11th exercise sheet on Relativity and Cosmology I
Winter term 2013/14

Deadline for delivery: Thursday, 16th January 2014 during the exercise class.

Exercise 28 (11 credit points): Polarization

Consider two coordinate systems \((t, x, y, z)\) and \((t', x', y', z)\) that can be transformed into each other by a rotation with the angle \(\theta\) around the \(z\)-axis.

28.1 Let \(\hat{e}_x, \hat{e}_y, \hat{e}_x', \) and \(\hat{e}_y'\) be the unit polarization vectors in both coordinate systems for an electromagnetic wave that propagates in the \(z\)-direction. Show that

\[
\hat{e}_x' = \hat{e}_x \cos(\theta) + \hat{e}_y \sin(\theta), \quad \hat{e}_y' = -\hat{e}_x \sin(\theta) + \hat{e}_y \cos(\theta).
\]

28.2 Analogously, let \(e_+, e_-, e_{\times}, e_{\times}'\) be the polarization tensors for a gravitational wave in the linearized theory. Show that

\[
e_{\times}' = e_+ \cos(2\theta) + e_- \sin(2\theta), \quad e_{\times}' = -e_+ \sin(2\theta) + e_- \cos(2\theta).
\]

28.3 Let \(|\uparrow\rangle\) and \(|\downarrow\rangle\) be the quantum-mechanical states of a neutrino whose spin is aligned parallelly or antiparallelly with respect to the \(x\)-direction, respectively, and analogously \(|\uparrow'\rangle\) and \(|\downarrow'\rangle\) with respect to the \(x'\)-direction. Show that

\[
|\uparrow'\rangle = |\uparrow\rangle \cos\left(\frac{\theta}{2}\right) + i|\downarrow\rangle \sin\left(\frac{\theta}{2}\right), \quad |\downarrow'\rangle = i|\uparrow\rangle \sin\left(\frac{\theta}{2}\right) + |\downarrow\rangle \cos\left(\frac{\theta}{2}\right).
\]

28.4 Determine the generalization for the basis states of linear polarization for a radiation field of arbitrary spin \(s\).

Exercise 29 (9 credit points): Gauge transformation

In the linear approximation to general relativity, we make the ansatz

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\psi_{\mu\nu}(x),
\]

where \(\psi_{\mu\nu}\) is ‘small’.

29.1 Show that under the infinitesimal transformation

\[
x'\mu = x^\mu - 2 f^\mu(\chi)
\]

one arrives at the following ‘gauge’ transformation law for \(\psi_{\mu\nu}\):

\[
\psi'_{\mu\nu}(x') = \psi_{\mu\nu}(x) + f_{\mu\nu}(x) + f_{\mu\nu}(x).
\]

29.2 Show that the Riemann tensor at the linearized level is invariant under this gauge transformation.