University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Manuel Krämer and Thaisa Guio

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www.thp.uni-koeln.de/gravitation/courses/rci13.html

4th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 14th November 2013 during the exercise class.

Exercise 9 (5 credit points): *Clocks*

Two atomic clocks are transported in two airplanes once around the Earth in either eastern or western direction. Calculate the respective time dilations the clocks exhibit after the airplanes have landed compared to a clock which stayed on the ground. For this purpose, assume for reasons of simplicity that the airplanes fly directly above the equator, where the rotation velocity of the Earth is about $v_{\mathrm{E}} \approx 1667$ km/h. Furthermore, use an average cruising speed of $v_{\mathrm{F}} \approx 800$ km/h and a mean flying altitude of 10 km.

Hint: Take into account the separate contributions due to the gravitational and the special relativistic velocity effect, respectively.

Exercise 10 (4 credit points): *Accelerated frame of reference and gravitational redshift*

10.1 Consider a frame of reference which is accelerated in x-direction as given in exercise 7. In this frame a photon is emitted from a source at the height x_1 and is absorbed at the height x_2.

Calculate the frequency shift compared to a non-accelerated reference frame with $g = 0$. Perform an expansion up to the first order in gx'/c^2.

10.2 A photon "falls" in a homogeneous gravitational field $\mathbf{g} = g\,\mathbf{e}_x$ for a distance Δx along $\mathbf{e}_x$.

Use the equivalence of mass and energy (Planck–Einstein relation) and the formula for the potential energy known from the Newtonian theory of gravity to derive the frequency shift of the photon. Discuss your results with regard to the equivalence principle.

Exercise 11 (6 credit points): *Motion in the gravitational field*

The equation of motion for a test particle in a gravitational field is given by

$$\ddot{x}^\mu + \Gamma^\mu{}_{\nu\kappa}\,\dot{x}^\nu \dot{x}^\kappa = 0\,, \tag{1}$$

where $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, τ is the proper time and $\Gamma^\mu{}_{\nu\kappa} = \frac{1}{2}\,g^{\mu\sigma}\,(\partial_\kappa\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\kappa} - \partial_\sigma\, g_{\nu\kappa})$.

11.1 Repeat briefly the derivation of (1) from the variational principle $\delta \int \mathrm{d}\tau = 0$ as presented in the lecture. Why can the derivation not be used for photons?

11.2 Derive (1) from the alternative variational principle

$$\delta \int g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu\,\mathrm{d}\lambda \equiv \delta \int \mathcal{K}\,\mathrm{d}\lambda = 0\,,$$

where λ is an affine parameter and $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$.
Show that this derivation also holds for photons and determine $\mathcal{K}$ for the solution of (1).

See reverse.

Exercise 12 (5 credit points): *Christoffel symbols*

Derive the transformation properties of the Christoffel symbols

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\lambda\mu,\nu} - g_{\nu\lambda,\mu}\right)$$

under a coordinate transformation $x^\mu \to x'^\mu(x^\alpha)$.

(The result shows that the Christoffel symbols do not form a tensor.)