

Light propagation in local and linear media: Fresnel-Kummer wave surfaces with 16 singular points

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It is known that the Fresnel wave surfaces of transparent biaxial media have 4 singular points, located on two special directions. We show that, in more general media, the number of singularities can exceed 4. In fact, a highly symmetric linear material is proposed whose Fresnel surface exhibits 16 singular points. Because, for every linear material, the dispersion equation is quartic, we conclude that 16 is the maximum number of singularities. The identity of Fresnel and Kummer surfaces, which holds true for media with a certain symmetry (zero skewon piece), provides an elegant interpretation of the results. We describe a metamaterial realization for our linear medium with 16 singular points. It is found that an appropriate combination of metal bars, split-ring resonators, and magnetized particles can generate the correct permittivity, permeability, and magnetoelectric moduli. Lastly, we discuss the arrangement of the singularities in terms of Kummer's 16_6 configuration of points and planes. An investigation parallel to ours, but in linear elasticity, is suggested for future research.

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I. INTRODUCTION

In the geometrical optics approximation, the Fresnel (wave) surface describes the propagation of light inside a transparent material. For biaxial media, such as Aragonite (orthorhombic CaCO_3) and Topaz (orthorhombic $\text{Al}_2\text{SiO}_4(\text{F},\text{OH})_2$), the surface consists of two shells that intersect at four locations [1]. Indeed, the property that the four singular points are aligned on two axes lends the name to this class of media. The singularities in the Fresnel surface of a biaxial crystal give rise to the phenomenon of internal conical refraction [2]. More precisely, a narrow light beam (in vacuum) striking a plate of crystal along the optical axis is refracted, upon incidence, into a cone and, upon exit, into a hollow cylinder. The geometry of wave propagation in biaxial media is a classical topic of research, first investigated by Fresnel [3] in 1821. As an aside, this explains why the wave surface is also termed Fresnel surface. Conical refraction in biaxial media was predicted as early as 1832 by Hamilton [4] on the basis of Fresnel's work. It was experimentally demonstrated just a year later by Lloyd, using a crystal of Aragonite [5].

The rise of metamaterial technology [6] prompts new questions in the traditional analysis of Fresnel surfaces. By designing suitable artificial materials, it is possible to control a wide range of electromagnetic medium parameters. As a matter of fact, one can tune with remarkable accuracy not only the permittivity, but also the perme-

ability and the magnetoelectric response. For comparison, it is worthwhile to recall that the familiar biaxial crystals have anisotropic permittivity, but are not magnetic or magnetoelectric. Typically, more complicated media give rise to more elaborate Fresnel surfaces, whose properties are still largely unexplored. Our work intends to answer general questions with respect to the number and arrangement of the singularities in a Fresnel surface. Most notably, we put forward a highly symmetric medium that exhibits 16 singular points, and discuss a metamaterial realization for it (Sec. II B and Sec. IV).

Two additional remarks are as follows: (i) the singular points examined in this article are more correctly termed 'ordinary double points' [7]; (ii) the papers [8–10] analyze conical refraction in bianisotropic media and are somewhat complementary to our investigation.

A. Biaxial medium

A good point of departure is the simple biaxial medium with electromagnetic response

$$\begin{aligned} D^a &= \varepsilon^{ab} E_b, & \varepsilon^{ab} &= \text{diag}(3, 4, 6), \\ H_a &= \mu_{ab}^{-1} B^b, & \mu_{ab}^{-1} &= \text{diag}(1, 1, 1). \end{aligned} \quad (1)$$

Here, ε^{ab} and μ_{ab}^{-1} are the permittivity and the inverse permeability. Moreover, Latin indices range from 1 to 3, and Einstein's summation convention is employed. The medium (1) gives rise to the dispersion equation, cf. [11,

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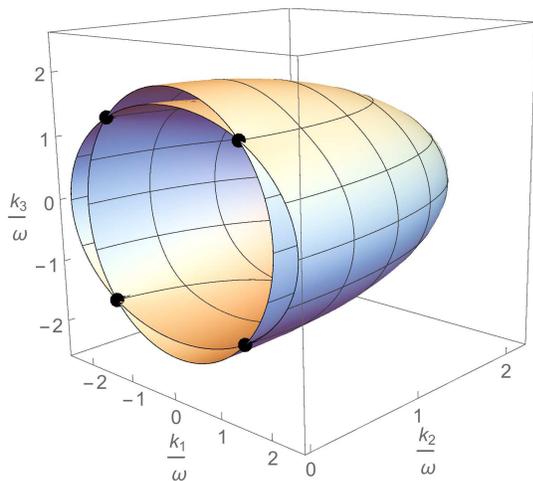


Figure 1. Cross section of the Fresnel surface for the biaxial medium (1). The remaining half of the surface is obtained via symmetry.

Eq. (D.2.44)],

$$f(\omega, k_a) = -72\omega^4 - 3k_1^4 - 4k_2^4 - 6k_3^4 + 30\omega^2k_1^2 + 36\omega^2k_2^2 + 42\omega^2k_3^2 - 10k_2^2k_3^2 - 9k_3^2k_1^2 - 7k_1^2k_2^2 = 0, \quad (2)$$

with ω being the angular frequency and $k_a = (k_1, k_2, k_3)$ the spatial wave covector. Fig. 1 illustrates the Fresnel surface corresponding to (2). This surface determines the inverse phase velocity k_a/ω of an electromagnetic wave traveling in a given direction. Since the medium is birefringent, there are typically two possible inverse phase velocities for each direction.

The solutions of the four simultaneous equations

$$\frac{\partial f(\omega, k_a)}{\partial \omega} = 0, \quad \frac{\partial f(\omega, k_b)}{\partial k_a} = 0, \quad (3)$$

are the singular points of the Fresnel surface [12, §79]. It may occur that a solution of (3) has $\omega = 0$, and therefore exhibits infinite k_a/ω . Hence, representing the singular points through the inverse phase velocity is, at times, impractical. In what follows, the singular points are specified via the 4-dimensional wave covector $q_\alpha = (-\omega, k_a)$, where $\alpha = 0, \dots, 3$. This representation does not suffer from the problem just mentioned.

As expected, the locations on the Fresnel surface in Fig. 1, where the two shells meet, are singular points. Indeed, they correspond to solutions of (3) with $f(\omega, k_a)$ being the function in (2). The locations, as represented through q_α , are

$$\begin{aligned} (1, +\sqrt{2}, 0, +\sqrt{2}), & \quad (1, -\sqrt{2}, 0, -\sqrt{2}), \\ (1, +\sqrt{2}, 0, -\sqrt{2}), & \quad (1, -\sqrt{2}, 0, +\sqrt{2}), \end{aligned}$$

whereby it is easy to check that the singular points form two optical axes. In turn, this verifies that the electromagnetic medium (1) is biaxial.

The Fresnel surface of the medium (1) has, in addition to the 4 real singular points listed above, 12 complex ones, such as:

$$(1, 0, i\sqrt{3}, \sqrt{6}).$$

More generally, the solutions of (3) do not need to be real. The physical interpretation of complex singular points is quite unclear—they seem to describe waves that are partly evanescent and whose exponential decay is independent of polarization.

The discussion so far motivates a question: Can the Fresnel surface of a linear medium exhibit more than 4 real singular points?

II. SIXTEEN REAL SINGULAR POINTS

A. Four singular points at infinity

A medium is magnetoelectric if an applied electric field induces a non-zero magnetization and, similarly, an applied magnetic field induces a non-zero polarization. We consider the medium

$$\begin{aligned} D^a &= \varepsilon^{ab} E_b + \alpha^a{}_b B^b, \\ H_a &= \mu_{ab}^{-1} B^b - \alpha^b{}_a E_b, \end{aligned} \quad (4)$$

whose magnetoelectric tensor, permittivity, and permeability are

$$\begin{aligned} \alpha^a{}_b &= \text{diag}\left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right), \\ \varepsilon^{ab} &= \text{diag}\left(1, 1, -\frac{1}{2}\right), \\ \mu_{ab}^{-1} &= \text{diag}\left(1, 1, -\frac{1}{2}\right). \end{aligned} \quad (5)$$

The dispersion equation specified by (4) with (5), namely

$$f(\omega, k_a) = -\frac{1}{2} \left(\omega^4 + k_1^4 + k_2^4 + k_3^4 - \omega^2k_1^2 - \omega^2k_2^2 - \omega^2k_3^2 - k_2^2k_3^2 - k_3^2k_1^2 - k_1^2k_2^2 \right) = 0, \quad (6)$$

leads to a Fresnel surface with 16 real singular points. Moreover, (6) is the equation, in homogeneous coordinates, for a highly symmetric Kummer surface [13, p. 140]. The general relationship between Fresnel surfaces and Kummer surfaces is examined in Sec. III.

By importing $f(\omega, k_a)$ from (6) into (3), one arrives at

$$\begin{aligned} \frac{\partial f}{\partial \omega} &= \omega (k_1^2 + k_2^2 + k_3^2 - 2\omega^2) = 0, \\ \frac{\partial f}{\partial k_1} &= k_1 (\omega^2 - 2k_1^2 + k_2^2 + k_3^2) = 0, \\ \frac{\partial f}{\partial k_2} &= k_2 (\omega^2 + k_1^2 - 2k_2^2 + k_3^2) = 0, \\ \frac{\partial f}{\partial k_3} &= k_3 (\omega^2 + k_1^2 + k_2^2 - 2k_3^2) = 0. \end{aligned} \quad (7)$$

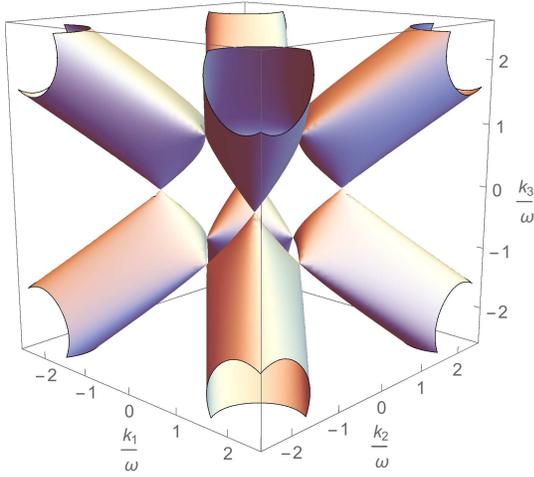


Figure 2. The Fresnel surface of the magnetolectric medium (4) with (5) has 12 singularities in the finite (and 4 at infinity).

The solutions to (7), that is, the 16 real singular points of the Fresnel surface for the medium (4) with (5) are [13, 14]:

1	(0, 1, 1, 1)	9	(0, 1, -1, 1)
2	(1, 0, 1, 1)	10	(1, 0, 1, -1)
3	(1, 1, 0, 1)	11	(-1, 1, 0, 1)
4	(1, 1, 1, 0)	12	(1, -1, 1, 0)
5	(0, -1, 1, 1)	13	(0, 1, 1, -1)
6	(1, 0, -1, 1)	14	(-1, 0, 1, 1)
7	(1, 1, 0, -1)	15	(1, -1, 0, 1)
8	(-1, 1, 1, 0)	16	(1, 1, -1, 0)

Fig. 2 displays the Fresnel surface generated by (6). A careful inspection of the plot establishes that it includes only 12 singular points. This is because 4 singular points are located at infinity. In other words, the electromagnetic waves described by the covectors **1**, **5**, **9** and **13** have $\omega = 0$. As a consequence, the inverse phase velocity k_a/ω for these is unbounded.

It is worthwhile to observe that the 12 singular points included in Fig. 2 lie on the edges of a cube, at the mid-points. The other 4 singularities are retrieved by extending the body diagonals of the cube to the plane at infinity. As explained in Sec. V below, this arrangement of 16 points has various remarkable properties.

The question arises: Can the Fresnel surface of a linear medium have 16 real singular points that are all finite?

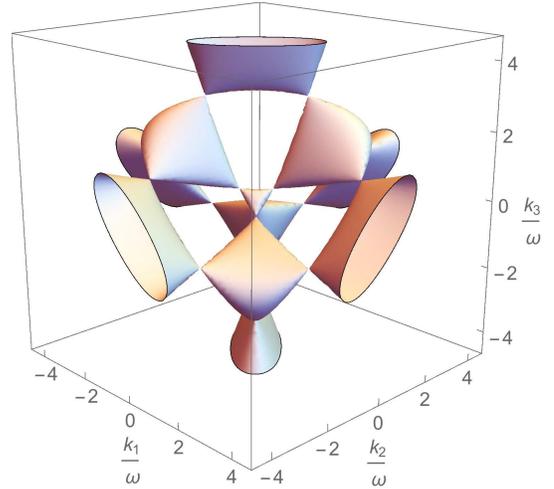


Figure 3. The Fresnel surface generated by (9). Some readers will note that the plot illustrates a Kummer surface.

B. No singular points at infinity

We consider the medium specified by (4) together with

$$\begin{aligned}\alpha^a_b &= \frac{1}{4}\text{diag}(3 + \sqrt{3}, -3 - \sqrt{3}, 0), \\ \varepsilon^{ab} &= \frac{1}{4}\text{diag}(-1 - \sqrt{3}, -1 - \sqrt{3}, -4 + 2\sqrt{3}), \\ \mu_{ab}^{-1} &= \frac{1}{4}\text{diag}(1 + \sqrt{3}, 1 + \sqrt{3}, 4 - 2\sqrt{3}),\end{aligned}\quad (8)$$

and formulate the accompanying dispersion equation as

$$f(\omega, k_a) = -\frac{1}{2}(\psi^2 - 6abcd) = 0, \quad (9)$$

where ψ, a, b, c, d are polynomials in the angular frequency and the spatial wave covector,

$$\begin{aligned}\psi(\omega, k_a) &= \frac{1}{2}(11\omega^2 - k_1^2 - k_2^2 - k_3^2), \\ a(\omega, k_a) &= k_1 + k_2 + k_3, \\ b(\omega, k_a) &= -\omega + k_2 + k_3, \\ c(\omega, k_a) &= -\omega + k_1 + k_3, \\ d(\omega, k_a) &= -\omega + k_1 + k_2.\end{aligned}$$

The Fresnel surface of the medium determined by (4) and (8) has 16 real singular points that are all finite. Accordingly, the plot in Fig. 3 exhibits 16 singularities, which one can view by applying different rotations. The singular points, as represented via $q_\alpha = (-\omega, k_a)$, are

1	$\frac{1}{2}(3, -1, -1, -1)$	9	$\frac{1}{2}(1, 1, -3, 1)$
2	$\frac{1}{2}(3, -1, 1, 1)$	10	$\frac{1}{2}(1, 1, 3, -1)$
3	$\frac{1}{2}(3, 1, -1, 1)$	11	$\frac{1}{2}(1, -1, -3, -1)$
4	$\frac{1}{2}(3, 1, 1, -1)$	12	$\frac{1}{2}(1, -1, 3, 1)$
5	$\frac{1}{2}(1, -3, 1, 1)$	13	$\frac{1}{2}(1, 1, 1, -3)$
6	$\frac{1}{2}(1, 1, -1, 3)$	14	$\frac{1}{2}(1, -3, -1, -1)$
7	$\frac{1}{2}(1, 3, 1, -1)$	15	$\frac{1}{2}(1, -1, 1, 3)$
8	$\frac{1}{2}(1, -1, -1, -3)$	16	$\frac{1}{2}(1, 3, -1, 1)$

At these locations, the partial derivatives of the function $f(\omega, k_a)$ in (9) vanish, as is appropriate. Moreover, because 1–16 have $\omega \neq 0$, the singular points of the Fresnel surface are indeed all bounded; to put it differently, the inverse phase velocities of the electromagnetic waves, which correspond to the 16 singularities, are finite.

Now, one may ask: Can the Fresnel surface of a linear medium exhibit more than 16 real singular points?

III. QUARTIC KUMMER SURFACES

The generic local and linear medium is

$$\begin{aligned} D^a &= \varepsilon^{ab} E_b + \alpha^a{}_b B^b, \\ H_a &= \mu_{ab}^{-1} B^b + \beta_a{}^b E_b, \end{aligned} \quad (10)$$

where $\{\varepsilon^{ab}, \mu_{ab}^{-1}, \alpha^a{}_b, \beta_a{}^b\}$ have real components and are independent of $\{\omega, k_a\}$, since the medium is nondispersive. No further conditions are imposed on these tensors. As a result, $\{\varepsilon^{ab}, \mu_{ab}^{-1}, \alpha^a{}_b, \beta_a{}^b\}$ contain $3 \times 3 = 9$ parameters each, and the medium $4 \times 9 = 36$ in total.

The dispersion equation for the generic local and linear response, as expressed in terms of 3-dimensional quantities, is unwieldy. A more convenient formula is obtained with the help of 4-dimensional electrodynamics. In this representation, the medium (10) becomes [11, 15]:

$$\mathcal{H}^{\alpha\beta} = \frac{1}{2} \chi^{\alpha\beta\mu\nu} F_{\mu\nu}, \quad (11)$$

where the excitation $\mathcal{H}^{\alpha\beta} = -\mathcal{H}^{\beta\alpha}$ and the field strength $F_{\alpha\beta} = -F_{\beta\alpha}$ summarize the fields (D^a, H_a) and (E_a, B^a) , respectively. Greek indices are assumed to take the values $\{0, 1, 2, 3\}$. Owing to the symmetries

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} = -\chi^{\alpha\beta\delta\gamma}, \quad (12)$$

the 4-dimensional medium tensor has 36 independent components, see the remark above.

One can prove [11, 16, 17] that the dispersion equation of the generic local and linear medium is quartic in ω and k_a . As a matter of fact, in 4 dimensions, it takes the form

$$\tilde{f}(q_\alpha) = \mathcal{G}^{\kappa\lambda\mu\nu} q_\kappa q_\lambda q_\mu q_\nu = 0, \quad (13)$$

with $q_\alpha = (-\omega, k_a)$ and $\tilde{f}(q_\alpha) = f(\omega, k_a)$. The Tamm-Rubilar tensor, $\mathcal{G}^{\alpha\beta\gamma\delta}$, may be defined as [18]:

$$\mathcal{G}^{\alpha\beta\gamma\delta} = \frac{1}{3!} \diamond \chi \diamond_{\kappa\lambda\mu\nu} \chi^{\kappa(\alpha\beta|\lambda\chi^{\mu|\gamma\delta)\nu}. \quad (14)$$

In particular, the indices enclosed by round brackets are symmetrized. Moreover, $\diamond \chi \diamond_{\alpha\beta\gamma\delta} = \frac{1}{4} \varepsilon_{\alpha\beta\kappa\lambda} \varepsilon_{\gamma\delta\mu\nu} \chi^{\kappa\lambda\mu\nu}$, with $\varepsilon_{\alpha\beta\gamma\delta}$ being the Levi-Civita symbol.

A surface that originates from a quartic homogeneous polynomial equation in 4 variables cannot have more than 16 isolated singular points [19]. The statement refers to the overall number of singular points—no distinction is

made between those that are real and those that are not. In view of (13), one readily concludes that the Fresnel surface of a local and linear medium never exhibits more than 16 singularities.

By considering index symmetries, one can split the medium tensor into a principal part, a skewon part, and an axion part [11]:

$$\chi^{\alpha\beta\mu\nu} = {}^{(1)}\chi^{\alpha\beta\mu\nu} + {}^{(2)}\chi^{\alpha\beta\mu\nu} + {}^{(3)}\chi^{\alpha\beta\mu\nu}. \quad (15)$$

The decomposition is valid in all reference frames, and is irreducible under the action of $\text{GL}(4, \mathbb{R})$. Moreover, the three elements contain, respectively, $20 + 15 + 1 = 36$ independent components. When

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\gamma\delta\alpha\beta}, \quad (16)$$

so that ${}^{(2)}\chi^{\alpha\beta\mu\nu} = 0$, the medium is called skewon-free. Expressing (16) in 3-dimensional form yields

$$\varepsilon^{ab} = \varepsilon^{ba}, \quad \mu_{ab}^{-1} = \mu_{ba}^{-1}, \quad \beta_a{}^b = -\alpha^b{}_a. \quad (17)$$

It is then simple to verify that the local linear media with the constitutive laws (1), (4) with (5), and (4) with (8) are all skewon-free. Furthermore, as the skewon part of (10) is arbitrary, this law includes two independent magnetoelectric tensors, $\alpha^a{}_b$ and $\beta_a{}^b$.

The Fresnel surfaces of skewon-free local and linear media are Kummer surfaces in the real projective space; conversely, every Kummer surface is the Fresnel surface of a local and linear medium with vanishing skewon part [18, 20]. We deduce that the Fresnel surface of a skewon-free medium has exactly 16 singular points, provided these are isolated. As a matter of fact, this property is known to be valid for the Kummer surfaces [13, 21]. Notably, the surfaces discussed in earlier sections had 16 singularities (over the complex numbers), in agreement with the general theory.

If the electromagnetic response is linear but not local, $\{\varepsilon^{ab}, \mu_{ab}^{-1}, \alpha^a{}_b, \beta_a{}^b\}$ can be complex and depend on $\{\omega, k_a\}$. The impact of generalizing the medium parameters to be complex is limited. As the dispersion equation remains quartic, homogeneous, and polynomial in nature, the Fresnel surface exhibits no more than 16 singular points. By contrast, allowing $\{\varepsilon^{ab}, \mu_{ab}^{-1}, \alpha^a{}_b, \beta_a{}^b\}$ to depend on $\{\omega, k_a\}$, that is, allowing the material to be dispersive, makes general questions very difficult.

As an aside, a medium has zero axion part, see (15), if in 3 dimensions it satisfies $\alpha^a{}_a - \beta_a{}^a = 0$. One can easily check that the media (1), (4) with (5), and (4) with (8) are axion-free. Because the axion part ${}^{(3)}\chi^{\alpha\beta\gamma\delta}$ drops out from the Tamm-Rubilar tensor (14), and thus from the dispersion equation (13), it is irrelevant for the propagation of electromagnetic waves [11].

IV. METAMATERIAL REALIZATIONS

It appears likely that media with 12 or 16 real and finite singular points can be realized as metamaterials.

More specifically, this section examines how (4) with (5), and (4) with (8) may be created as artificial materials.

The permittivity and permeability matrices in (5) and (8) have the structure $\varepsilon^{ab} = \text{diag}(\varepsilon_{\perp}, \varepsilon_{\perp}, \varepsilon_{\parallel})$ and $\mu^{ab} = \text{diag}(\mu_{\perp}, \mu_{\perp}, \mu_{\parallel})$, where the parameters $\{\varepsilon_{\perp}, \varepsilon_{\parallel}, \mu_{\perp}, \mu_{\parallel}\}$ are allowed to be negative. According to [22], permittivity and permeability matrices of this type can be achieved through a suitable arrangement of metal rods and splitting resonators. As an aside, the metamaterial will reflect the property that ε^{ab} and μ^{ab} exhibit a preferred direction (they are uniaxial). Obtaining the numerical values for $\{\varepsilon_{\perp}, \varepsilon_{\parallel}, \mu_{\perp}, \mu_{\parallel}\}$, as given in (5) and (8), requires, almost certainly, a detailed optimization. More explicitly, the individual resonant particles must be tuned and their density adjusted. The optimization is easier when the particles are widely spaced, so that their mutual interactions may be neglected. Because one takes advantage of resonant effects, the desired values of $\{\varepsilon_{\perp}, \varepsilon_{\parallel}, \mu_{\perp}, \mu_{\parallel}\}$ are expected to be achieved only in a narrow band of frequencies ω .

To realize magnetoelectric matrices of the type $\alpha^a_b = \text{diag}(A, -A, 0)$, see (5) and (8), we suggest two alternative metamaterial designs. The first design relies on the properties of Chromium Sesquioxide (Cr_2O_3), a magnetoelectric crystal [23]. More in detail, a mixture of perpendicularly arranged Cr_2O_3 pieces can be used to generate a matrix α^a_b with the correct structure. The second design, which is practical at microwave frequencies, involves a suitable arrangement of magnetostatic wave resonators [24, 25]. These are ferrite elements attached to metal strips and magnetized by a static external magnetic field. As in the case of the permittivity and the permeability, obtaining a specific numerical value for the magnetoelectric parameter A is likely to require accurate optimization. Moreover, one can perform a successful tuning across a narrow frequency band only.

V. KUMMER'S 16_6 CONFIGURATION

As noted in Sec. II A, the singular points of the Fresnel surface generated by the medium (4) with (5) are very symmetric. In fact, it is possible to show [26, 27] that the 4-dimensional covectors $\mathbf{q} = (-\omega, \mathbf{k})$ representing these singular locations are mapped into each other by Dirac γ -matrices:

$$\begin{aligned} \underline{\underline{\gamma}}^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \underline{\underline{\gamma}}^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \underline{\underline{\gamma}}^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \underline{\underline{\gamma}}^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{aligned}$$

and $\underline{\underline{\gamma}}^5 = i\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^1\underline{\underline{\gamma}}^2\underline{\underline{\gamma}}^3$. Equivalently, one can start from the 4-dimensional wave covector that describes a singular

point, and retrieve all other singularities by matrix multiplication with $\underline{\underline{\gamma}}^0, \dots, \underline{\underline{\gamma}}^5$. For instance, if $\mathbf{q}_1, \dots, \mathbf{q}_{16}$ are the 16 singular locations listed in Sec. II A, we have that

$$\begin{aligned} \mathbf{q}_1 &= (0, 1, 1, 1), & i\underline{\underline{\gamma}}^2\underline{\underline{\gamma}}^3\mathbf{q}_1 &= \mathbf{q}_2, \\ -\underline{\underline{\gamma}}^0\mathbf{q}_1 &= \mathbf{q}_5, & -\underline{\underline{\gamma}}^3\underline{\underline{\gamma}}^1\mathbf{q}_1 &= \mathbf{q}_{10}, \\ \underline{\underline{\gamma}}^1\mathbf{q}_1 &= \mathbf{q}_{16}, & -i\underline{\underline{\gamma}}^1\underline{\underline{\gamma}}^2\mathbf{q}_1 &= \mathbf{q}_9, \\ -i\underline{\underline{\gamma}}^2\mathbf{q}_1 &= \mathbf{q}_8, & \underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^5\mathbf{q}_1 &= \mathbf{q}_7, \\ \underline{\underline{\gamma}}^3\mathbf{q}_1 &= \mathbf{q}_{15}, & -\underline{\underline{\gamma}}^1\underline{\underline{\gamma}}^5\mathbf{q}_1 &= \mathbf{q}_{14}, \\ \underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^1\mathbf{q}_1 &= \mathbf{q}_4, & i\underline{\underline{\gamma}}^2\underline{\underline{\gamma}}^5\mathbf{q}_1 &= \mathbf{q}_6, \\ i\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^2\mathbf{q}_1 &= \mathbf{q}_{12}, & -\underline{\underline{\gamma}}^3\underline{\underline{\gamma}}^5\mathbf{q}_1 &= \mathbf{q}_{13}, \\ -\underline{\underline{\gamma}}^0\underline{\underline{\gamma}}^3\mathbf{q}_1 &= \mathbf{q}_{11}, & \underline{\underline{\gamma}}^5\mathbf{q}_1 &= \mathbf{q}_3. \end{aligned} \tag{18}$$

Here, the extra factors of -1 and $\sqrt{-1}$ are immaterial, because the equations (7), which define the singular points, are invariant under scaling of $\mathbf{q} = (-\omega, \mathbf{k})$ by any nonzero constant, and the same holds true for the dispersion equation (6). The striking property (18) is a manifestation of the fact that the singularities give rise to a 16_6 configuration [13]. More explicitly, the singular points determine 16 planes, with each plane containing 6 points; in addition, the planes meet at the 16 singular points, with each point lying on 6 of the planes.

It is well-known [14, 19] that the singularities of a Kummer surface always identify a 16_6 configuration. Moreover, as explained in Sec. III, the Fresnel surfaces of skewon-free local and linear media are Kummer surfaces. We deduce that, for any medium in this wide class, isolated singular points give rise to a 16_6 configuration.

VI. CONCLUSION

We established that rather simple local and linear media can exhibit 16 real singular points. It was found that practical realizations of these media are within the current technical abilities. To achieve the required permittivity, permeability, and magnetoelectric moduli, metamaterial designs were put forward consisting of metal rods, split-ring resonators, and magnetized inclusions.

On the basis of the general dispersion equation, we discovered that the Fresnel surface of a local and linear medium cannot exhibit more than 16 singularities. It was in fact recognized that the Fresnel surfaces of media with no skewon part have exactly 16 singular points (assumed isolated). To reach this conclusion, we made use of an interesting link to the classical projective geometry of Kummer surfaces.

Finally, it was observed that, for the medium (4) with (5), the singular points are mapped into each other by the Dirac γ -matrices. We related this property to the Kummer 16_6 configuration and described the generalization to all local and linear media with vanishing skewon part.

An idea for future work is to conduct an investigation similar to the above in linear elasticity. For instance, it appears desirable to establish how many singularities can the wave surface of a linear anisotropic elastic medium have. Deriving a general answer to this question is likely to be more difficult than in electrodynamics. An upper bound on the number of singularities can however be readily attained. Because the dispersion equation for anisotropic elastic media is sextic [28], the resulting wave surface cannot display more than 65 singular points; this

follows from algebraic geometry, see [29].

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