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ver. 1.1

Ninth exercise sheet on Relativity and Cosmology I

Winter term 2018/19

Release: Mon, Dec. 17th **Submit**: Mon, Jan. 7th in lecture **Discuss**: Thu, Jan. 10th

Exercise 28 (4 points): *Geodesic deviation*

Consider two neighbouring geodesics with worldlines $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$, where $\xi^\mu(\tau)$ is considered to be small, so that quadratic and higher-order terms can be neglected. Let $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ be the velocity. Show that the relative acceleration satisfies

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}\tau^2} = R^\mu{}_{\nu\kappa\lambda} u^\nu u^\kappa \xi^\lambda \,,$$

which is known as the *geodesic deviation equation*.

Exercise 29 (7 points): *Dust and ideal fluid*

In curved spacetime, the energy-momentum tensors of dust and ideal fluid are given by

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + P\,(u^\mu u^\nu + g^{\mu\nu})\,,$$

where u^μ is the four-velocity field, ρ the energy density, P the pressure; for dust, $P = 0$.

29.1 Argue briefly that ρ and P are *scalars*.

29.2 For dust, show that dust particles move on geodesics.

29.3 Derive the continuity and the Euler equations of an ideal fluid by contracting $\nabla_\nu T^{\mu\nu} = 0$ with u_μ and $g_{\mu\nu} + u_\mu u_\nu$, respectively.

29.4 Consider the spatially-flat (*Friedmann–Lemaître–*)*Robertson–Walker metric*, defined by

$$\mathrm{d}s^2 = -N(t)^2\,\mathrm{d}t^2 + a(t)^2\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \qquad N > 0,\ a > 0.$$

Write down the continuity equation for an ideal fluid.

Hint: the spatial homogeneity and isotropy have to be used.

See overleaf.

Exercise 30 (3+8 points): *Relativistic charged particle I*

Consider a charged massive test particle in special relativity, described by the action ($c = 1$)

$$S\left[x^i\right] = \int_A^B \mathrm{d}t\, L\left(x^i, \dot{x}^j\right) := \int_A^B \mathrm{d}t \left\{-m\sqrt{1-\left(\dot{x}^i\right)^2} - q\Phi\left(x^j\right) + q\dot{x}^j A_i\left(x^k\right)\right\},$$

where $\dot{x}^i := \mathrm{d}x^i/\mathrm{d}t$, m and q are the mass and the electric charge, Φ and A_i the electric and vector potentials.

30.1 Calculate the *canonical* momentum $P_i = P_i\left(x^j, \dot{x}^k\right) := \partial L/\partial \dot{x}^i$. Derive its partial inverse $\dot{x}^i = v^i\left(x^j, P_k\right)$.

Remark. If such an inverse exists, the system is called *regular*, and there is *no constraint*.

30.2 (bonus) Calculate the *canonical* Hamiltonian $H = H\left(x^i, P_j\right)$. Derive the canonical equations of motion

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial P_i}, \qquad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}.$$

30.3 (bonus) From the results in **30.2**, find the relativistic Lorentz force in terms of the three-velocity $\dot{x}^i$, *kinematic* momentum $p_i := P_i - qA_i$, electric field $E_i := -\partial_i\Phi - \partial_t A_i$ and magnetic B-field $B^i := \epsilon^{ijk}\partial_j A_k$.

Exercise 31 (6 points): *Relativistic charged particle II: parametrised formulation*

Consider a charged massive test particle in special relativity, described by the action ($c = 1$)

$$S[x^\mu] = \int_A^B \mathrm{d}\lambda\, L(x^\mu, \dot{x}^\nu) := \int_{\lambda_A}^{\lambda_B} \mathrm{d}\lambda \left\{-m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + q\dot{x}^\mu A_\mu\right\}, \qquad \mu,\nu,\rho = 0,1,2,3,$$

where $\dot{x}^\mu := \mathrm{d}x^\mu/\mathrm{d}\lambda$, A_μ is the four-potential.

31.1 Calculate the action under $\lambda \mapsto \lambda_f = f(\lambda)$, where the boundaries are fixed, $f(\lambda_{A,B}) = \lambda_{A,B}$, and $f'(\lambda) > 0$. Can one impose $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ before deriving the equations of motion?

Remark. Such a system is called *parametrised*, and belongs to a subset of all *singular systems*. The Einstein–Hilbert action is also parametrised.

31.2 Calculate the canonical four-momentum $P_\mu = P_\mu(x^\nu, \dot{x}^\rho) := \partial L/\partial \dot{x}^\mu$ of the particle. Show that its partial inverse $\dot{x}^\mu = v^\mu\left(x^\nu, P_\rho\right)$ does *not* exist.

Remark. Such a non-existence is the defining property of a *singular system*, which is often a synonym for *constrained system*. The Maxwell theory is also singular, but not parametrised.

31.3 Calculate $\dot{x}^\mu P_\mu(x^\nu, \dot{x}^\rho) - L(x^\nu, \dot{x}^\rho)$.

Remark. This result can be shown to be universal for all parametrised systems.