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www.thp.uni-koeln.de/gravitation/courses/rci13.html

7th exercise sheet on Relativity and Cosmology I

Winter term 2013/14

Deadline for delivery: Thursday, 5th December 2013 during the exercise class.

Exercise 20 (12 credit points): *Algebraic properties of the curvature tensor*

The first algebraic identity of the curvature tensor,

$$R_{\mu\nu\kappa\lambda} = -R_{\mu\nu\lambda\kappa} \quad \text{or} \quad R_{\mu\nu(\kappa\lambda)} = 0\,,$$

reflects the fact that the latter two indices of the curvature tensor can be associated with a surface element (*bi-vector*), which is always antisymmetric. This antisymmetry can be seen directly from the definition and also holds for arbitrary asymmetric connections.

The second algebraic identity,

$$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} \quad \text{or} \quad R_{(\mu\nu)\kappa\lambda} = 0\,,$$

can be verified rather quickly in the case of a symmetric connection (i. e. Christoffel symbol), but also holds in more general cases like in a *Riemann–Cartan space*, in which also torsion appears in addition to curvature. Can you give an illustrative interpretation of the second algebraic identity?

20.1 The third algebraic identity is a particularity of Riemannian space. Show that in the case that the connection is given by the Christoffel symbol, the following relation holds:

$$R^{\mu}{}_{[\nu\kappa\lambda]} = 0\,.$$

20.2 How many independent components does the curvature tensor have in *Riemann–Cartan space*, i. e. if one only considers the first and second algebraic identity? How can curvature be represented by a matrix in this case?

20.3 Show the following equivalence relation for an arbitrary tensor of rank 4:

$$\left.\begin{array}{rcl} R_{\mu\nu(\kappa\lambda)} &=& 0 \\ R_{(\mu\nu)\kappa\lambda} &=& 0 \\ R_{\mu[\nu\kappa\lambda]} &=& 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} R_{\mu\nu(\kappa\lambda)} &=& 0 \\ R_{(\mu\nu)\kappa\lambda} &=& 0 \\ R_{\mu\nu\kappa\lambda} &=& R_{\kappa\lambda\mu\nu} \\ R_{[\mu\nu\kappa\lambda]} &=& 0 \end{array}\right.$$

Make yourself aware of the consistency by counting the number of conditional equations.

Hint: One example of how to prove this is by using the following formula, which holds for an arbitrary tensor T of rank 4 that obeys the first and second algebraic identity:
$T_{\mu\nu\kappa\lambda} - T_{\kappa\lambda\mu\nu} = \frac{3}{2}\left(T_{\nu[\lambda\kappa\mu]} + T_{\kappa[\lambda\nu\mu]} + T_{\lambda[\nu\kappa\mu]} + T_{\mu[\kappa\lambda\nu]}\right).$

Exercise 21 (8 credit points): *Geodesic deviation equation*

Consider two adjacent geodesics (“freely falling particles”) with paths $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$.
$\xi^\mu(\tau)$ is considered to be “small” in the sense that terms of quadratic and higher order can be neglected.

Show that

$$\frac{\mathrm{D}^2\xi^\mu}{\mathrm{D}\,\tau^2} = R^{\mu}{}_{\nu\kappa\lambda}\, u^\nu\, u^\kappa\, \xi^\lambda\,,$$

where $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$.

Hints: At first, formulate the geodesic equations for $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu(\tau)$, then take the difference and expand up to the first order in $\xi^\mu(\tau)$; the result is an equation that contains $\mathrm{d}^2\xi^\mu/\mathrm{d}\tau^2$. Afterwards compute the general expression for $\mathrm{D}^2\xi^\mu/\mathrm{D}\tau^2$ and replace the term $\mathrm{d}^2\xi^\mu/\mathrm{d}\tau^2$ appearing therein by means of the equation found before.