10th exercise sheet on Relativity and Cosmology II
Summer term 2013

Deadline for delivery: Thursday, 27th June 2013 during the exercise class.

Exercise 23 (20 bonus points): Derivation of the Friedmann equations

The aim of this exercise is to derive the Friedmann equations using the Cartan formalism.

23.1 Start with the Friedmann–Lemaître–Robertson–Walker line element in the form

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]

and find the orthonormal cobasis \( \vartheta^\mu \) to rewrite this line element as

\[ ds^2 = \eta_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu. \]

For convenience, use the definition \( w := \sqrt{1 - kr^2} \).

23.2 Calculate the exterior derivatives \( d\vartheta^\mu \).

23.3 Determine the connection forms \( \omega^{\mu\nu} \).

Hint: Use the metricity condition as well as the first Cartan structure equation.

23.4 Calculate the curvature 2-forms \( \Omega^{\mu\nu} \) and deduce the components of the Riemann curvature tensor \( R^{\mu\nu}_{\lambda\chi} \) by means of the second Cartan structure equation.

23.5 Determine the components of the Ricci tensor \( R_{\mu\nu} \) as well as the Ricci scalar \( R \).

23.6 Calculate the components of the Einstein tensor \( G_{\mu\nu} \) and derive the Friedmann equations by using the Einstein equations with the energy–momentum tensor \( T_{\mu\nu} = \text{diag}(-\rho(t), P(t), P(t), P(t)) \), where \( \rho \) is the energy density and \( P \) the pressure of an ideal fluid filling the universe.

Why do we use mixed components here?