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www.thp.uni-koeln.de/gravitation/courses/rci12.html

6th exercise sheet on Relativity and Cosmology I

Winter term 2012/13

Deadline for delivery: Thursday, 22nd November 2012 during the exercise class.

Exercise 16 (5 credit points): *Curvature II*

Consider a family of Gaussian curves $z = \exp(-a^2 r^2)$ with $r^2 = x^2 + y^2$, embedded into a flat 3-dimensional space. Determine the metric on a Gaussian curve using polar coordinates (r, φ) and calculate the curvature at the apex using two different methods:

16.1 Use the two formulae given in the lecture (comparison of circumference and comparison of area).

16.2 Approximate a spherical shell and use the known curvature of a sphere with radius R.

Exercise 17 (10 credit points): *On the covariant derivative*

17.1 In order to define the general covariant derivative $\widetilde{\nabla}_\mu$, one only needs a connection $\widetilde{\Gamma}^{\nu}{}_{\mu\lambda}$, where at first the connection is not linked to the metric in any way:

$$\widetilde{\nabla}_\mu T^{\nu}{}_{\kappa} = \partial_\mu T^{\nu}{}_{\kappa} + \widetilde{\Gamma}^{\nu}{}_{\mu\lambda} T^{\lambda}{}_{\kappa} - \widetilde{\Gamma}^{\lambda}{}_{\mu\kappa} T^{\nu}{}_{\lambda} .$$

Show that the two conditions

$$Q_{\mu\nu\kappa} \equiv -\widetilde{\nabla}_\mu g_{\nu\kappa} = 0 , \qquad T^{\mu}{}_{\nu\kappa} \equiv 2 \widetilde{\Gamma}^{\mu}{}_{[\nu\kappa]} = 0$$

are equivalent to the condition that the connection $\widetilde{\Gamma}^{\mu}{}_{\nu\kappa}$ is the Christoffel symbol of second kind $\Gamma^{\mu}{}_{\nu\kappa}$.

The quantities $Q_{\mu\nu\kappa}$ und $T^{\mu}{}_{\nu\kappa}$ are called *non-metricity* and *torsion*, respectively. Are they tensors?

17.2 Let V^μ be a vector field and $\mathcal{V}^\mu \equiv \sqrt{-g}\, V^\mu$ be the corresponding vector density. Show that

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, V^\mu \right) \quad \text{and} \quad \nabla_\mu \mathcal{V}^\mu = \partial_\mu \mathcal{V}^\mu .$$

17.3 The covariant wave operator for a scalar field ϕ is given by

$$\Box \phi \equiv \nabla^\mu \nabla_\mu \phi .$$

Rewrite this by means of **17.2** such that the resulting expression only contains partial derivatives.

As an example, calculate the wave operator in 3-dimensional spherical coordinates.

Exercise 18 (5 credit points): *Riemannian normal coordinates*

Show that near the origin of a Riemannian normal coordinate system ($\xi^\mu \ll 1$) the following holds:

$$g_{\mu\nu}(0 + \xi) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\kappa\lambda\nu}(0)\, \xi^\kappa \xi^\lambda + \dots$$

Give a physical interpretation.

See reverse.

Exercise B_1 (8 **bonus** points): *Stereographic projection*

Consider a unit 2-sphere.

B_1.1 Introduce stereographic coordinates $(x_\mathrm{N}, y_\mathrm{N})$ and $(x_\mathrm{S}, y_\mathrm{S})$ by projecting the north and south pole, respectively, onto the equatorial plane and express these coordinates in terms of 3-dimensional cartesian coordinates (x, y, z). What is the domain of definition of these coordinate mappings?

B_1.2 Determine the coordinates (x, y, z) on the sphere as a function of $(x_\mathrm{N}, y_\mathrm{N})$ and $(x_\mathrm{S}, y_\mathrm{S})$, respectively, and thereby calculate the 2-dimensional metric which is induced on the sphere by the flat metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ in terms of both coordinate systems.

B_1.3 Determine explicitly the transfer function on the overlap of the domains of definition and show that the unit sphere combined with both coordinate mappings forms a differentiable manifold.

B_1.4 Verify the transformation formula for the metric by inserting the expressions obtained above:

$$g^\mathrm{S}_{jl} = \frac{\partial x^i_\mathrm{N}}{\partial x^j_\mathrm{S}} \frac{\partial x^k_\mathrm{N}}{\partial x^l_\mathrm{S}} g^\mathrm{N}_{ik} \,.$$