6th exercise sheet on Relativity and Cosmology II
Summer term 2014

Deadline for delivery: Wednesday, 21st May 2014 during the exercise class.

Exercise 10 (5 credit points): Kruskal coordinates

Derive the line element of the Schwarzschild metric in Kruskal coordinates as given in the lecture. For this purpose, introduce a new radial coordinate (for \( r > 2M \)) as follows

\[
rs = r + 2M \ln \left( \frac{r}{2M} - 1 \right).
\]

Then perform the coordinate transformation:

\[
X = \exp \left( \frac{rs}{4M} \right) \cosh \left( \frac{t}{4M} \right), \quad T = \exp \left( \frac{rs}{4M} \right) \sinh \left( \frac{t}{4M} \right).
\]

Exercise 11 (6 credit points): Another coordinate system

Construct a coordinate system for the Schwarzschild metric that is singularity-free at the event horizon by transforming the Schwarzschild time \( t \) according to

\[
t \rightarrow T = t + f(r).
\]

Determine \( f(r) \) by imposing that the prefactor of \( dr^2 \) is equal to +1 in the transformed line element. Write out the transformed line element. Is it still static? Which parts of the Kruskal diagram are covered by these coordinates?

Exercise 12 (9 credit points): Penrose diagrams

12.1 Express the line element for Minkowski spacetime in terms of spherical coordinates \( (t, r, \theta, \phi) \). Then perform a coordinate transformation

\[
u = t - r, \quad v = t + r.
\]

Write out the transformed line element. How can one interpret the coordinates \( u \) and \( v \)?

12.2 Perform another coordinate transformation \((u, v) \rightarrow (u', v')\) according to

\[
u' = \arctan(u) =: t' - r', \quad v' = \arctan(v) =: t' + r'.
\]

Draw a \((t', r')\) diagram and hatch the area covered by these coordinates. Then draw a radial light ray in this diagram that goes from infinity (in the original coordinates) to \( r = 0 \) and back to infinity.

In a second \((t', r')\) diagram, sketch the areas \( t = \text{const.} \) and \( r = \text{const.} \).

12.3 Calculate the line element in the primed coordinates and show that it is conformal to the line element

\[
ds^2 = -4 \left( dt'^2 - dr'^2 \right) + \sin^2(2r') \, d\Omega^2.
\]