7th exercise sheet on Relativity and Cosmology I
Winter term 2012/13

Deadline for delivery: Thursday, 29th November 2012 during the exercise class.

Exercise 19 (10 credit points): Algebraic properties of the curvature tensor

The first algebraic identity of the curvature tensor,
\[ R_{\mu\nu\kappa\lambda} = -R_{\mu\nu\lambda\kappa}, \]
or
\[ R_{\mu\nu}(\kappa\lambda) = 0, \]
reflects the fact that the latter two indices of the curvature tensor can be associated with a surface element (bi-vector), which is always antisymmetric. This antisymmetry can be seen directly from the definition and also holds for arbitrary asymmetric connections.

The second algebraic identity,
\[ R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda}, \]
or
\[ R_{(\mu\nu)}\kappa\lambda = 0, \]
can be verified rather quickly in the case of a symmetric connection (i.e. Christoffel symbol), but also holds in more general cases like in a Riemann–Cartan space, in which also torsion appears in addition to curvature. Can you give an illustrative interpretation of the second algebraic identity?

19.1 The third algebraic identity is a particularity of Riemannian space. Show that in the case that the connection is given by the Christoffel symbol, the following relation holds:
\[ R_{\mu\nu\kappa\lambda} = 0. \]

19.2 How many independent components does the curvature tensor have in Riemann–Cartan space, i.e. if one only considers the first and second algebraic identity? How can curvature be represented by a matrix in this case?

19.3 Show the following equivalence relation for an arbitrary tensor of rank 4:

\[
\begin{align*}
R_{\mu(\kappa\lambda)} &= 0 \\
R_{(\mu\nu)}\kappa\lambda &= 0 \\
R_{\mu[\kappa\lambda]} &= 0 \\
\end{align*}
\]
\[
\begin{align*}
R_{\mu\nu(\kappa\lambda)} &= 0 \\
R_{\mu\nu\kappa\lambda} &= R_{\kappa\lambda\mu\nu} \\
R_{[\mu\nu][\kappa\lambda]} &= 0 \\
\end{align*}
\]

Make yourself aware of the consistency by counting the number of conditional equations.

Hint: One example of how to prove this is by using the following formula, which holds for an arbitrary tensor \( T \) of rank 4 that obeys the first and second algebraic identity:
\[ T_{\mu\nu\kappa\lambda} - T_{\kappa\lambda\mu\nu} = \frac{3}{2} \left( T_{v[\lambda\kappa\mu]} + T_{\kappa[\lambda\nu\mu]} + T_{\lambda[\nu\kappa\mu]} + T_{\mu[\kappa\lambda\nu]} \right). \]

Exercise 20 (10 credit points): Geodesic deviation equation

Consider two adjacent geodesics ("freely falling particles") with paths \( x^\mu(\tau) \) and \( x^\mu(\tau) + \xi^\mu(\tau) \).
\( \xi^\mu(\tau) \) is considered to be "small" in the sense that terms of quadratic and higher order can be neglected.

Show that
\[ \frac{D^2 \xi^\mu}{D\tau^2} = R^\mu_{\nu\kappa\lambda} u^\nu u^\kappa \xi^\lambda, \]
where \( u^\mu = dx^\mu/d\tau \).

Hints: At first, formulate the geodesic equations for \( x^\mu(\tau) \) and \( x^\mu(\tau) + \xi^\mu(\tau) \), then take the difference and expand up to the first order in \( \xi^\mu(\tau) \); the result is an equation that contains \( d^2\xi^\mu/d\tau^2 \). Afterwards compute the general expression for \( D^2 \xi^\mu/D\tau^2 \) and replace the term \( d^2\xi^\mu/d\tau^2 \) appearing therein by means of the equation found before.