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17th June 2014

www.thp.uni-koeln.de/gravitation/courses/rcii14.html

10th exercise sheet on Relativity and Cosmology II

Summer term 2014

Deadline for delivery: Wednesday, 25th June 2014 during the exercise class.

Exercise 20 (20 credit points): *Derivation of the Friedmann equations in Cartan calculus*

The aim of this exercise is to derive the Friedmann equations using the Cartan formalism.

We start with the Robertson–Walker line element in coordinates that is given by:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]. \tag{1}$$

Remember that in terms of the pseudo-orthogonal coframe basis $\{\vartheta^i\}$, $i = 0,\dots,3$, the metric takes the form

$$\mathrm{d}s^2 = \eta_{ij}\,\vartheta^i \otimes \vartheta^j = -\,\vartheta^0 \otimes \vartheta^0 + \vartheta^1 \otimes \vartheta^1 + \vartheta^2 \otimes \vartheta^2 + \vartheta^3 \otimes \vartheta^3\,. \tag{2}$$

Like in exercise **7**, Latin letters are used for anholonomic frame indices, whereas Greek letters are used for holonomic coordinate indices.

20.1 Write out the components of the coframe basis. For convenience, use the definition $w := \sqrt{1-kr^2}$.

20.2 Calculate the exterior derivatives $\mathrm{d}\vartheta^i$. Insert these into the first Cartan structure equation

$$\mathrm{d}\vartheta^i + \omega^i{}_j \wedge \vartheta^j = 0 \tag{3}$$

to determine the 1-form-valued components $\omega^i{}_j$ of the connection.

20.3 Calculate the curvature 2-forms $\Omega^i{}_j$ by using the second Cartan structure equation

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_a \wedge \omega^a{}_j =: \frac{1}{2}\,R^i{}_{jkl}\,\vartheta^k \wedge \vartheta^l \tag{4}$$

and read off the anholonomic components $R^i{}_{jkl}$ of the Riemann curvature tensor.

Intermediate result: The non-vanishing anholonomic components of the Riemann curvature tensor read

$$R^r{}_{ttr} = -\,R^r{}_{trt} = R^\theta{}_{tt\theta} = -\,R^\theta{}_{t\theta t} = R^\phi{}_{tt\phi} = -\,R^\phi{}_{t\phi t} = \frac{\ddot a}{a}\,, \tag{5}$$

$$R^\theta{}_{r\theta r} = -\,R^\theta{}_{rr\theta} = R^\phi{}_{r\phi r} = -\,R^\phi{}_{rr\phi} = R^\phi{}_{\theta\phi\theta} = -\,R^\phi{}_{\theta\theta\phi} = \frac{\dot a^2 + k}{a^2}\,. \tag{6}$$

20.4 Determine the anholonomic components of the Ricci tensor $R_{ij} = R^a{}_{iaj}$ as well as the Ricci scalar $R = \eta^{ij}R_{ij}$. Note that for the contraction of anholonomic indices the Minkowski metric has to be used.

20.5 Calculate the mixed components of the Einstein tensor

$$G^i{}_j \stackrel{*}{=} G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\,\delta^\mu_\nu\,R\,. \tag{7}$$

20.6 Use the energy–momentum tensor of an ideal fluid with energy density ρ and pressure P given by

$$T^\mu{}_\nu = \mathrm{diag}(-\rho(t), P(t), P(t), P(t)) \tag{8}$$

to write out the Einstein equations $G^\mu{}_\nu = 8\pi G\,T^\mu{}_\nu$, which are called Friedmann equations in this case.