Exercise 24 (5 points): Geometry and topology

In Euclidean space $\mathbb{R}^3$, consider the cylindrical coordinates $(\rho, \varphi, z)$. Define a surface of revolution $\mathcal{A}$ via
\[
z^2 = |f(\rho)|^2, \quad \rho^2 = x^2 + y^2,
\]
where $f$ is a strictly positive smooth function on $[0, a]$ with $f(a) = 0$, $f'(a^-) \to -\infty$ and $f'(0^+) = 0$.

24.1 Determine the induced metric $d\mathcal{s}^2 = h_{ij} \, dy^i \, dy^j$ on $\mathcal{A}$.

24.2 Let $\mathcal{R}$ be the Ricci scalar of $\mathcal{A}$. Evaluate the following integral explicitly
\[
\int_{\mathcal{A}} \, d^2y \, \sqrt{\mathcal{R}}.
\]

24.3 (bonus) Argue that in $(1+1)$ dimensions, the Einstein–Hilbert action does not give any dynamics.

Exercise 25 (3 points): Contracted Bianchi identity from action

Let $R_{\mu\nu}$ and $R$ be the Ricci tensor and scalar, respectively. Derive the contracted Bianchi identity
\[
G^{\mu\nu}_{;\mu} := \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{;\mu} = 0
\]
by demanding the Einstein–Hilbert action be invariant under infinitesimal coordinate transformations.

In the following two exercises, the Hilbert or symmetric energy-momentum tensor is defined as
\[
T_{\mu\nu} := -\frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}}.
\]

Exercise 26 (6 points): Klein–Gordon theory of scalar field

Consider the action of a neutral Klein–Gordon field $\phi = \phi(x)$ with mass parameter $m$ and potential $V(\phi)$,
\[
S_{\text{KG}}[\phi] := \int d^4x \sqrt{|g|} \left\{ -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \frac{m^2}{2} \phi^2 - V(\phi) \right\}.
\]

26.1 Derive the Klein–Gordon field equation by the action principle.

26.2 Derive the Hilbert energy-momentum tensor and calculate its trace.

Exercise 27 (6 points): Maxwell theory of electromagnetic field

Consider the action of an electromagnetic field in vacuum (in Gaussian units),
\[
S_{\text{M}}[A_{\mu}] := \int d^4x \sqrt{|g|} \left\{ -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \right\}.
\]

28.1 Derive the covariant Maxwell field equations by the action principle.

28.2 Derive the Hilbert energy-momentum tensor and prove the continuity equations $\nabla_{\mu} T^{\mu\nu} = 0$. 