Exercise 17 (7 points): Covariant derivative I: Metricity and torsion

An affine connection $\hat{\nabla}$, also known as a covariant derivative $\hat{\nabla}$, can be defined by its connection coefficients $\tilde{\Gamma}^i_{jk}$, which transform in the same way as the Christoffel symbols. The covariant derivative of a tensor $A^i_j$ is then

$$\hat{\nabla}_j A^i_k := \partial_j A^i_k + \tilde{\Gamma}^i_{jd} A^d_k - \tilde{\Gamma}^i_{dk} A^j_k.$$

$\hat{\nabla}$ can be characterised by its non-metricity and torsion, defined as

$$Q_{ijk} (\hat{\nabla}) := -\hat{\nabla}_i g_{jk}, \quad T^i_{jk} (\hat{\nabla}) := 2 \tilde{\Gamma}^i_{[jk]} - \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj}.$$

17.1 Let $\tilde{T}^{\cdot \cdot \cdot}$ be a $(q, p)$-tensor. Argue that $\hat{\nabla}_i \tilde{T}^{\cdot \cdot \cdot}$ is a $(q, p + 1)$-tensor. Are $Q_{ijk}$ and $T^i_{jk}$ tensors?

17.2 Let $\nabla$ be the Levi-Civita connection in (pseudo-)Riemannian space, whose coefficients are given by the Christoffel symbols of the second kind $\Gamma^i_{jk}$. Show that the metric is covariantly constant, i.e. $\nabla_k g_{ij} = -Q_{kij} (\nabla) = 0$, $\nabla_k g^{ij} = 0$; furthermore, show that $T^i_{jk} (\nabla) = 0$.

17.3 Let the non-metricity and torsion of $\hat{\nabla}$ be zero. Show that $\hat{\nabla}$ is necessarily Levi-Civita.

Exercise 18 (7 points): Derivative of a determinant

Let $g = \det g_{ij}$ be the metric determinant, $d$ the dimension of the manifold; $\epsilon^{ij\ldots m}$ and $\epsilon_{ij\ldots m}$ the Levi-Civita symbols, taking values $+1$ (or $-1$) for even (or odd) permutations of the indices, denoted in the lecture as $\epsilon(ij\ldots m)$.

18.1 From the definition of $g$, argue that

$$g = \frac{1}{d!} \epsilon^{jk\ldots m \ldots gn} g_{ij} g_{kl} \ldots g_{mn}, \quad \frac{1}{g} = \frac{1}{d!} \epsilon_{jk\ldots m \ldots gn} g^{ij} g^{kl} \ldots g^{mn},$$

which implies $\epsilon^{ij\ldots k}$ ($\epsilon_{ij\ldots k}$) is a tensor density of weight $+1$ ($-1$).

18.2 Express $g_{ij}$ (and $g^{ij}$) in terms of $d$, $g$, $\epsilon_{ij\ldots k}$ (or $\epsilon^{ij\ldots k}$) and $g^{ij}$ (or $g_{ij}$). Show that the determinant variation can be given by

$$\delta g = +g g^{ij} \delta g_{ij} = -g g^{ij} \delta g_{ij}.$$

18.3 Express $\partial g/\partial x^i$ in terms of $g$ and the Christoffel symbols.

Exercise 19 (6 points): Covariant derivative II: Vector densities

19.1 Let $V^i$ be a vector field and $\tilde{V}^i := \sqrt{|g|} V^i$ be the corresponding vector density of weight 1. Show that

$$\nabla_i \tilde{V}^i = \frac{1}{\sqrt{|g|}} \tilde{\nabla}_i \left( \sqrt{|g|} V^i \right), \quad \text{and} \quad \nabla_i \tilde{V}^i = \tilde{\nabla}_i \tilde{V}^i.$$

19.2 The Laplace–Beltrami operator for a scalar field $\phi$ is given by $\Box \phi := \nabla_i \nabla^i \phi$.

- Rewrite $\Box \phi$ such that the result only contains partial derivatives, instead of covariant derivatives.
- As an example, calculate the operator in spherical coordinates of a 3-dimensional Euclidean space.