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http://www.thp.uni-koeln.de/gravitation/courses/rcii18.html

12th April 2018

1st exercise sheet on Relativity and Cosmology II

Summer term 2018

Deadline for delivery: Thursday, 19th April 2018 during the first exercise class.

Exercise 31 (14 credit points): *The Schwarzschild metric in isotropic coordinates*

In the previous semester course we learned that a given metric can be expressed in another coordinate system while still describing the same solution to the Einstein equations.

31.1 What makes this possible? In how many different sets of coordinates can a given metric be expressed?

31.2 In practice it is often convenient to use this freedom to simplify calculations, adapt the form of the metric tensor to describe a desired situation, or/and to reveal its features and symmetries. In this exercise we are going to express the following metric

$$\mathrm{d}s^2 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \qquad \mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\,,$$

describing the Schwarzschild solution, in other coordinates in order to familiarize ourselves with this freedom and obtain more insight into the Schwarzschild solution. For this purpose, use the coordinate transformation

$$t = \bar{t}\,, \qquad r = \left(1+\frac{GM}{2\bar{r}}\right)^2 \bar{r}$$

to express the metric in terms of the coordinates $\bar{t}$, $\bar{r}$.

How does the metric behave at the horizon? These coordinates are called *isotropic coordinates*. Why?

31.3 Use the Schwarzschild geometry in isotropic coordinates derived above to calculate the surface of an equatorial circular ring that ranges from the Schwarzschild radius to a fixed radius $r = R > 2GM$, as well as the volume of a spherical shell between these radii.

Compare your results to those in Euclidean space with the metric $\mathrm{d}\bar{r}^2 + \bar{r}^2\,\mathrm{d}\Omega^2$.

Exercise 32 (6 credit points): *Wormholes*

Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \left(b^2+r^2\right)\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right),$$

where b is a constant with the dimension of length. Illustrate this geometry by embedding it into a flat space.

To do so, choose the slicings $t = \text{const.}$ and $\vartheta = \pi/2$. Why does this suffice?
Map the resulting 2-dimensional geometry with the line element

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + \left(b^2+r^2\right)\mathrm{d}\varphi^2$$

onto a surface in $\mathbb{R}^3$ having the same geometry. Use cylindrical coordinates with the line element

$$\mathrm{d}\ell^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\psi^2 + \mathrm{d}z^2\,.$$

Find the function $z(r(\rho))$ and draw a sketch of the rotation surface of the curve described by this function.