10th exercise sheet on Relativity and Cosmology I
Winter term 2012/13

Deadline for delivery: Thursday, 20th December 2012 during the exercise class.

Exercise 25 (14 credit points): Maxwell theory

Consider the Lagrange density of the electromagnetic field in the vacuum:

\[ \mathcal{L} = - \frac{\sqrt{-g}}{16\pi} F_{\mu\nu} F^{\mu\nu}, \quad \text{where} \quad F_{\mu\nu} := 2 \partial_{[\mu} A_{\nu]} . \]

25.1 Derive the field equations by means of the principle of least action.

25.2 Calculate the energy–momentum tensor \( T_{\mu\nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \).

25.3 Show by direct calculation that the covariant divergence of the energy–momentum tensor \( \nabla_\mu T^{\mu\nu} \) vanishes.

Exercise 26 (6 credit points): Conformal transformations

Two metrics \( g \) and \( \bar{g} \) are defined to be conformal to each other if there is a non-vanishing differentiable function \( \Omega(x) \) such that

\[ \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) . \]

26.1 Show that angles between two vectors are conserved under a conformal transformation.

26.2 Check that the Christoffel symbol behaves under a conformal transformation as follows:

\[ \bar{\Gamma}^{\mu}_{\nu\kappa} = \Gamma^{\mu}_{\nu\kappa} + S^{\mu}_{\nu\kappa} , \quad \text{where} \quad S^{\mu}_{\nu\kappa} := 2 \delta^{\mu}_{(\nu} \sigma_{\kappa)} - g_{\nu\kappa} \sigma^{\mu} \quad \text{and} \quad \sigma_{\mu} := \partial_{\mu} \log \Omega . \]

Is \( S^{\mu}_{\nu\kappa} \) a tensor?

26.3 Show that lightlike geodesics with respect to a metric \( g_{\mu\nu} \) are also lightlike geodesics with respect to a conformally transformed metric \( \bar{g}_{\mu\nu} \).