3rd exercise sheet on Relativity and Cosmology II
Summer term 2019

Release: Mon, Apr. 15th  
Submit: Mon, Apr. 29th in lecture  
Discuss: Thu, May 2nd

Please note that you have two weeks for this exercise sheet. It gives twice the points as usual.

Exercise 44 (15 points): Differential forms

44.1 Consider an $n$-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal co-basis of 1-forms, and let $\omega$ be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^n$.

Show that, in an arbitrary coordinate system $\{x^k\}$, the following holds:

$$\omega = \sqrt{|g|} \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,$$

(1)

where $g$ denotes the determinant of the metric, whose components $g_{ij}$ are given in these coordinates.

44.2 The contraction of a $p$-form $\omega$ (with components $\omega_{i_1,\ldots,i_p}$) with a vector $v$ (with components $v^i$) is given by $[\omega(v)]_{i_1,\ldots,i_k} := \omega_{i_1,\ldots,i_k} v^i$. Consider the $n$-form $\omega = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$.

Show that, with a given vector field $v$, the following holds:

$$d[\omega(v)] = v^i, i \omega.$$

(2)

44.3 Define $(\text{div}_v \omega) := d[\omega(v)]$.

Show that, by using coordinates in which $\omega = f \, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, the following holds:

$$\text{div}_v \omega = \frac{1}{f} \left( f v^i \right),$$

(3)

44.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = dx \wedge dy \wedge dz$.

Show that, in spherical coordinates, this volume form is given by $\omega = r^2 \sin \theta \, dr \wedge d\theta \wedge d\phi$.

Use the result of 44.3 to show that the divergence of a vector field $v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi}$

(4)

is given by

$$\text{div} v = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v^r \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta v^\theta \right) + \frac{\partial v^\phi}{\partial \phi}.$$

(5)

Exercise 45 (25 points): Electrodynamics in flat space-time

Differential forms are a convenient tool for field theories, as we will show in this exercise on the example of Maxwell electrodynamics in Minkowski space-time. We know that the electromagnetic field strength is given by the Faraday antisymmetric tensor $F_{\mu \nu}$, and the current is given by $j^\mu$. In the language of exterior calculus, the $F_{\mu \nu}$ are components of the Faraday 2-form $F$ describing an arbitrary electromagnetic field, given by

$$F := \frac{1}{2} F_{\mu \nu} \, dx^\mu \wedge dx^\nu = -E_x \, dt \wedge dx - E_y \, dt \wedge dy - E_z \, dt \wedge dz + B_x \, dy \wedge dz + B_y \, dx \wedge dy + B_z \, dx \wedge dy,$$

(6)

while the $j^\mu$ are the components of the current 1-form $j$, which is given by

$$j := j_\mu \, dx^\mu = -\rho \, dt + j_x \, dx + j_y \, dy + j_z \, dz.$$

(7)

See reverse.
45.1 The Hodge star operator \( \star \) maps \( p \)-forms to \( (4-p) \)-forms. Therefore, 2-forms are mapped to 2-forms by this operator. The 2-form dual to the Faraday 2-form is the Maxwell 2-form, defined by \( G := \star F \).

The Hodge star operator acts on the 2-form basis as follows
\[
\star (dx^\mu \wedge dx^\nu) = \frac{1}{2} \eta^{\mu\alpha} \eta^{\nu\beta} \varepsilon_{\alpha\beta\gamma\delta} dx^\gamma \wedge dx^\delta,
\]
where \( \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1) \) is the inverse Minkowski metric and \( \varepsilon_{\alpha\beta\gamma\delta} \) the totally anti-symmetric Levi-Civita tensor density (\( \varepsilon_{0123} = +1 \)). For example,
\[
\star (dt \wedge dx) = -dy \wedge dz, \quad \text{etc.}\]

Show that the Maxwell 2-form is given by
\[
G = B_x dt \wedge dx + B_y dt \wedge dy + B_z dt \wedge dz + E_t dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy.
\]

Which formal relation between the components of \( F \) and \( G \) holds?

45.2 With these definitions, the Maxwell equations (in Gaussian units) can be written in a compact form: \( dF = 0 \) and \( dG = 4\pi \star j \).

a) Show that the equation \( dF = 0 \) corresponds to the two homogeneous Maxwell equations
\[
\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0.
\]

b) Calculate \( \star j \) (a 3-form), which is the dual of the current 1-form \( j \). For that purpose use the following relations
\[
\star dx^\mu = \frac{1}{3!} \eta^{\mu\alpha} \eta^{\nu\beta} \varepsilon_{\alpha\beta\gamma\delta} dx^\gamma \wedge dx^\delta,
\]
for example
\[
\star dt = -dx \wedge dy \wedge dz, \quad \text{etc.}
\]

c) Show that the equation \( dG = 4\pi \star j \) corresponds to the two inhomogeneous Maxwell equations
\[
\nabla \cdot \vec{E} = 4\pi \rho \quad \text{and} \quad \nabla \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{j}.
\]

d) Which formal manipulations can you perform to turn eq. (11) into eq. (14) in the vacuum case?

45.3 The exterior derivative is nilpotent, i.e. the relation \( d(\star \omega) = 0 \) holds for any \( p \)-form \( \omega \).

Choosing \( \omega = \star j \), show that this yields the continuity equation \( \partial_\mu \vec{j} - \nabla \cdot j = 0 \).

45.5 Derive the inhomogeneous Maxwell equations from a variational principle. The action is given by
\[
S[A] = \int \left( \frac{1}{8\pi} \star F \wedge F + A \wedge \star j \right),
\]
where \( F = dA \) and \( A = A_\mu dx^\mu \) is the electromagnetic 1-form potential.

Show in addition that the continuity equation implies that the action is invariant under the gauge transformation \( A \rightarrow A' = A + df \), where \( f \) is a 0-form.

*Hint:* You can make use of the fact that \( \star a \wedge \beta = a \wedge \star \beta \), where \( a \) and \( \beta \) are both \( k \)-forms.

45.6 *Bonus exercise.* Let \( \omega \) be a \( k \)-form. We define the codifferential \( \delta \) via
\[
\delta \omega := (-1)^k \star d \star \omega.
\]

This allows us to define the d’Alembert operator
\[
\Box := d \delta + \delta d.
\]

Show the following statements:
- \( \delta A = 0 \) is the well-known Lorenz gauge condition \( \partial_\mu A^\mu = 0 \)
- The Lorenz gauge can always be realized by a suitable gauge transformation.
- In the Lorenz gauge the inhomogeneous Maxwell equation can be written as \( \Box A = 4\pi j \).

45.7 *Bonus exercise.* Explain in your own words why a covariant derivative is needed on a curved background, and give special attention to the connection. What is the intuitive interpretation of the connection? Why can it be chosen to be zero in Minkowski space-time?