if \( \omega = d\theta \) we can let \( Q = P \times S^1 \) and \( \alpha = \theta + \hbar ds \) for some suitable \( s \in C^\infty(Q) \) and \( \hbar \) is a constant, ultimately to be defined with **Plank’s constant**.

(b) For more details view Souriau 1970: Consider the two sphere \( S^2 \) in \( \mathbb{R}^3 \) with radius \( mK \) with the symplectic form

\[
\omega = \frac{-i}{\sqrt{8me}} \left( \frac{dx_1 \wedge dx_2}{x_3} \right),
\]

where \( m \) is the mass, \( K \) is the attractive constant and \( e \) is the energy. Then \( S^2 \) is quantizable if and only if \( e = -2\pi^2 mK^2 / N^2 \) for an integer \( n \in \mathbb{N} \).