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Second exercise sheet on Relativity and Cosmology I

Winter term 2018/19

Release: Mon, Oct. 22nd **Submit**: Mon, Nov. 5th in lecture **Discuss**: Thu, Nov. 8th

Note that exercises 3 – 7 are for two weeks, see above.

Exercise 3 (9 points): *Inertial frames*

A rocket with a rest length L_0 moves with constant velocity radially away from Earth. From Earth a light pulse is emitted, which is then reflected by mirrors at the front as well as at the rear of the rocket. The first signal is received after the time t_A, the second after the time t_B.

3.1 Calculate the velocity at which the rocket moves in terms of L_0, t_A and t_B.

3.2 Determine at which distance from Earth the rocket is located when the first signal reaches Earth.

Exercise 4 (9 points): *Addition of velocities*

Consider a mass point moving with velocity $\vec{w}'$ with respect to the inertial system $\mathcal{I}'$.

4.1 What is its velocity $\vec{u}$ with respect to an inertial system $\mathcal{I}$ if $\mathcal{I}'$ moves with velocity $\vec{v}$ against $\mathcal{I}$? Show that the result can be written as (in units where $c = 1$)

$$\vec{u} = \frac{\vec{v} + \vec{w}'_{\parallel} + \frac{\vec{w}'_{\perp}}{\gamma(v)}}{1 + \vec{v} \cdot \vec{w}'} \,,$$

where $\vec{w}'_{\parallel}$ and $\vec{w}'_{\perp}$ denote the parallel and orthogonal components of $\vec{w}'$ with respect to $\vec{v}$, respectively. Discuss the special cases $\vec{v} \parallel \vec{w}'$ and $\vec{v} \perp \vec{w}'$.

4.2 Show that

$$\vec{u}^2 = 1 - \frac{\left(1 - \vec{w}'^2\right)\left(1 - \vec{v}^2\right)}{\left(1 + \vec{v} \cdot \vec{w}'\right)^2} \leq 1 \,.$$

When does the equality hold? Discuss the limiting case $|\vec{w}'| \to 1^-$.

Exercise 5 (6 points): *Aberration*

Consider an inertial system $\mathcal{I}'$ that moves with velocity $\vec{v}$ against an inertial system $\mathcal{I}$. Consider a ray of light which arrives in $\mathcal{I}$ at an angle θ with respect to $\vec{v}$ (in units where $c = 1$).
Under which angle θ' does this light ray arrive in $\mathcal{I}'$? Show that this relation can be written in the form

$$\tan\frac{\theta}{2} = \sqrt{\frac{1+v}{1-v}} \tan\frac{\theta'}{2} \,.$$

Hint: Draw a picture of the angle. Use the law for the addition of velocities from exercise **4**.

See overleaf.

Exercise 6 (10 points): *Covariant Maxwell equations*

Recall from classical electromagnetism the Maxwell equations (in Gaussian units with $c = 1$):

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho\,, \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi\vec{J}\,; \qquad \vec{\nabla} \cdot \vec{B} = 0\,, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0\,. \tag{1}$$

In terms of the scalar (Φ) and vector ($\vec{A}$) potentials, the electric and magnetic fields are $\vec{E} = -\vec{\nabla}\Phi - \partial_t\vec{A}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. In components, eq. (1) reads

$$\sum_{i=1}^{3} \partial_i E^i \equiv \partial_i E^i = 4\pi\rho\,, \qquad \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon^{ijk}\partial_j B_k - \partial_t E^i \equiv \epsilon^{ijk}\partial_j B_k - \partial_t E^i = 4\pi J^i\,; \tag{2a}$$

$$\partial_i B^i = 0\,, \qquad \epsilon^{ijk}\partial_j E_k + \partial_t B^i = 0\,, \tag{2b}$$

where ϵ^{ijk} is the Levi-Civita pseudo-tensor, $i, j, \ldots = 1, 2, 3$, and Einstein notation is assumed, as has been explained in eq. (2a). Let $A^\mu := (\Phi, \vec{A})$ be the four-potential, $j^\mu := (\rho, \vec{J})$ the four-current, $\mu, \nu, \ldots = 0, 1, 2, 3$, and define the field-strength tensor

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3}$$

6.1 Show that the expressions

$$\sum_{\nu=0}^{4} \partial_\nu F^{\mu\nu} \equiv \partial_\nu F^{\mu\nu} = 4\pi j^\mu\,, \tag{4a}$$

$$\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} = 0 \tag{4b}$$

correspond to eqs. (2a) and (2b) respectively.

Hint: $\epsilon_{ijk}\epsilon^{ilm} = \delta_j{}^l\delta_k{}^m - \delta_j{}^m\delta_k{}^l$.

6.2 Show that eq. (4a) leads to continuity equation $0 = \partial_t\rho + \vec{\nabla} \cdot \vec{J} \equiv \partial_\mu j^\mu$. How does the continuity equation look like in a Lorentz-boosted reference frame?

Exercise 7 (6 points): *Covariant Lorentz force*

Let $p^\mu := m u^\mu$ be the kinematic four-momentum, u^μ the four-velocity, τ the proper time; $\vec{p} := m\vec{v}$, $\vec{v} := \mathrm{d}\vec{x}/\mathrm{d}t$, and t the coordinate time.

7.1 Show that the spatial components of the covariant Lorentz four-force

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} = f_\mu = qF_{\mu\nu}u^\nu \tag{5}$$

give in the non-relativistic limit the Lorentz force

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).$$

7.2 What is the physical meaning of the time component f^0 of the covariant four-force in eq. (5)?