6th exercise sheet on Relativity and Cosmology I
Winter term 2015/16

Deadline for delivery: Thursday, 3rd December 2015 during the exercise class.

Exercise 15 (5 credit points): Curvature II

Consider a family of Gaussian curves \( z = \exp(-a^2 r^2) \) with \( r^2 = x^2 + y^2 \), embedded into a flat 3-dimensional space. Determine the metric on the surface formed by these Gaussian curves using polar coordinates \((r, \varphi)\) and calculate the curvature at the apex using three different methods:

15.1 Use the two formulae given in the lecture: a) comparison of circumference and b) comparison of area.
15.2 Find the spherical shell with radius \( R \) that approximates the given surface best around the apex and use the known curvature of a sphere with radius \( R \).

Exercise 16 (6 credit points): Christoffel symbols

Derive the transformation properties of the Christoffel symbols \( \Gamma_{\mu\nu\lambda} = \frac{1}{2} (g_{\mu\nu,\lambda} + g_{\lambda\mu,\nu} - g_{\nu\lambda,\mu}) \) under a coordinate transformation \( x^\mu \to x'^\nu(x^\alpha) \).
(The result shows that the Christoffel symbols do not form a tensor.)

Exercise 17 (9 credit points): Metricity

It was stated in the lecture that the metric is covariantly constant for Riemannian spaces, meaning that its covariant derivative vanishes,
\[ \nabla_\alpha g_{\mu\nu} = 0 \quad \text{and} \quad \nabla_\alpha g^{\mu\nu} = 0. \]

17.1 Prove the above two statements.
17.2 How does \( \nabla_\alpha g_{\mu\nu} = 0 \) transform under a coordinate transformation \( x^\mu \to x'^\mu(x^\alpha) \)?