University of Cologne
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Manuel Krämer and Jens Boos

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www.thp.uni-koeln.de/gravitation/courses/rcii14.html

5th exercise sheet on Relativity and Cosmology II

Summer term 2014

Deadline for delivery: Wednesday, 14th May 2014 during the exercise class.

Exercise 8 (12 credit points): *Reissner–Nordström solution*

The aim of this exercise is to calculate the gravitational field outside of a static, spherically symmetric charge distribution with mass M and charge Q.

For this purpose, use the ansatz

$$\mathrm{d}s^2 = -\,\mathrm{e}^{\nu(t,r)}\,\mathrm{d}t^2 + \mathrm{e}^{\lambda(t,r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{1}$$

for the metric and calculate the electro-magnetic field strength tensor $F^{\mu\nu}$, for example, by using the Maxwell equations in source-free regions

$$F^{\mu\nu}{}_{;\nu} = 0\,.$$

(*Check:* $F^{0r} = -F^{r0} = \mathrm{e}^{-(\lambda+\nu)/2}\,Q/r^2$, otherwise $F^{\mu\nu} = 0$.)

Afterwards, compute the components of the energy-momentum tensor given by

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\lambda}\,F_{\nu}{}^{\lambda} - \frac{1}{4}\,g_{\mu\nu}\,F_{\kappa\sigma}\,F^{\kappa\sigma}\right).$$

Now calculate the metric functions ν and λ by means of the Einstein equations $G^{\mu}{}_{\nu} = \kappa\,T^{\mu}{}_{\nu}$.
(Use the components of the Einstein tensor $G^{\mu}{}_{\nu}$ arising from the ansatz (1) given in the lecture course.)

Choose the integration constant appropriately such that you obtain the correct Newtonian limit and compare the resulting metric to the Schwarzschild solution.

Exercise 9 (8 credit points): *Classical tests of GR for the Reissner–Nordström solution*

Find a coordinate transformation $\bar{r}(r)$ to transform the Reissner–Nordström metric into its isotropic form

$$\mathrm{d}s^2 = -\,A(\bar{r})\,\mathrm{d}t^2 + B(\bar{r})\left(\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2\right),$$

where

$$A(\bar{r}) = \left[\left(1+\frac{M}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right]^{-2}\left[2+\frac{M}{\bar{r}} - \left(1+\frac{M}{2\bar{r}}\right)^2 + \frac{Q^2}{4\bar{r}^2}\right]^2 \quad \text{and} \quad B(\bar{r}) = \left[\left(1+\frac{M}{2\bar{r}}\right)^2 - \frac{Q^2}{4\bar{r}^2}\right]^2,$$

and calculate the coefficients α_1, α_2, β_1, β_2 introduced in the lecture, which are the coefficients of an expansion of $1/A(\bar{r})$ and $B(\bar{r})$ with respect to $u := 1/\bar{r}$:

$$\frac{1}{A(\bar{r})} = 1 + \alpha_1\,Mu + \alpha_2\,M^2u^2 + \mathcal{O}(u^3)\,, \quad B(\bar{r}) = 1 + \beta_1\,Mu + \beta_2\,M^2u^2 + \mathcal{O}(u^3)\,.$$

Use this result to analyze how a charge Q of the Sun would influence the classical tests of General Relativity in the solar system.