Exercise 18: Arnowitt–Deser–Misner formalism: gauge transformation

So far we have seen the gauge generators for electromagnetic field (Ex. 14) and the quadratic description of a charged relativistic particle (Ex. 17), which were based on somewhat ad hoc derivations. A systematic algorithm for finding generic gauge generators of first-class systems had been established in [1] by Castellani. The Arnowitt–Deser–Misner formulation in the absence of matter is governed by the Hamiltonian action with primary constraints

$$ SP = \int dt \, d^4x \left\{ p^{ij} h_{ij} + \Pi I + \Pi I \dot{N}^i - N \delta_{\perp} - N^i \delta_i - V - V^i \Pi_i \right\} + \text{surface terms}, \quad (1a) $$

where \( \{ N, N^i, h_{ij} \} \) are the canonical positions; \( \{ \Pi, \Pi_i, p^{ij} \} \) are the corresponding conjugate momenta, which are all densities of weight 1; \( \delta_{\perp} \) and \( \delta_i \) are called the Hamiltonian and momentum constraints, given by (e.g. [2, sec. 4.2.2, 3, sec. E.2])

$$ \delta_{\perp} := 2 \varepsilon G_{ijkl} p^{ij} p^{kl} - \frac{\hbar}{2 \varepsilon} R[h] = 2 \varepsilon F_{ijkl} h_{ij} h_{kl} - \frac{\hbar}{2 \varepsilon} R[h], \quad \varepsilon := 8 \pi G, \quad (1b) $$

$$ \delta_i := -2 p^{ij} \dot{p}_{ij} := -2 h_{ij} \left( p^{ij} + \Pi^{ij} p^k \right), \quad (1c) $$

where \( R[h] \) is the induced Ricci scalar on hypersurfaces, and

$$ G_{ijkl} := \frac{1}{2 \hbar} \left( h_{ik} h_{lj} + h_{il} h_{kj} - h_{ij} h_{kl} \right) = -\Delta \left( \frac{\hbar}{2} h_{ij} \right), \quad (1d) $$

$$ F_{ijkl} := \frac{1}{\hbar} \left( p^{ik} p^{lj} + p^{il} p^{kj} - p^{ij} p^{kl} \right). \quad (1e) $$

Applying the Castellani algorithm to the Arnowitt–Deser–Misner formulation gives the gauge generator [4]

$$ G = -\int d^4x \left\{ \left[ \xi^\perp \left( \delta_{\perp} + N_i \Pi^i \right) + \left( N^i \Pi_i \right) \right] + \xi^\perp \Pi \right\} + \left[ \xi^i \left( \delta_i + N^j \Pi_j + \left( N^j \Pi_j \right)^i - N_j \Pi_i \right) + \xi_i \Pi_i \right]. \quad (2a) $$

1. Show that the gauge transformations for the lapse and shift functions, as well as their conjugate momenta, are

$$ \delta N = \xi^\perp_i N^i - \xi^\perp - \xi_i N_i, \quad \delta N^i = -\xi^\perp_i N_j h_{ij} + \xi^\perp_i - \xi_i h_{ij} - \xi^\perp_i N^i_j + \xi_i N^i_j - \xi^\perp_i ; \quad (3a) $$

$$ \delta \Pi = -\left( \xi^\perp_i \Pi^i \right) - \xi^\perp_i \Pi^i - \left( \xi^\perp_i \Pi_i \right), \quad \delta \Pi_i = -\xi^\perp_i \Pi - \left( \xi^\perp_i \Pi_i \right)^j - \xi^\perp_i \Pi_j. \quad (3b) $$

Remark. Although the lapse and shift functions are not dynamical so that their time evolution cannot be solved and have to be imposed, they do have a definite rule under gauge transformations.

2. Show that the gauge transformations for the metric on hypersurfaces are

$$ \delta h_{ij} = -H_{ij} \left( \xi^\perp, \xi^i \right), \quad (4a) $$

$$ H_{ij}(N, N_i) := 2 \left( 2 \varepsilon N G_{ijkl} p^{ij} + N_{(ij)} \right). \quad (4b) $$

Remark. One may refer to [2, sec. 4.2.2, 3, sec. E.2, 5, sec. 4.2.7] for the results.
3. Argue that the gauge transformations for the momenta conjugate to $h_{ij}$ are

$$\delta p^{ij} = -P^{ij} \left( \xi^\perp, \xi^\parallel \right),$$  \hspace{1cm} (5a)

$$P^{ij}(N, N^i) := 2N \left\{ \frac{1}{2} h^{ij} \tilde{F}^{klnm} h_{mn} - 2 \tilde{F}^{ijkl} h_{kl} \right\} + \frac{1}{2N} \left( -N^{1/2} \tilde{G}^{ij} \left[ h \right] + \tilde{G}^{ijkl} N_{[kl]} \right) - \left\{ \left( h^{ki} p^i j + h^{kj} p^i i - h^{kl} p^{ji} \right) N_k \right\}_{[i]}.$$  \hspace{1cm} (5b)

where $G^{ij}[h] := R^{ij}[h] - \frac{1}{2} R[h] h^{ij}$ is the induced $(2, 0)$ Einstein tensor on the hypersurfaces,

$$\tilde{G}^{ijkl} := \frac{1}{2} h^{1/2} \left( h^{ki} h^{lj} + h^{ki} h^{lj} - 2 h^{ij} h^{kl} \right) = -h^{-1/2} \frac{\delta}{\delta h_{kl}} (\tilde{h}^{ij}),$$  \hspace{1cm} (5c)

**Remark.** One may refer to [3, sec. E.2, 5, sec. 4.2.7] for the argument.

4. One may wonder how the ‘gauge’ transformations in eqs. (3a) to (5c) in the Arnowitt–Deser–Misner formulation, generated by the first-class constraints, are related to those generated by diffeomorphism. Here is how one begins with.

Remember that

$$\delta_{\text{diffeo}} g_{\mu \nu} = g_{\mu \nu} \epsilon^\rho - g^{\rho \sigma} e^\sigma_{\mu} - g^{\mu \rho} e^\nu_{\lambda},$$  \hspace{1cm} (6)

where $e^\mu$ is the ‘gauge’ parameter of diffeomorphism. Furthermore, $g^{00} = -N^{-2}$.

a) Show that

$$\delta_{\text{diffeo}} N = N e^0 + N_j e^i_j + N^i e^0_i - N N^i e^0_{,i}$$

$$= \tilde{\epsilon}_i \left( N e^0 \right) + \left( e^i + N^i e^0 \right) N_{,j} - N^i \left( N e^0 \right)_{,i}.$$  \hspace{1cm} (7a)

b) Compare eq. (7a) with eq. (3a) by setting $\delta N = \delta_{\text{diffeo}} N$ and show that

$$\xi^\perp = -Ne^0, \quad \xi^\parallel = -e^i - N_i e^0.$$  \hspace{1cm} (7b)

**References:**


