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www.thp.uni-koeln.de/gravitation/courses/rci1718.html

29th November 2017

7th exercise sheet on Relativity and Cosmology I

Winter term 2017/18

Deadline for delivery: Thursday, 7th December 2017 during the exercise class.

Exercise 19: *On the covariant derivative*

19.1 In order to define the general covariant derivative $\widetilde{\nabla}_\mu$ one only needs *a connection* $\widetilde{\Gamma}^\nu{}_{\mu\lambda}$, where at first the connection is not linked to the metric in any way:

$$\widetilde{\nabla}_\mu A^\nu{}_\kappa := \partial_\mu A^\nu{}_\kappa + \widetilde{\Gamma}^\nu{}_{\mu\lambda} A^\lambda{}_\kappa - \widetilde{\Gamma}^\lambda{}_{\mu\kappa} A^\nu{}_\lambda .$$

1. Show that the two conditions

$$Q_{\mu\nu\kappa} := -\widetilde{\nabla}_\mu g_{\nu\kappa} = 0 , \qquad T^\mu{}_{\nu\kappa} := 2\,\widetilde{\Gamma}^\mu{}_{[\nu\kappa]} = 0$$

are equivalent to the condition that $\widetilde{\Gamma}^\mu{}_{\nu\kappa}$ is the Christoffel symbol of the second kind $\Gamma^\mu{}_{\nu\kappa}$.

2. The quantities $Q_{\mu\nu\kappa}$ and $T^\mu{}_{\nu\kappa}$ are called *non-metricity* (see Exercise 18) and *torsion*, respectively. Are they tensors?

19.2 Let V^μ be a vector field and $\mathfrak{V}^\mu := \sqrt{|g|}\, V^\mu$ be the corresponding vector density. Show that

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\,\partial_\mu\left(\sqrt{|g|}\,V^\mu\right) \quad \text{and} \quad \nabla_\mu \mathfrak{V}^\mu = \partial_\mu \mathfrak{V}^\mu .$$

19.3 The covariant wave operator for a scalar field ϕ is given by

$$\Box\phi := \nabla^\mu \nabla_\mu \phi .$$

1. Rewrite this by means of **19.2** such that the resulting expression only contains partial derivatives.
2. As an example, calculate the wave operator in 3-dimensional spherical coordinates.

Exercise 20: *Riemannian normal coordinates*

1. Show that near the origin of a Riemannian normal coordinate system ($\xi^\mu \ll 1$) the following holds:

$$g_{\mu\nu}(0+\xi) = \eta_{\mu\nu} + \frac{1}{3} R_{\mu\kappa\lambda\nu}(0)\,\xi^\kappa \xi^\lambda + \ldots$$

2. Give a physical interpretation.

Exercise 21: *Algebraic identities of Riemann tensor*

1. Show that, in Riemannian spaces, i.e. where the connection $\widetilde{\Gamma}^\mu{}_{\nu\kappa}$ is given by the Christoffel symbol of the second kind $\Gamma^\mu{}_{\nu\kappa}$, the Riemann tensor satisfies the following algebraic identities:

$$R_{\mu\nu(\alpha\beta)} = 0 , \quad R_{(\mu\nu)\alpha\beta} = 0 , \quad R_{\mu[\nu\alpha\beta]} = 0 .$$

2. Can you give an illustrative interpretation of the first two identities?