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www.thp.uni-koeln.de/gravitation/courses/rci13.html

13th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 30th January 2014 during the exercise class.

Exercise 32 (9 credit points): *The Schwarzschild metric in isotropic coordinates*

Consider the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,.$$

32.1 Use the coordinate transformation

$$t = \bar{t}\,, \qquad r = \left(1+\frac{M}{2\bar{r}}\right)^2 \bar{r}$$

to express the metric in terms of the so-called *isotropic coordinates* $\bar{t}$, $\bar{r}$.

How does the metric behave at the horizon?

32.2 Use the Schwarzschild geometry in isotropic coordinates derived above to calculate the surface of an equatorial circular ring that ranges from the Schwarzschild radius to a fixed radius R, as well as the volume of a spherical shell between these radii.

Compare your results to those in a Euclidean space.

Exercise 33 (5 credit points): *Lemaître coordinates*

Find a suitable coordinate transformation to show that the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \frac{4}{9}\left[\frac{9M}{2\,(r-t)}\right]^{\frac{2}{3}}\mathrm{d}r^2 + \left[\frac{9M}{2}(r-t)^2\right]^{\frac{2}{3}}\mathrm{d}\Omega^2\,,$$

which seems to be dynamical, is in fact the static Schwarzschild metric.

Exercise 34 (6 credit points): *Wormholes*

Consider the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + \left(b^2+r^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right),$$

where b is a constant of dimension length. Illustrate this geometry by embedding it into a flat space.

To do so, choose the slicings $t = \text{const.}$ and $\vartheta = \frac{\pi}{2}$. Why does this suffice?
Map the resulting 2-dimensional geometry with the line element

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + \left(b^2+r^2\right)\mathrm{d}\varphi^2$$

onto a surface in $\mathbb{R}^3$ having the same geometry. Use cylindrical coordinates with the line element

$$\mathrm{d}\ell^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2\,.$$

Find the function $z(r(\rho))$ and draw a sketch of the rotation surface of the curve described by this function.