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www.thp.uni-koeln.de/gravitation/courses/rci12.html

13th exercise sheet on Relativity and Cosmology I

Winter term 2012/13

Deadline for delivery: Monday, 28th January 2013 at the end of the lecture.

Exercise 32 (20 credit points): *Differential forms*

32.1 Consider an n-dimensional manifold with a metric. Let $\{\omega^i\}$ be an orthonormal basis of 1-forms, and let ω be the preferred volume form $\omega = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n$.
Show that in an arbitrary coordinate system $\{x^k\}$ the following holds:

$$\omega = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n ,$$

where g denotes the determinant of the metric whose components g_{ij} are given in these coordinates.

32.2 The contraction of a p-form ω (with components $\omega_{ij\ldots k}$) with a vector v (with components v^i) is given by $[\omega(v)]_{j\ldots k} = \omega_{ij\ldots k}\, v^i$. Consider the n-form $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$.
Show that with a given vector field v the following holds:

$$\mathrm{d}[\omega(v)] = v^i{}_{,i}\, \omega .$$

32.3 We define $(\mathrm{div}_\omega v)\, \omega := \mathrm{d}[\omega(v)]$.
Show that by using coordinates in which ω has the form $\omega = f\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n$ the following holds:

$$\mathrm{div}_\omega v = \frac{1}{f} \left(f v^i \right)_{,i} .$$

32.4 In three-dimensional Euclidean space, the preferred volume form is given by $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
Show that in spherical coordinates this volume form is given by $\omega = r^2 \sin\theta\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$.
Use the result of **32.3** to show that the divergence of a vector field

$$v = v^r \frac{\partial}{\partial r} + v^\theta \frac{\partial}{\partial \theta} + v^\phi \frac{\partial}{\partial \phi}$$

is given by

$$\mathrm{div}\, v = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v^r \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta\, v^\theta \right) + \frac{\partial v^\phi}{\partial \phi} .$$