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www.thp.uni-koeln.de/gravitation/courses/rcii14.html

1st exercise sheet on Relativity and Cosmology II

Summer term 2014

Deadline for delivery: Wednesday, 16th April 2014 during the first exercise class.

Exercise 1 (20 credit points): *Effective Schwarzschild potential*

The aim of this exercise is to analyze certain properties of the movement of massive test particles in the Schwarzschild spacetime.
For this purpose, consider the equation of motion on the plane $\vartheta = \frac{\pi}{2}$ with an effective potential V_{eff} that results from the geodesic equation:

$$\frac{\dot{r}^2}{2} + V_{\text{eff}}(r) = E\,, \qquad V_{\text{eff}}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}\,.$$

Here ℓ and E indicate constants of motion.

In the following, express radial distances in terms of the Schwarzschild radius $r_{\text{S}} = 2M$.

1.1 **a)** Which conditions does a test particle approaching from infinity ($r = \infty$) have to fulfill in order to fall into the center of the effective potential?

b) Under which circumstances does a particle that starts from rest at infinity fall into the center?

Compare your results to the situation in Newtonian gravity.

1.2 **a)** In which cases do bound particle orbits exist?

b) For which values of the radial coordinate r are there stable orbits?

c) Under which conditions are there instable orbits and what happens to a particle which is slightly deflected from such an orbit?

d) Sketch the potential V_{eff} qualitatively for the cases $\ell/M < 2\sqrt{3}$, $\ell/M = 2\sqrt{3}$ and $\ell/M > 2\sqrt{3}$ and show the following statements:

i. For $\ell/M < 2\sqrt{3}$ every infalling particle falls towards the event horizon $r = 2M$.

ii. The most strongly bound stable orbit is located at $r = 6M$ with $\ell/M = 2\sqrt{3}$ and it possesses a relative binding energy of $1 - \sqrt{8/9}$.

1.3 Consider a massive test particle initially being at rest at the radial coordinate $R > r_{\text{S}}$ ($\ell = 0$) that falls radially into the center.

a) Show that its orbit is given by the parametrisation of a cycloid orbit

$$r = \frac{R}{2}\left(1 + \cos\eta\right), \quad \tau = \frac{R}{2}\sqrt{\frac{R}{2M}}\left(\eta + \sin\eta\right),$$

where τ denotes the proper time of the particle. At which proper time τ_0 does the particle reach the center of the potential?

b) Show that the time an observer at infinity measures diverges as the particle approaches the Schwarzschild radius.