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www.thp.uni-koeln.de/gravitation/courses/rci13.html

5th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 21st November 2013 during the exercise class.

Exercise 13 (7 credit points): *Rotating reference frame*

Calculate the Christoffel symbols for a system which rotates with constant angular velocity ω around the z-axis in the Newtonian approximation and formulate the geodesic equation for this case.
Identify the centrifugal force and the Coriolis force in the resulting equation of motion.

Exercise 14 (4 credit points): *Freely falling observer*

The equation of motion of a point mass in a (flat) (1+1)-dimensional Minkowski space is given by

$$m\,\ddot{x} - m\,g = 0\,.$$

We can obtain this equation from the equation of motion $\ddot{x}^\mu + \Gamma^\mu{}_{\nu\kappa}\,\dot{x}^\nu\dot{x}^\kappa = 0$ by setting $\Gamma^1{}_{00} = -g$ and $\Gamma^\mu{}_{\nu\kappa} = 0$ otherwise. On physical grounds it is obvious that there should exist a reference frame in which all the Christoffel symbols vanish and the equation of motion for a free point mass therefore reads $m\,\ddot{x} = 0$.

Find such a coordinate system by integrating the respective transformation rules of the Christoffel symbols.

Exercise 15 (4 credit points): *Curvature*

The ratio of the Scharzschild radius of a body to its radius is a heuristic measure for the deviation of the geometry in the neighbourhood of the considered body from a flat Minkowski spacetime.

15.1 Compare this ratio for a globular cluster of stars ($M \approx 10^6 M_\odot$, $R \approx 20$ pc), the Sun, the Earth, a neutron star ($M \approx M_\odot$, $R \approx 10$ km), a White Dwarf ($M \approx M_\odot$, $R \approx 10^4$ km) as well as for a proton and an electron. For the latter two, use their Compton wavelengths $\hbar/mc$ as the (effective) radius.

15.2 Which mass would an elementary particle need to have, such that its Compton wavelength would be as large as its Schwarzschild radius? What size would its Schwarzschild radius then be?

15.3 The quantities appearing in these considerations are often expressed in terms of the so-called Planck units, which result from a unique combination of the natural constants G, c and $\hbar$. Calculate the Planck mass, the Planck length, the Planck time and the Planck energy in CGS units or SI units.

Exercise 16 (5 credit points): *Derivative of a determinant*

Show that the partial derivative of the determinant of a non-singular matrix M with respect to a coordinate x^μ is given by the formula:

$$\frac{\partial}{\partial x^\mu}\det(M) = \det(M)\,\mathrm{tr}\left(M^{-1}\frac{\partial}{\partial x^\mu}M\right). \tag{1}$$

Hint: Every square matrix A can be transformed into the Jordan normal form via the similarity transformation $A \to BAB^{-1}$.

Use (1) to give a proof for the following formula known from the lecture, where $g_{\mu\nu}$ is the metric tensor and $g \equiv \det(g_{\mu\nu})$:

$$\frac{\partial g}{\partial x^\nu} = g^{\mu\kappa}\,g\,\frac{\partial g_{\mu\kappa}}{\partial x^\nu} = -g_{\mu\kappa}\,g\,\frac{\partial g^{\mu\kappa}}{\partial x^\nu}$$