**12th exercise sheet on Relativity and Cosmology I**  
*Winter term 2015/16*

**Deadline for delivery:** Thursday, 4th February 2016 during the exercise class.

### Exercise 29 (9 credit points): The Schwarzschild metric in isotropic coordinates

Consider the Schwarzschild metric

\[ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. \]

#### 29.1 Use the coordinate transformation

\[ t = \bar{t}, \quad r = \left( 1 + \frac{M}{2\bar{r}} \right)^2 \bar{r} \]

to express the metric in terms of the so-called isotropic coordinates \( \bar{t}, \bar{r} \).

How does the metric behave at the horizon?

#### 29.2 Use the Schwarzschild geometry in isotropic coordinates derived above to calculate the surface of an equatorial circular ring that ranges from the Schwarzschild radius to a fixed radius \( R \), as well as the volume of a spherical shell between these radii.

Compare your results to those in a Euclidean space.

### Exercise 30 (5 credit points): Lemaître coordinates

Find a suitable coordinate transformation to show that the metric

\[ ds^2 = - dt^2 + \frac{4}{9} \left[ \frac{9M}{2(r-t)} \right]^\frac{2}{3} dr^2 + \left[ \frac{9M}{2}(r-t)^2 \right]^\frac{2}{3} d\Omega^2, \]

which seems to be dynamical, is in fact the static Schwarzschild metric.

### Exercise 31 (6 credit points): Wormholes

Consider the metric

\[ ds^2 = - dt^2 + dr^2 + \left( b^2 + r^2 \right) \left( d\theta^2 + \sin^2(\theta) d\phi^2 \right), \]

where \( b \) is a constant of dimension length. Illustrate this geometry by embedding it into a flat space.

To do so, choose the slicings \( t = \text{const.} \) and \( \theta = \frac{\pi}{2} \). Why does this suffice?

Map the resulting 2-dimensional geometry with the line element

\[ d\Sigma^2 = dr^2 + \left( b^2 + r^2 \right) d\phi^2 \]

onto a surface in \( \mathbb{R}^3 \) having the same geometry. Use cylindrical coordinates with the line element

\[ d\ell^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2. \]

Find the function \( z(r(\rho)) \) and draw a sketch of the rotation surface of the curve described by this function.