Exercise 55 (7 credit points): Hawking temperature

In the lecture it was mentioned that a Schwarzschild black hole radiates with the so-called Hawking temperature

\[ T_H = \frac{\hbar c^3}{8\pi k_B G M}. \]

Assume that only photons are emitted and that they have a perfect Planck spectrum. Find a relation between the initial mass of the black hole and its lifetime and analyse this relation for several interesting masses and time intervals.

Exercise 56 (6 credit points): Accretion disks

Give an estimate for the characteristic energy that is emitted by an accretion disk with radius \( R \) around a compact spherically symmetric object. For simplicity (even though this is not totally realistic), assume that the luminosity is that of a black body of radius \( R \) and temperature \( T \) and that it amounts to a given fraction \( \varepsilon \) of the Eddington luminosity. (At the end, use \( \varepsilon \approx 0.5 \).)

Exercise 57 (7 credit points): Redshift in case of a gravitational collapse

Consider an observer on the surface of a collapsing spherical star who emits radial light signals in short proper time intervals \( \Delta s \), i.e. with a constant frequency \( \omega_s = 2\pi/\Delta s \). These signals are received by a stationary observer at large distance \( r = r_R \), i.e. with a frequency \( \omega_R = 2\pi/\Delta t_R \), where \( \Delta t_R \) refers to the Schwarzschild time.

Calculate the dependence of the frequency ratio \( \omega_R/\omega_s \) on \( t_R \). Indicate the time scale of the redshift in terms of seconds if you measure \( M \) in solar masses.

Hint: Use Eddington–Finkelstein coordinates (as discussed in the lecture) and assume that the emitting observer is already located near the Schwarzschild radius.