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Tenth exercise sheet on Relativity and Cosmology I

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In the following two exercises, consider *linearised general relativity* in a flat background with the ansatz

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\,\psi_{\mu\nu}(x)\,,$$

where $\psi_{\mu\nu}$, $\psi_{\mu\nu,\rho}$ and $\psi_{\mu\nu,\rho,\sigma}$ are small perturbations of the same order.

Exercise 32 (9 points): *Linearised general relativity I: redundancy transformation*

Consider the infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} - 2\,f^{\mu}(x)\,,$$

where f^{μ} and $f^{\mu}{}_{,\nu}$ are of the same order as ψ.

32.1 Show that $\psi_{\mu\nu}$ transform as $\psi'_{\mu\nu}(x') = \psi_{\mu\nu}(x) + f_{\mu,\nu}(x) + f_{\nu,\mu}(x)$.

32.2 The *de Donder* or *harmonic condition* reads

$$\psi_{\mu\nu,}{}^{\nu} = \frac{1}{2}\psi^{\nu}{}_{\nu,\mu}\,.$$

Show that it can be realised by applying such a transformation.

32.3 Show that the linearised Riemann tensor is invariant under this transformation.

Exercise 33 (5+4 points): *Linearised general relativity II: Fierz–Pauli action*

Linearised general relativity can also be derived from an action, which has been given by *Fierz and Pauli*,

$$\begin{aligned} S_{\mathrm{FP}}[\psi_{\mu\nu}] &:= \int \mathrm{d}^4x\left\{\frac{1}{2\varkappa}\left(-\psi^{\mu\nu,\sigma}\,\psi_{\mu\nu,\sigma} + 2\,\psi^{\mu\nu,\sigma}\,\psi_{\sigma\nu,\mu} + \psi^{\mu}{}_{\mu,\nu}\,\psi^{\rho}{}_{\rho,}{}^{\nu} - 2\,\psi^{\rho\nu}{}_{,\nu}\,\psi^{\sigma}{}_{\sigma,\rho}\right) - T_{\mu\nu}\,\psi^{\mu\nu}\right\} \\ &=: \int \mathrm{d}^4x\left\{\mathcal{L}_{\mathrm{FP}} - T_{\mu\nu}\,\psi^{\mu\nu}\right\}, \end{aligned}$$

where $\varkappa := 8\pi G$; $T_{\mu\nu}$ is the *symmetric* energy-momentum tensor of matter, here playing the role of source, and is of the same order as $\psi_{\mu\nu}$.

33.1 Derive the equations of motion for $\psi_{\mu\nu}$ from S_{FP}, and show that they are equivalent to the linearised Einstein equations given in the lecture.

Hint. It might be quicker to apply the variational method directly, instead of using the Euler–Lagrange equations.

33.2 (bonus) Discard the source term. Calculate the *canonical* energy-momentum tensor of $\psi_{\mu\nu}$, defined by

$$t_{\mu\nu} := \frac{\delta S_{\mathrm{FP}}}{\delta \psi_{\rho\sigma,}{}^{\nu}}\,\psi_{\rho\sigma,\mu} - \eta_{\mu\nu}\,\mathcal{L}_{\mathrm{FP}}\,.$$

Remark. S_{FP} can be derived by expanding the Einstein–Hilbert action to the quadratic order, but the calculation is tedious by hand.

See overleaf.

Exercise 34 (6 points): *Fermi–Walker transport*

Let $x^i = x^i(s)$ be a curve, s its arc-length, and $u^i = \mathrm{d}x^i/\mathrm{d}s$ its tangent vector field. A vector v^i is called *Fermi–Walker transported* along the curve iff

$$\frac{\mathrm{D}v^i}{\mathrm{D}s} = v_k \left(u^k \frac{\mathrm{D}u^i}{\mathrm{D}s} - \frac{\mathrm{D}u^k}{\mathrm{D}s} u^i \right) .$$

34.1 If the curve is a geodesic, show that the Fermi–Walker transport is identical to the parallel transport.

34.2 Show that the tangent vector u^i is Fermi–Walker transported.

34.3 If v^i and w^i are Fermi–Walker transported, show that $v^i w_i$ is constant along the curve.

Remark. In practice, one might desire to describe the motion of objects, which are more than simple point particles. The Fermi–Walker transport is convenient for describing non-rotating motions in the 3-dimensional sense. For instance, if spatial basis vectors were attached to a free gyroscope, they would be Fermi–Walker transported. This will later be employed in the discussion of *geodetic precession* and the so-called *Thirring–Lense effect*.