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www.thp.uni-koeln.de/gravitation/courses/rci1819.html

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Eighth exercise sheet on Relativity and Cosmology I

Winter term 2018/19

Release: Mon, Dec. 10th **Submit**: Mon, Dec. 17th in lecture **Discuss**: Thu, Dec. 20th

Exercise 24 (5 points): *Geometry and topology*

In Euclidean space $\mathbb{R}^3$, consider the cylindrical coordinates (ρ, φ, z). Define a surface of revolution $\mathcal{A}$ via

$$z^2 = [f(\rho)]^2, \qquad \rho^2 = x^2 + y^2,$$

where f is a strictly positive smooth function on $[0, a]$ with $f(a) = 0$, $f'(a^-) \to -\infty$ and $f'(0^+) = 0$.

24.1 Determine the induced metric $\mathrm{d}s^2 = h_{ij}\,\mathrm{d}y^i\,\mathrm{d}y^j$ on $\mathcal{A}$.

24.2 Let R be the Ricci scalar of $\mathcal{A}$. Evaluate the following integral explicitly

$$\int_{\mathcal{A}} \mathrm{d}^2 y \sqrt{h}\, R\,.$$

24.3 (bonus) Argue that in $(1+1)$ dimensions, the Einstein–Hilbert action does not give any dynamics.

Exercise 25 (3 points): *Contracted Bianchi identity from action*

Let $R_{\mu\nu}$ and R be the Ricci tensor and scalar, respectively. Derive the contracted Bianchi identity

$$G^{\mu\nu}{}_{;\mu} := \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0$$

by demanding the Einstein–Hilbert action be invariant under infinitesimal coordinate transformations.

In the following two exercises, the *Hilbert* or *symmetric energy-momentum tensor* is defined as

$$T_{\mu\nu} := -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}\,.$$

Exercise 26 (6 points): *Klein–Gordon theory of scalar field*

Consider the action of a neutral Klein–Gordon field $\phi = \phi(x)$ with mass parameter m and potential $V(\phi)$,

$$S_{\mathrm{KG}}[\phi] := \int \mathrm{d}^4 x \sqrt{|g|} \left\{ -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{m^2}{2} \phi^2 - V(\phi) \right\}.$$

26.1 Derive the Klein–Gordon field equation by the action principle.

26.2 Derive the Hilbert energy-momentum tensor and calculate its trace.

Exercise 27 (6 points): *Maxwell theory of electromagnetic field*

Consider the action of an electromagnetic field in vacuum (in Gaussian units),

$$S_{\mathrm{M}}[A_\mu] := \int \mathrm{d}^4 x \sqrt{|g|} \left\{ -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right\}.$$

28.1 Derive the covariant Maxwell field equations by the action principle.

28.2 Derive the Hilbert energy-momentum tensor and prove the continuity equations $\nabla_\mu T^{\mu\nu} = 0$.