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www.thp.uni-koeln.de/gravitation/courses/rci13.html

11th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 16th January 2014 during the exercise class.

Exercise 28 (11 credit points): *Polarization*

Consider two coordinate systems (t, x, y, z) and (t, x', y', z) that can be transformed into each other by a rotation with the angle θ around the z-axis.

28.1 Let $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_{x'}$, and $\hat{\mathbf{e}}_{y'}$ be the unit polarization vectors in both coordinate systems for an electromagnetic wave that propagates in the z-direction. Show that

$$\hat{\mathbf{e}}_{x'} = \hat{\mathbf{e}}_x \, \cos(\theta) + \hat{\mathbf{e}}_y \, \sin(\theta) \,, \qquad \hat{\mathbf{e}}_{y'} = -\, \hat{\mathbf{e}}_x \, \sin(\theta) + \hat{\mathbf{e}}_y \, \cos(\theta) \,.$$

28.2 Analogously, let $\mathbf{e}_+$, $\mathbf{e}_\times$, $\mathbf{e}_{+'}$, $\mathbf{e}_{\times'}$ be the polarization tensors for a gravitational wave in the linearized theory. Show that

$$\mathbf{e}_{+'} = \mathbf{e}_+ \, \cos(2\,\theta) + \mathbf{e}_\times \, \sin(2\,\theta) \,, \qquad \mathbf{e}_{\times'} = -\, \mathbf{e}_+ \, \sin(2\,\theta) + \mathbf{e}_\times \, \cos(2\,\theta) \,.$$

28.3 Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the quantum-mechanical states of a neutrino whose spin is aligned parallelly or anti-parallelly with respect to the x-direction, respectively, and analogously $|\uparrow'\rangle$ and $|\downarrow'\rangle$ with respect to the x'-direction. Show that

$$|\uparrow'\rangle = |\uparrow\rangle \, \cos\left(\frac{\theta}{2}\right) + \mathrm{i}\,|\downarrow\rangle \, \sin\left(\frac{\theta}{2}\right) \,, \qquad |\downarrow'\rangle = \mathrm{i}\,|\uparrow\rangle \, \sin\left(\frac{\theta}{2}\right) + |\downarrow\rangle \, \cos\left(\frac{\theta}{2}\right) .$$

28.4 Determine the generalization for the basis states of linear polarization for a radiation field of arbitrary spin s.

Exercise 29 (9 credit points): *Gauge transformation*

In the linear approximation to general relativity, we make the ansatz

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2\,\psi_{\mu\nu}(x) \,,$$

where $\psi_{\mu\nu}$ is 'small'.

29.1 Show that under the infinitesimal transformation

$$x'^{\mu} = x^{\mu} - 2\,f^{\mu}(x^{\nu})$$

one arrives at the following 'gauge' transformation law for $\psi_{\mu\nu}$:

$$\psi'_{\mu\nu}(x') = \psi_{\mu\nu}(x) + f_{\mu,\nu}(x) + f_{\nu,\mu}(x) \,.$$

29.2 Show that the Riemann tensor at the linearized level is invariant under this gauge transformation.