Exercise 62 (10 credit points): Friedmann III

Current observations indicate that we live in a flat ($\mathcal{K} = 0$) universe with positive cosmological constant, in which the contribution of radiation to the total energy density can be neglected, i.e. the only contributions come from non-relativistic matter (dust) and the cosmological constant.

Solve the Friedmann equation for this model. (Hint: The substitution $x^2 = (1/\Omega_{m,0}) - 1/a^3$ could be helpful.) Determine the age of the universe as a function of $H_0$ and $\Omega_{m,0}$. How does $a(t)$ behave for large and small values of $t$?

According to well-established models for stellar evolution, several globular clusters in our galaxy are at least 12 billion years old. Draw a ($h - \Omega_m$)-diagram ($h$ is the parameter in the definition of $H_0$) and sketch the contour lines for a constant age of the universe. Determine which parameter range is compatible with the above-mentioned observation. In doing so, only consider values $0.4 < h < 1$.

Current observations by the Planck satellite indicate that in the present universe $\Omega_{m,0} \approx 0.31$ and $\Omega_\Lambda \approx 0.69$.

Calculate the redshift at which the energy density of matter was equal to that of the vacuum. Compare this to the redshift at which $\ddot{a}$ was equal to zero.

Exercise 63 (10 credit points): Dark energy

One way to simulate a cosmological constant is by means of a homogeneous scalar field $\phi$ with a suitable potential $V(\phi)$. For this purpose, consider the action

$$ S = \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). $$

63.1 Derive the equation of motion for a homogeneous field $\phi(t)$ in a Friedmann universe.

63.2 Calculate the energy–momentum tensor of the scalar field by means of a variation with respect to the metric. Specialize this calculation to a homogeneous field in a Friedmann universe and identify its energy–momentum tensor with that of an ideal fluid. That way, determine the energy density $\rho_\phi$ and the pressure $p_\phi$. For which idealization does $\phi$ describe a cosmological constant?

63.3 In a concrete model one considers the potential $V(\phi) = \kappa/\phi^4$ with at first arbitrary parameters $\kappa$ and $a$. The scale factor shall obey the time evolution $a(t) \propto t^n$ (universe with $\mathcal{K} = 0$; $n = 2$ during matter domination, $n = 3/2$ during radiation domination).

Look for a solution for $\phi$ of the form $\phi(t) \propto t^A$. Determine $A$ and find the relation that has to be imposed between $\kappa$ and $a$. Finally, calculate the energy density $\rho_\phi$ and compare this to the density $\rho$ of matter (or radiation, respectively).