University of Cologne http://www.thp.uni-koeln.de/gravitation/courses/rcii18.html
Institute for Theoretical Physics
Prof. Dr. Claus Kiefer
Branislav Nikolic, Nick Kwidzinski and Yi-Fan Wang 14th June 2018

8th exercise sheet on Relativity and Cosmology II

Summer term 2018

Deadline for delivery: Thursday, 21st June 2018 during the exercise class.

Exercise 44 (16 credit points + 4 bonus): *Kerr–Newman metric*

The most general solution for a stationary black hole is given by the *Kerr–Newman metric*, which describes a black hole with angular momentum $J = M\,a$ and charge q. The line element expressed in Boyer–Lindquist coordinates takes the following form:

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2(\theta)\,d\phi\right)^2 + \frac{\sin^2(\theta)}{\rho^2}\left[\left(r^2+a^2\right)d\phi - a\,dt\right]^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2\,,$$

where

$$\rho^2 = r^2 + a^2\cos^2(\theta)\,,\qquad \Delta = r^2 - 2Mr + q^2 + a^2\,,\qquad q^2 + a^2 \leqslant M^2\,.$$

44.1 How can one obtain this line element from the Kerr metric by means of a simple substitution of certain parameters (without calculation)?

44.2 For $\Delta = 0$ the metric exhibits coordinate singularities. Determine their radial coordinates $r_\pm$.

The surface $r_+ =$ const. (with r_+ being the radial coordinate with a larger value) represents the event horizon. Calculate its surface area for $t =$ const.

44.3 Analogously to the Kerr metric, consider an observer with $r =$ const., $\theta = \pi/2$, whose tangent vector is parallel to the Killing field $\chi^\mu = \xi^\mu + \Omega\,\Psi^\mu$.

Which values can Ω take for given $r \geqslant r_+$? Show that at the horizon only one value Ω_H is possible and determine this value.

44.4 Consider the Killing field $\chi^\mu = \xi^\mu + \Omega\,\Psi^\mu$ evaluated at the event horizon.

Show that this Killing field is light-like on the entire horizon. Furthermore, show that the surface gravity κ defined by means of $\left[\nabla^\mu(\chi_\nu\chi^\nu)\right]_H = -2\,\kappa\,\chi^\mu\big|_H$ is a well-defined quantity.

Calculate the Lie derivative of the defining equation for κ with respect to χ^μ and thereby show that κ is constant along the integral curves of χ.

Remark: After a rather long calculation one obtains $\kappa = (r_+ - M)/(r_+^2 + a^2)$. (Not to be shown here.)

44.5 ***Bonus***. Consider the null geodesics defined at the horizon, whose tangent vectors k^μ are proportional to χ^μ. Find the functional relationship between the affine parameter λ of these null geodesics and the Killing parameter ν of the integral curves of χ^μ, i.e. $\chi^\mu = (\partial/\partial\nu)^\mu$.

Exercise 45 (4 credit points): *Hawking temperature*

In the lecture it was mentioned that a Schwarzschild black hole radiates with the so-called *Hawking temperature*

$$T_H = \frac{\hbar\,c^3}{8\pi\,k_B\,G\,M}\,.$$

Assume that only photons are emitted and that they have a perfect Planck spectrum. Find a relation between the initial mass of the black hole and its lifetime and analyze this relation for several interesting masses and time intervals.