University of Cologne www.thp.uni-koeln.de/gravitation/courses/rcii19.html
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11th exercise sheet on Relativity and Cosmology II

Summer term 2019

Release: Mon, June 24th **Submit**: Mon, July 1st in lecture **Discuss**: July 4th/5th

Exercise 59 (7 credit points): *Ideal fluids in Cosmology*

Consider an ideal fluid in a Friedmann–Lemaître model.

59.1 Show that comoving observers move on geodesics.

59.2 Evaluate the covariant conservation of the energy–momentum tensor of the ideal fluid for this kind of observers and show that this yields only *one* non-trivial equation, which on the other hand can also be deduced directly from the Friedmann equations.

59.3 Consider an equation of state of the form $p = w\rho$ with $w = \text{const.}$

Calculate the function $\rho(a)$. For which values of w does $\ddot{a} > 0$ hold? In which cases is the strong energy condition fulfilled?
Calculate $\rho(a)$ for a so-called "Chaplygin gas" whose equation of state is $p = -A/\rho$ ($A = \text{const.} > 0$) and discuss the extremal cases $a \to 0$ and $a \to \infty$.

59.4 Consider a flat Friedmann universe that satisfies $\Omega_\text{m} + \Omega_\text{x} = 1$, where Ω_m refers to pressureless matter and $\Omega_\text{x} = \rho_\text{x}/\rho_\text{c}$ denotes a hypothetical form of energy with density ρ_x and equation of state

$$p_\text{x} = w_\text{x}\rho_\text{x}\,.$$

Which condition has w_x depending on Ω_m to fulfill such that there is an accelerated expansion?
Calculate the Hubble parameter as a function of redshift, $H(z)$.

Exercise 60 (6 credit points): *Friedmann I*

Consider a Friedmann–Lemaître model with $\mathcal{K} \neq 0$ and present density parameters $\Omega_{\text{m},0}$, $\Omega_{\text{r},0}$ and $\Omega_{\text{v},0}$ as well as $\Omega := \Omega_{\text{m},0} + \Omega_{\text{r},0} + \Omega_{\text{v},0}$. Furthermore let

$$\rho_\text{c}(a) = \frac{3\,\dot{a}^2}{8\pi G\,a^2}$$

be the critical density at a time when the scale parameter had the value a, and let $\Omega_\text{m}(a) = \rho_\text{m}(a)/\rho_\text{c}(a)$ etc. be the corresponding relative densities.

Determine the quantity $\Omega(a) - 1$ as a function of $\Omega_{\text{m},0}$, $\Omega_{\text{r},0}$, $\Omega_{\text{v},0}$ and a. This quantity indicates how much the considered model "deviates" from a flat model at a certain time. What kind of problem with regard to the deviation from flatness at earlier times arises for a Friedmann–Lemaître model whose density parameter Ω differs only slightly from unity today?

Exercise 61 (7 credit points): *Friedmann II*

Solve the Friedmann equation for a universe that contains radiation as well as non-relativistic matter (dust). For this purpose, rewrite the Friedmann equation as a differential equation with respect to the conformal time η, solve this equation for the three possible values of $\mathcal{K}$ and write out the result in the form $(a(\eta), t(\eta))$.

Determine the t-dependence of $a(t)$ for early ($t \to 0$) and late ($t \to \infty$) times for all the possible values of $\mathcal{K}$.