University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Manuel Krämer and Jens Boos

15th April 2014

www.thp.uni-koeln.de/gravitation/courses/rcii14.html

2nd exercise sheet on Relativity and Cosmology II

Summer term 2014

Deadline for delivery: Thursday, 24th April 2014 at the end of the lecture, since there will be **no** exercise class on 23rd April.

Exercise 2 (10 credit points): *ADM energy*

Assume that the metric of a certain spacetime can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ are functions that vanish at infinity. In this case the total energy of the system is given by the surface integral

$$E = \frac{1}{16\pi G} \int \sum_{i,j} \left(g_{ij,j} - g_{jj,i} \right) \, \mathrm{d}^2 S_i \, ,$$

which has to be taken over a surface far away from any mass distribution. The Latin indices denote spatial coordinates. Calculate this so-called ADM energy for the Schwarzschild metric.

Hint: The calculation is most easily done in isotropic coordinates, cf. exercise **32** from last semester's course.

Exercise 3 (5 credit points): *Redshift and conserved quantities in the Schwarzschild spacetime*

3.1 Consider a stationary* observer $\mathcal{A}$ at $r = R$, $R \geq 2M$ in the Schwarzschild spacetime of mass M and an observer $\mathcal{B}$ at infinity. The timelike Killing vector shall be denoted by $\xi^\mu = (1, 0, 0, 0)$. Furthermore, we define the quantity $V^2 := -\xi_\mu \xi^\mu$. Observer $\mathcal{A}$ emits energy with frequency ω_R (measured in his rest frame) which is measured by observer $\mathcal{B}$ as being ω_∞.

- **a)** Express the four-velocity u^μ of observer $\mathcal{A}$ in terms of ξ^μ and V and use this to derive the relation between the frequencies ω_R and ω_∞.
- **b)** What does observer $\mathcal{B}$ measure when observer $\mathcal{A}$ reaches the Schwarzschild radius $r = 2M$? What does this mean for the redshift?

3.2 When discussing the movement of particles in the Schwarzschild spacetime, it was shown that the angular momentum

$$\ell := r^2 \sin^2(\vartheta) \, \frac{\mathrm{d}\varphi}{\mathrm{d}s}$$

is a conserved quantity. Derive this result from the existence of a Killing vector $\eta^\mu = (0, 0, 0, 1)$, where the last component corresponds to the φ-component.

Exercise 4 (5 credit points): *Time dilation in the Schwarzschild spacetime*

Show that the proper time $\mathrm{d}\tau$ on a circular geodesic in the Schwarzschild geometry of mass M obeys the relation:

$$\mathrm{d}\tau = \sqrt{1 - \frac{3M}{r}} \, \mathrm{d}t \, .$$

Use this to give an estimate for the time dilation of a satellite flying in a low orbit around the Earth.

---

*A *stationary* observer is an observer in a stationary spacetime whose 4-velocity u^μ is proportional to the given timelike Killing vector.