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Seventh exercise sheet on Relativity and Cosmology I

Winter term 2018/19

Release: Mon, Dec. 3rd **Submit**: Mon, Dec. 10th in lecture **Discuss**: Thu, Dec. 13th

In the following four exercises, consider a (pseudo-)Riemannian space with the Levi-Civita connection ∇.

Exercise 20 (6 points): *Killing equations*

20.1 Show that $2\nabla_{(\mu}\, v_{\nu)} = \pounds_{\underline{v}}\, g_{\mu\nu}$. What does $\pounds_{\underline{v}}\, g_{\mu\nu} = 0$ mean from a geometrical point of view?

20.2 Consider a Minkowski space in Cardesian coordinates, $\mathrm{d}s^2 = \eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu$. Find all Killing vector fields.

20.3 Consider a Poincaré half-plane $\mathrm{d}s^2 = (l/z)^2 \big(\mathrm{d}x^2 + \mathrm{d}z^2\big)$, where l is a constant length, $z > 0$. Write down the Killing equations and (bonus) find all the Killing vector fields.

Exercise 21 (5 points): *Killing vector fields*

Let ξ^μ be a Killing vector field.

21.1 Let ξ^μ be time-like. Show that there exists a coordinate system (t, x, y, z), in which t is temporal, and the metric does not depend on time t, i.e. $\partial g_{\mu\nu} / \partial t = 0$ holds.

21.2 Let $u^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau$ be the tangent vector of an affine-parametrised geodesic. Show that $u^\mu \xi_\mu$ is constant along the geodesic. Interpret the physical meaning of the Killing vector fields from **20.2**.

21.3 Let $T^{\mu\nu}$ be a symmetric tensor field with vanishing covariant divergence. Calculate $\big(\xi_\mu\, T^{\mu\nu}\big)_{;\nu}$.

21.4 (bonus) By using the first Bianchi identity and the Killing equations, prove

$$\xi_{\lambda;\kappa\nu} = -\,\xi_\mu\, R^\mu{}_{\nu\lambda\kappa}\,,$$

where $R^\mu{}_{\nu\lambda\kappa}$ is the Riemann tensor. Argue that the equation serves as an integrability condition.

Exercise 22 (6 points): *Riemannian normal coordinates*

Consider Riemannian normal coordinates y^μ at a point P, which is set as the coordinate origin $y^\mu = 0$.

22.1 Let $R_{\mu\kappa\lambda\nu}$ be the Riemann tensor. Show that

$$g_{\mu\nu}(y) = \eta_{\mu\nu} + \frac{1}{3}\, R_{\mu\kappa\lambda\nu}(0)\, y^\kappa\, y^\lambda + O\Big(|y|^3\Big)\,.$$

22.2 Give a physical interpretation of the equation.

Exercise 23 (3 points): *Algebraic identities of the Riemann tensor*

23.1 Let R_{ijkl} be the Riemann tensor. Show that the following algebraic identities hold

$$R_{ij(kl)} = 0\,, \quad R_{(ij)kl} = 0\,; \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0\,.$$

The last one is also called the *first* or the *algebraic Bianchi identity*.

23.2 (bonus) Give an interpretation of the first two identities in terms of parallel transport.