Exercise 42 (14 points): Redshift in the Schwarzschild spacetime

Consider a stationary observer \( A \) at \( r = R, R \geq 2GM \) in the Schwarzschild spacetime of mass \( M \) and an observer \( B \) at infinity. The timelike Killing vector shall be denoted by \( \xi^\mu = (1,0,0,0) \). Furthermore, we define the quantity \( V^2 := -\xi_\mu \xi^\mu \). Observer \( A \) emits energy with frequency \( \omega_R \) (measured in her/his rest frame) which is measured by observer \( B \) as being \( \omega_\infty \).

42.1 Express the four-velocity \( u^\mu \) of observer \( A \) in terms of \( \xi^\mu \) and \( V \) and use this to derive the relation between the frequencies \( \omega_R \) and \( \omega_\infty \).

42.2 What does observer \( B \) measure when observer \( A \) reaches the Schwarzschild radius \( r = 2GM \)? What does this mean for the redshift?

Exercise 43 (6 points): Time dilation in the Schwarzschild spacetime

Show that the proper time \( d\tau \) on a circular geodesic in the Schwarzschild geometry of mass \( M \) obeys the relation:

\[
d\tau = \sqrt{1 - \frac{3GM}{r}} \, dt.
\]

Use this to give an estimate for the time dilation of a satellite flying in a low orbit around the Earth.

*A stationary observer is an observer in a stationary spacetime whose 4-velocity \( u^\mu \) is proportional to the given timelike Killing vector.*