On the change in shape of Maxwell's equations during the last 150 years
(Über die Gestaltsänderung der Maxwellschen Gleichungen während der letzten 150 Jahre)

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Oberseminar "History of Electrodynamics"

See also Y. Itin et al., arXiv:0911.5175 *file MaxGestalt/MaxwellEqs6.tex*

The Maxwell equations 1862-1868

Five decisive papers of Maxwell (1831-1879) + his Treatise (see C.W.F. Everitt, Maxwell, 1975)

1. "On Faraday's Lines of Force" (1855-1856): Analogies between lines of force and streamlines in an incompressible fluid, electrotonic function $\mathbf{A}$, with $\mathbf{B} =$ curl $\mathbf{A}$ (the latter formula was used earlier by Gauss)
2. "On Physical Lines of Force" (1861-1862): Molecular vortices and electric particles, induced electromotive force $\mathbf{E} = (-)\partial\mathbf{A}/\partial t$
3. "On the Elementary Relations of Electrical Quantities" (1863, missing in his scientific papers): electromagnetic quantities and their physical dimensions, forces and fluxes
4. "A Dynamical Theory of the Electromagnetic Field" (1865): Provides a new theoretical framework for the subject; systematic overview given of all equations, first clear formulation of his system of eqs.
5. "Note on the Electromagnetic Theory of Light" (1868): integral form *without* $\mathbf{A}$, four basic theorems provided: *MaxwellEqsP1.pdf,* later Murnaghan 1921, Kottler 1922, Cartan 1924, de Rham 1931...

We will provide some *spotlights* on the subsequent development of these eqs.

In Maxwell: " A Treatise on Electricity and Magnetism" (2nd edition, 1881) he gave his electromagnetic field equations their most compact form.

Part A: On the history of Maxwell's equations (this seminar)

1. In components: Maxwell 1862-1865
2. In quaternions (Hamilton 1843): Maxwell 1873
3. In symbolic vector calculus: Heaviside 1885-1888, Gibbs 1901, Föppl
4. In components (compact): Hertz 1890, ansatz for moving bodies
5. In components à la Maxwell-Hertz + Lorentz transf.: Einstein 1905
6. In symbolic 4d calculus: Minkowski 1907-1908
7. In 4d generally covariant tensor calculus: Einstein 1916
8. In premetric/integral formulation: (Maxwell), Murnaghan, Kottler, Cartan (formulated in differential forms), van Dantzig, Schrödinger, Schouten, Truesdell-Toupin, Post (2 books), Bopp's axiomatics
9. In spinor calculus: From 1929, Weyl, Fock, van der Waerden,...
10. In algebraic/discrete formulation, Tonti $\sim$1972 as an example

[**Part B: Maxwell's equations today (supplementary material)**

1. 3d vector and 4d tensor calculus $\Rightarrow$ Jackson, Landau-Lifshitz
2. 4d Clifford algebra formalism (vacuum) $\Rightarrow$ Baylis
3. 4d spinor calculus (vacuum) $\Rightarrow$ Penrose & Rindler
4. Discrete formulation in terms of (co)chains $\Rightarrow$ Bossavit, Tonti, Zirnbauer
5. 3d and 4d exterior calculus, premetric topological form of Maxwell's eqs. $\Rightarrow$ Kovetz, Russer, Lindell, H. & Obukhov]

A.1 In components: Maxwell 1862-1865

See the *original* of 1865 where for the first time the “Maxwell equations” appeared systematically ordered: file *Maxwell1865_73.pdf,* see lecture of Christian Schell

A.2 In quaternions (Hamilton 1843): Maxwell 1873

• Quaternion

The quaternions are a set of symbols of the form

$$\underbrace{a}_{\text{scalar p.}} + \underbrace{bi + cj + dk}_{\text{vector part}}, \qquad (1)$$

where a, b, c, d are real numbers. They multiply using the rules

$$i^2 = j^2 = k^2 = -1 \qquad \text{and} \qquad ij = k\,. \qquad (2)$$

They form a *non-commutative* division algebra.

• Hamilton 1843: The quotient of two vectors is generally a quaternion.
The name *vector* originates from Hamilton ($\Rightarrow$ Struik), also *nabla* ∇ (Assyrian harp)

• Quaternions: the most simple associative number system with more than 2 units (complex number has 2 units)

• Supporters of Hamilton against those of Grassmann (theory of extensions, exterior product, Grassmann algebra with anticommuting numbers)

• Clifford: Biquaternions: Quaternions the coefficients of which are a system of complex numbers $a + be$, with $e^2 = \pm 1$ or 0. Clifford algebra.

• Maxwell's equations in quaterionic form (Treatise, 2nd edition, 1881, Vol. II, p. 239–240; S = scalar and V vector part of quaternion) $\mathfrak{G}$ = velocity, ψ, Ω = scalar el./mg. pot., eq. numbers 1st column from 1865, 2nd one from 1881

(B_1)	(A)	$\mathfrak{B} = V\nabla\mathfrak{A} \qquad (S\nabla\mathfrak{A} = 0 \;\Rightarrow\; \mathfrak{B} = \nabla\mathfrak{A})$	eq. of mg. induction
(D)	(B)	$\mathfrak{E} = V\mathfrak{G}\mathfrak{B} - \dot{\mathfrak{A}} - \nabla\psi$	eq. of el.*motive* force
	(C)	$\mathfrak{F} = V\mathfrak{C}\mathfrak{B} - e\nabla\psi - m\nabla\Omega$	eq. of el.magn. force
	(D)	$\mathfrak{B} = \mathfrak{H} + 4\pi\mathfrak{J}$	eq. of magnetization
(C)	(E)	$4\pi\mathfrak{C} = V\nabla\mathfrak{H}$	eq. of el. currents
(F)	[G]	$\mathfrak{K} = C\mathfrak{E}$	eq. of conductiv. (Ohm)
(E)	[F]	$\mathfrak{D} = \frac{1}{4\pi}K\mathfrak{E}$	eq. of el. displacement
(A)	[H]	$\mathfrak{C} = \mathfrak{K} + \dot{\mathfrak{D}}$	eq. of true currents
(B_2)	[L]	$\mathfrak{B} = \mu\mathfrak{H}$	eq. of ind. magnetiz.
(G)	[J]	$e = S\nabla\mathfrak{D}$	[Coulomb-Gauss law]
		$m = S\nabla\mathfrak{J}$	
		$\mathfrak{H} = -\nabla\Omega$	
(H)		Number of eqs.(A) to (H) $= 20$	cont. eq. missing here

First Three Pairs. ⇐ from Treatise, Vol.II, pp. 241/242

Electrostatic Pair.

1. Quantity of electricity e
2. Line-integral of electromotive force, or electric potential E

Magnetic Pair.

3. Quantity of free magnetism, or strength of a pole m
4. Magnetic potential Ω

Electrokinetic Pair.

5. Electrokinetic momentum of a circuit p
6. Electric current C

Second Three Pairs.

Electrostatic Pair.

7. Electric displacement (measured by surface-density) $\mathfrak{D}$
8. Electromotive force at a point $\mathfrak{E}$

Magnetic Pair.

9. Magnetic induction $\mathfrak{B}$
10. Magnetic force $\mathfrak{H}$

Electrokinetic Pair.

11. Intensity of electric current at a point $\mathfrak{C}$
12. Vector potential of electric currents $\mathfrak{A}$

A.3 In symbolic vector calculus: Heaviside 1885/88, Föppl 1894, Gibbs 1901

Heaviside's 'duplex system' of 1888 (see the *original Heaviside1888.pdf* in Phil. Mag. Ser. 5, 25: 153, pp. 130–156 (1888)) $\mathbf{e}, \mathbf{h}$ = impressed fields

$$\mathbf{B} = \mu\mathbf{H}, \quad \mathbf{C} = k\mathbf{E}, \quad \mathbf{D} = (c/4\pi)\mathbf{E}$$

$$\text{curl}\,(\mathbf{H} - \mathbf{h}) = 4\pi\boldsymbol{\Gamma}$$
$$\text{curl}\,(\mathbf{e} - \mathbf{E}) = 4\pi\mathbf{G}$$

$$\boldsymbol{\Gamma} = \mathbf{C} + \dot{\mathbf{D}}, \quad \mathbf{G} = \dot{\mathbf{B}}/4\pi$$
$$\text{div}\,\mathbf{B} = 0$$

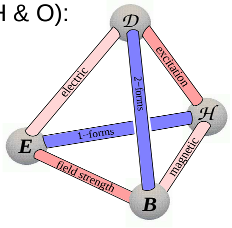
[Energy:

$$U = \frac{1}{2}\mathbf{E}\mathbf{D}\,, \quad T = \frac{1}{2}\mathbf{H}\mathbf{B}/4\pi\,, \quad Q = \mathbf{E}\mathbf{C}\,,$$

$$\mathbf{W} = V(\mathbf{E} - \mathbf{e})(\mathbf{H} - \mathbf{h})/4\pi \quad \Leftarrow \text{Poynting}$$

$$\mathbf{e}\boldsymbol{\Gamma} + \mathbf{h}\mathbf{G} = Q + \dot{U} + \dot{T} + \text{div}\,\mathbf{W}$$]

The electromagnetic field (H & O):



• Heaviside + Grassmann + Gibbs $\Rightarrow$ vector analysis: Hamilton's vectors, Grassmann's exterior product, Gibbs' dyadics; see also J. Crowe, A History of Vector Analysis, Dover

• 1872 Erlangen Program of Klein $\Rightarrow$ group theory + geometry: "Let be given a manifold and a transformation group in it. Develop the theory of invariants with respect to this group." [Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man entwickle die auf die Gruppe bezügliche Invariantentheorie."] 3d Euclidean group $T^3 \otimes SO(3) \Rightarrow$ Poincaré group (4d translations $\otimes$ Lorentz) $T^4 \otimes SO(1,3) \Rightarrow$ diffeomorphism group

• Around 1900: Ricci + Levi-Civita $\Rightarrow$ absolute differential calculus, tensor analysis (tensor Voigt 1900) $\Rightarrow$ Einstein 1916, see history of Karin Reich

• In textbooks, Abraham-Föppl is a leading example (Einstein learned from Föppl), see *original AbrahamFoeppl1904.pdf,* 2nd edition

• Recommended textbooks: Sommerfeld Vol.III (theoretical), Bergmann-Schaefer, Vol.2 (experimental)

• Bamberg + Sternberg: "...the most suitable framework for geometrical analysis is the exterior differential calculus of Grassmann and Cartan." (topological in constrast to metrical concepts are stressed)

A.4 In components (compact): Hertz 1890, ansatz for moving bodies
Hertz's system in vacuum, see the *original* Ann. Phys. 1890, $[A^{-1}]$ = velocity; file *Hertz1890a.pdf*

$$A\frac{dL}{dt} = \frac{dZ}{dy} - \frac{dY}{dz} \qquad A\frac{dX}{dt} = \frac{dM}{dz} - \frac{dN}{dy}$$
$$A\frac{dM}{dt} = \frac{dX}{dz} - \frac{dZ}{dx} \qquad A\frac{dY}{dt} = \frac{dN}{dx} - \frac{dL}{dz}$$
$$A\frac{dN}{dt} = \frac{dY}{dx} - \frac{dX}{dy} \qquad A\frac{dZ}{dt} = \frac{dL}{dy} - \frac{dM}{dx}$$
$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dx} = 0\,, \qquad \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0$$

[$\mathbf{H} = (L, M, N)$, $\mathbf{E} = (X, Y, Z)$. For the first time we see all 4 Maxwell vacuum equations together, cf. Darrigol, p.254 et seq.:

$$A\frac{d\mathbf{H}}{dt} = -\text{curl}\,\mathbf{E}\,, \qquad A\frac{d\mathbf{E}}{dt} = \text{curl}\,\mathbf{H}$$
$$\text{div}\,\mathbf{H} = 0\,, \qquad \text{div}\,\mathbf{E} = 0$$

Note: By differentiating with respect to the time t, we find the wave equation

$$A\frac{d^2\mathbf{H}}{dt^2} = -\text{curl}\,\frac{d\mathbf{E}}{dt} = -\frac{1}{A}\text{curl}\,\text{curl}\,\mathbf{H} = \frac{1}{A}(\Delta\,\mathbf{H} - \text{grad}\,\underbrace{\text{div}\,H}_{=0})\,.]$$

Hertz (continued). Anisotropic conductor:

$$A\left(\mu_{11}\frac{dL}{dt}+\mu_{12}\frac{dM}{dt}+\mu_{13}\frac{dN}{dt}\right)=\frac{dZ}{dy}-\frac{dY}{dz}\,,$$

$$A\left(\mu_{12}\frac{dL}{dt}+\mu_{22}\frac{dM}{dt}+\mu_{32}\frac{dN}{dt}\right)=\frac{dX}{dz}-\frac{dZ}{dx}\,,$$

$$A\left(\mu_{13}\frac{dL}{dt}+\mu_{23}\frac{dM}{dt}+\mu_{33}\frac{dN}{dt}\right)=\frac{dY}{dx}-\frac{dX}{dy}\,,$$

$$A\left(\varepsilon_{11}\frac{dX}{dt}+\varepsilon_{12}\frac{dY}{dt}+\varepsilon_{13}\frac{dZ}{dt}\right)=\frac{dM}{dz}-\frac{dN}{dy}$$
$$-4\pi A\left\{\lambda_{11}(X-X')+\lambda_{12}(Y-Y')+\lambda_{13}(Z-Z')\right\}\,,$$

$$A\left(\varepsilon_{12}\frac{dX}{dt}+\varepsilon_{22}\frac{dY}{dt}+\varepsilon_{23}\frac{dZ}{dt}\right)=\frac{dN}{dx}-\frac{dL}{dz}$$
$$-4\pi A\left\{\lambda_{21}(X-X')+\lambda_{22}(Y-Y')+\lambda_{23}(Z-Z')\right\}\,,$$

$$A\left(\varepsilon_{13}\frac{dX}{dt}+\varepsilon_{23}\frac{dY}{dt}+\varepsilon_{33}\frac{dZ}{dt}\right)=\frac{dL}{dy}-\frac{dM}{dx}$$
$$-4\pi A\left\{\lambda_{31}(X-X')+\lambda_{32}(Y-Y')+\lambda_{33}(Z-Z')\right\}\,.$$

tensorial permittivity, permeability, conduct. (λ), several typos of Hertz corr.

Hertz (continued): Ansatz for moving bodies by substituting the convective derivative (of Helmholtz). Let a (electric of magnetic) flux $\mathbf{F}$ be given; then, with the velocity $\mathbf{v}$ of the medium (at the time of Hertz d meant ∂),

$$\frac{d\mathbf{F}}{dt} \qquad \Longrightarrow \qquad \frac{D\mathbf{F}}{Dt} = \frac{d\mathbf{F}}{dt} - [\nabla \times (v \times \mathbf{F}) - v(\nabla \cdot \mathbf{F})] \ .$$

Substitute this in the l.h.s. of the 2 Maxwell equations containing a time derivative. Turned out to be unsuccessful, but it brought the electrodynamics of moving bodies under way $\Rightarrow$ Einstein 1905.

A.5 In components à la Maxwell-Hertz + Lorentz transf.: original *Einstein1905.pdf*

In the kinematical part of his "On the Electrodynamics of Moving Bodies" he proves for a standard Lorentz transformation (boost in x-direction)

$$\begin{aligned} \tau &= \beta(t - vx/c^2)\,, \\ \xi &= \beta(x - vt)\,, \\ \eta &= y\,, \\ \zeta &= z\,, \end{aligned}$$

where $\beta = 1/\sqrt{1 - v^2/c^2}$.

Einstein 1905 (continued): He just took the Maxwell-Hertz equations for vacuum (with switched sign), electric field (X, Y, Z), magnetic field (L, M, N),

$$\frac{1}{V}\frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} \qquad \frac{1}{V}\frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}$$

$$\frac{1}{V}\frac{\partial Y}{\partial t} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} \qquad \frac{1}{V}\frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}$$

$$\frac{1}{V}\frac{\partial Z}{\partial t} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \qquad \frac{1}{V}\frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}$$

Then he referred the electromagnetic process to the coordinate system above and uses the corresponding transformation formulas:

$$\frac{1}{V}\frac{\partial X}{\partial \tau} = \frac{\partial \beta\left(N - \frac{v}{V}Y\right)}{\partial \eta} - \frac{\partial \beta\left(M + \frac{v}{V}Z\right)}{\partial \zeta} \qquad \text{etc.}$$

Because of the relativity principle, we have

$$\frac{1}{V}\frac{\partial X'}{\partial \tau} = \frac{\partial N'}{\partial \eta} - \frac{\partial M'}{\partial \zeta} \qquad \frac{1}{V}\frac{\partial L'}{\partial \tau} = \frac{\partial Y'}{\partial \zeta} - \frac{\partial Z'}{\partial \eta}$$

$$\frac{1}{V}\frac{\partial Y'}{\partial \tau} = \frac{\partial L'}{\partial \zeta} - \frac{\partial N'}{\partial \xi} \qquad \frac{1}{V}\frac{\partial M'}{\partial \tau} = \frac{\partial Z'}{\partial \xi} - \frac{\partial X'}{\partial \zeta}$$

$$\frac{1}{V}\frac{\partial Z'}{\partial \tau} = \frac{\partial M'}{\partial \xi} - \frac{\partial L'}{\partial \eta} \qquad \frac{1}{V}\frac{\partial N'}{\partial \tau} = \frac{\partial X'}{\partial \eta} - \frac{\partial Y'}{\partial \xi}$$

Consequently,

$$X' = X\,, \qquad L' = L\,,$$

$$Y' = \beta\left(Y - \frac{v}{V}N\right)\,, \qquad M' = \beta\left(M + \frac{v}{V}Z\right)\,,$$

$$Z' = \beta\left(Z + \frac{v}{V}M\right)\,, \qquad N' = \beta\left(N - \frac{v}{V}Y\right)\,,$$

are the transformation formulas for the components of the electromagnetic field.

Similar derivations were given (partly earlier) by Poincaré and by Lorentz.

See also the books of von Laue (1911), Silberstein (quaternions! 1914), Pauli (1921), Einstein (1922),..., Møller (1952),...

A.6 In symbolic 4d calculus: Minkowski's way to: lor $\mathbf{f} = -\mathbf{s}$, lor $\mathbf{F}^* = \mathbf{0}$

Minkowski introduced fields f and F in Cartesian coordinates x, y, z and with imaginary time coo. it ($c = 1$); moreover, $x_1 := x$, $x_2 := y$, $x_3 := z$, $x_4 := it$. Euclidean metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = g_{hk} dx_h dx_k$, with $g_{hk} =$ diag$(1, 1, 1, 1)$; there is no need to distinguish contravariant (upper) from covariant (lower) indices. The Maxwell equations in component form:

$$\begin{aligned}
\frac{\partial f_{12}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4} &= s_1 \,, \\
\frac{\partial f_{21}}{\partial x_1} + \frac{\partial f_{23}}{\partial x_3} + \frac{\partial f_{24}}{\partial x_4} &= s_2 \,, \\
\frac{\partial f_{31}}{\partial x_1} + \frac{\partial f_{32}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_4} &= s_3 \,, \\
\frac{\partial f_{41}}{\partial x_1} + \frac{\partial f_{42}}{\partial x_2} + \frac{\partial f_{43}}{\partial x_3} &= s_4 \,,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_{34}}{\partial x_2} + \frac{\partial F_{42}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_4} &= 0 \,, \\
\frac{\partial F_{43}}{\partial x_1} + \frac{\partial F_{14}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_4} &= 0 \,, \\
\frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_4} &= 0 \,, \\
\frac{\partial F_{32}}{\partial x_1} + \frac{\partial F_{13}}{\partial x_2} + \frac{\partial F_{21}}{\partial x_3} &= 0 \,.
\end{aligned}$$

Minkowski (cont.): Modern notation and summ. conv. $h, k, ... = 1, 2, 3, 4$,

$$\frac{\partial f_{hk}}{\partial x_k} = s_h \qquad \text{and} \qquad \frac{\partial F_{hk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_h} + \frac{\partial F_{lh}}{\partial x_k} = 0 \qquad (\text{or } \partial_{[l} F_{hk]} = 0)\,.$$

Excitation f and the *field strength* F (in Maxwell's nomenclature[1])

$$(f_{hk}) = -(f_{kh}) = \begin{pmatrix} 0 & H_z & -H_y & -iD_x \\ -H_z & 0 & H_x & -iD_y \\ H_y & -H_x & 0 & -iD_z \\ iD_x & iD_y & iD_z & 0 \end{pmatrix}$$

$$(F_{hk}) = -(F_{kh}) = \begin{pmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix},$$

The 4d electric current denoted by s_h.

[1] Minkowski took $\mathbf{D}=\mathbf{e}$, $\mathbf{H}=\mathbf{m}$; $\mathbf{E}=\mathbf{E}$, $\mathbf{B}=\mathbf{M}$, that is, for the excitation $f \sim (\mathbf{e}, \mathbf{m})$ and for the field strength $F \sim (\mathbf{E}, \mathbf{M})$.

Minkowski (cont.): He introduced the *dual* of F_{hk}, namely $F^*_{hk} := \frac{1}{2}\hat{\epsilon}_{hklm} F_{lm}$, with the Levi-Civita symbol $\hat{\epsilon}_{hklm} = \pm 1, 0$ and $\hat{\epsilon}_{1234} = +1$. Thus,

$$F^* = (F^*_{hk}) = \begin{pmatrix} 0 & -iE_z & iE_y & B_x \\ iE_z & 0 & -iE_x & B_y \\ -iE_y & iE_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{pmatrix}.$$

Then both Maxwell equations read

$$\frac{\partial f_{hk}}{\partial x_k} = s_h\,, \qquad\qquad \frac{\partial F^*_{hk}}{\partial x_k} = 0\,.$$

Subsequently Minkowski developed a 4-dimensional type of Cartesian tensor calculus with a 4d differential operator called 'lor' (abbreviation of Lorentz). He introduces ordinary (co)vectors *(space-time vectors of the 1st kind)*, like x_h and lor$_h := \frac{\partial}{\partial x_h}$, and antisymmetric 2nd rank tensors *(space-time vectors of the 2nd kind)*, like f_{hk} and F_{hk}. Then, symbolically he wrote

$$\boxed{\text{lor}\, f = -s\,, \qquad\qquad \text{lor}\, F^* = 0\,.}$$

Using his Cartesian tensor calculus, Minkowski has shown that these eqs. are *covariant under Poincaré transformations.* In vacuum, $f \sim F$. Compare with exterior calculus version with $dH = J,\ dF = 0$ and, in vacuum, $H \sim {}^\star F$.

• Minkowski also discovered 1907 the energy-momentum tensor for the electromagnetic field: densities of energy/momentum and their fluxes.

A.7 In 4d generally covariant tensor calculus: Einstein 1916/1922

The next step occurred immediately after Einstein's fundamental 1915 paper on general relativity. Now Einstein was in command of tensor calculus in arbitrary coordinate systems. By picking suitable variables, he found ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ with signature $(1,-1,-1,-1)$, here $\mu,\nu,... = 0,1,2,3$)

$$\boxed{\frac{\partial F_{\rho\sigma}}{\partial x^\tau} + \frac{\partial F_{\sigma\tau}}{\partial x^\rho} + \frac{\partial F_{\tau\rho}}{\partial x^\sigma} = 0\,, \qquad \mathfrak{F}^{\mu\nu} = \sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}\,, \qquad \frac{\partial \mathfrak{F}^{\mu\nu}}{\partial x^\nu} = \mathcal{J}^\mu\,.}$$

The field strength $F_{\rho\sigma}$ is a tensor, the excitation $\mathfrak{F}^{\mu\nu}$ a tensor density. Einstein's identifications, which were only worked out by him for vacuum, read

$$\mathfrak{F} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & H_z & -H_y \\ -E_y & -H_z & 0 & H_x \\ -E_z & H_y & -H_x & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_y & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix}.$$

• These Maxwellian eqs. are generally covariant *and* metric independent. The gravitational potential only enters the "spacetime relation"—it is the 'constitutive law' of the vacuum.
Here *no* comma goes to semicolon rule " , $\to$; " (MTW) is necessary for eldyn.

For a mathematically precise presentation see Schouten "Tensor Analysis for Physicists" (Oxford 1951, Dover 1989).

A.8 In premetric/integral formulation, in tensor and exterior diff. calculus

Already initiated by Maxwell in his paper 5 (similar in Sommerfeld). Élie Cartan (1924) as an example. In special relativity:

$$\begin{aligned}\Omega &= B_x[dy\,dz] + B_y[dz\,dx] + B_z[dx\,dy] \\ &\quad + E_x[dx\,dt] + E_y[dy\,dt] + E_z[dz\,dt] \\ \overline{\Omega} &= D_x[dy\,dz] + D_y[dz\,dx] + D_z[dx\,dy] \\ &\quad + H_x[dx\,dt] + H_y[dy\,dt] + H_z[dz\,dt] \\ S &= \rho[dx\,dy\,dz] - I_x[dy\,dz\,dt] - I_y[dz\,dt\,dx] - I_z[dx\,dy\,dt]\end{aligned}$$

$$\Omega' = 0\,, \qquad \overline{\Omega}' = -4\pi S \qquad \Rightarrow\; S' = 0$$

Generalization:

$$\iint \Omega = 0\,, \qquad \iint \overline{\Omega} = 4\pi \iiint S\,,$$

where the integral on the right-hand-side extends over any 3-dimensional volume of spacetime and those on the left-hand-sides over the 2-dimensional boundary of this volume.

Isn't that a beautiful representation?

In premetric/integral formulation (continued)

Post (Quantum Reprogramming... [1995], p. 105), in our version (Hehl & Obukhov, Foundations of Classical Eldyn., Boston 2003): Notions of de Rham 1931; for any cycle C_3 with $\partial C_3 = 0$ and any cycle C_2 with $\partial C_2 = 0$, we have

$$\oint_{C_3} J = 0\,, \qquad f_\alpha = (e_\alpha \rfloor F) \wedge J\,, \qquad \oint_{C_2} F = 0\,.$$

This contains Maxwell's equations *in nuce*! The first axiom governs matter and its conserved electric *charge,* the second axiom links the notion of that charge and the concept of a *mechanical force* to an operational definition of the electromagnetic *field strength.* The third axiom determines the flux of the field strength as sourcefree.

Differential version of electrodynamics:

$$\begin{aligned} dJ &= 0\,, \qquad f_\alpha = (e_\alpha \rfloor F) \wedge J\,, \qquad dF = 0\,, \\ J &= dH\,, \qquad\qquad\qquad\qquad\qquad\quad F = dA\,. \end{aligned}$$

Because of the existence of conductors and superconductors, *we can measure the excitation* H. Thus, even if H emerges as a kind of potential for the electric current, it is more than that: It is measurable. This is in clear contrast to the potential A that is *not* measurable.

In premetric/integral formulation (continued)

The physical interpretation of the Maxwell equations can be found via the $(1 + 3)$-decomposition (signs embody the Lenz rule)

$$\begin{aligned} J &= -j \wedge dt + \rho\,, \\ H &= -\mathcal{H} \wedge dt + \mathcal{D}\,, \\ F &= E \wedge dt + B\,, \\ A &= -\varphi\, dt + \mathcal{A}\,, \end{aligned}$$

Then, by substitutions, the (1 + 3)-decomposition of the Maxwell eqs. read

$$dH = J \quad \begin{cases} \underline{d}\,\mathcal{D} = \rho & \text{(1 constraint eq.)}\,, \\ \dot{\mathcal{D}} = \underline{d}\,\mathcal{H} - j & \text{(3 time evol. eqs.)}\,, \end{cases}$$

$$dF = 0 \quad \begin{cases} \underline{d}\,B = 0 & \text{(1 constraint eq.)}\,, \\ \dot{B} = -\underline{d}\,E & \text{(3 time evol. eqs.)}\,. \end{cases}$$

Accordingly, we have $2 \times 3 = 6$ time evolution equations for the $2 \times 6 = 12$ variables $(\mathcal{D}, B, \mathcal{H}, E)$ of the electromagnetic field. Thus the Maxwellian structure is underdetermined. We need, in addition, an electromagnetic spacetime relation that expresses the excitation $H = (\mathcal{H}, \mathcal{D})$ in terms of the field strength $F = (E, B)$, i.e., $H = H[F]$.

A.9 In spinor calculus: Spinors as *semi*vectors, tensor with rank $\frac{1}{2}$.Weyl 1928–29, Fock 1929, van der Waerden 1929, Schrödinger 1930, systematically: Infeld & van der Waerden, Sitzungsber. Preuss. Akad. Wiss. Physik.-Math. Klasse, p. 380 (1933). *E.M. Corson,* Tensors, Spinors and Rel. Wave Eqs., 1953. We take as example, Laporte & Uhlenbeck Phys. Rev. **37** (1931) 1380–1397: Group $SL(2, C)$ with transformations

$$\xi_1' = \alpha_{11}\xi_1 + \alpha_{12}\xi_2\,, \qquad \overline{\xi_1'} = \overline{\alpha}_{11}\overline{\xi_1} + \overline{\alpha}_{12}\overline{\xi_2}\,,$$
$$\xi_2' = \alpha_{21}\xi_1 + \alpha_{22}\xi_2\,, \qquad \overline{\xi_2'} = \overline{\alpha}_{21}\overline{\xi_1} + \overline{\alpha}_{22}\overline{\xi_2}\,,$$

and $\det\alpha = 1$, simple covering group of the proper orthochronous Lorentz group $SO_0(1,3)$. Fundamental objects are the spinors a_k and $b_{\dot r}$; higher order objects $a_{kl}, b_{\dot r\dot s}, c_{\dot r k}, ...$ Spinor 'metric' $\epsilon^{kl} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\epsilon_{kl} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Relation between spinors and world vectors:

$$\frac{1}{2}(a_{\dot 21} + a_{\dot 12}) = A^1 = A_1\,, \qquad a_{\dot 21} = A_1 + iA_2\,,$$
$$\frac{1}{2i}(a_{\dot 21} - a_{\dot 12}) = A^2 = A_2\,, \qquad a_{\dot 12} = A_1 - iA_2\,,$$
$$\frac{1}{2}(a_{\dot 11} - a_{\dot 22}) = A^3 = A_3\,, \qquad a_{\dot 11} = A_3 - A_4\,,$$
$$\frac{1}{2}(a_{\dot 11} + a_{\dot 22}) = A^4 = -A_4\,, \qquad -a_{\dot 22} = A_3 + A_4\,.$$

In spinors (cont.) Definition of self-dual tensor $F^*_{kl} := \frac{i}{2}\epsilon_{kl\alpha\beta}F^{\alpha\beta}$. Introduce complex electromagnetic field strength (here for vacuum) and find the Maxwell equation

$$G^{kl} := F^{kl} + F^{*kl}\,, \qquad \frac{\partial G^{k\lambda}}{\partial x^\lambda} = S^k\,.$$

G^{kl} is an antisymmetric self-dual 2nd rank tensor with 6 independent components. We can assign to G^{kl} a *symmetric 2nd rank* spinor $g_{\dot{l}\dot{m}}$ with 3 complex components, that is, with 6 independent components. Then we find Maxwell's eqs. in spinor form (field strength g, current s, potential ϕ):

$$\boxed{\partial^{\dot\rho}{}_l\, g_{\dot\rho\dot m} = 2s_{\dot m l}}\,, \qquad g_{\dot r\dot s} = \partial_{\dot r\lambda}\phi_{\dot s}{}^{\lambda} + \partial_{\dot s\lambda}\phi_{\dot r}{}^{\lambda}\,,$$

$$\partial_{\dot\rho\alpha}\partial^{\dot\rho\alpha}\phi_{\dot m l} = 2s_{\dot m l} \qquad \text{with Lorenz condition} \qquad \partial^{\dot\mu\lambda}\,\phi_{\dot\mu\lambda} = 0\,.$$

The covariant version of this 'Maxwell equation' is being used for the analysis of propagating electromagnetic waves in GR. Einstein's equation can also be put in spinor form: The curvature tensor becomes a totally symmetric 4th rank curvature spinor $\psi_{mnrs} = \psi_{(mnrs)}$; this uniform formalism facilitates sometimes the investigations on electromagnetic and gravitational waves.

A.10 In algebraic/discrete formulation, Tonti's vers. of 2009, ACE'09 in Rome

$$\Psi := \int_S \mathbf{D}\cdot\mathbf{n}\,dS \text{ (electric flux)} \qquad \Phi := \int_S \mathbf{B}\cdot\mathbf{n}\,dS \text{ (magnetic flux)}$$

$$E := \int_L \mathbf{E}\cdot\mathbf{t}\,dL \qquad F := \int_L \mathbf{H}\cdot\mathbf{t}\,dL$$

$$I := \int_S \mathbf{J}\cdot\mathbf{n}\,dS \text{ (current)} \qquad Q^c := \int_V \rho\,dV \text{ (charge)}$$

Forget these defs., take the *global* quantities Ψ, Φ, E, F, I, Q^c as fundamental. Consider instant (of time) I, volume V and its boundary ∂V; furthermore, time interval T, surface S and its boundary ∂S. Inner $\_$ and outer orientation $\widetilde{\ }$:

$$\Psi\left[\overline{I}, \partial\widetilde{V}\right] = Q^c\left[\overline{I}, \widetilde{V}\right], \quad \mathcal{F}\left[\overline{T}, \partial\widetilde{S}\right] = +\left\{\Psi\left[\overline{I}^+, \widetilde{S}\right] - \Psi\left[\overline{I}^-, \widetilde{S}\right]\right\} + Q^f\left[\overline{T}, \widetilde{S}\right],$$

$$\Phi\left[\widetilde{I}, \partial\overline{V}\right] = 0, \qquad \mathcal{E}\left[\widetilde{T}, \partial\overline{S}\right] = -\left\{\Phi\left[\widetilde{I}^+, \overline{S}\right] - \Phi\left[\widetilde{I}^-, \overline{S}\right]\right\}.$$

$\mathcal{F}$ = impulse of magnetomotive force, $\mathcal{E}$ = impulse of electromotive force. All laws refer to the *boundaries* of the space elements V and S. Compare with

$$\underline{d}\,\mathcal{D} = \rho, \qquad \underline{d}\,\mathcal{H} = +\dot{\mathcal{D}} + j,$$

$$\underline{d}\,B = 0, \qquad \underline{d}\,E = -\dot{B}.$$

Can be used for computer calculations. Start with global/discrete structures; don't discretize the differential Maxwell equations!

Tonti (2009), see `http://www.dica.units.it/perspage/tonti/`
The *differential* formulation requires the *field* vectors $\mathbf{B}$, $\mathbf{E}$, ρ, $\mathbf{J}$, $\mathbf{D}$, $\mathbf{H}$ as *point* functions, the algebraic formulation requires the global scalar variables Φ, $\mathcal{E}$, Q^c, Q^f, Ψ, $\mathcal{F}$ as domain variables. Avoiding the field vectors, we don't need to perform integrations, better for computational electromagnetism,

• Which system of Maxwell's equations should be taught to physics and engineering students? I opt for A.8 and A.10, in contrast to Jackson, Landau-Lifshitz,...

I used the *original* articles mentioned above, for secondary literature, see

- E. Whittaker, *A History of the Theories of Aether and Electricity,* 2 volumes, Humanities Press, NY, 1973 [1951].
- C.W.F. Everitt, *James Clerk Maxwell,* Physicis and Natural Philosopher, Charles Scibner's Sons, NY, 1975.
- J.L. Heilbron, *Electricity in the 17th and 18th Centuries,* A Study of Early Modern Physics, UC Press, Berkeley, 1979.
- O. Darrigol, *Electrodynamics from Ampère to Einstein,* Oxford, 2000.
- F. Steinle, *Exporative Experiments,* Ampère, Faraday und die Ursprünge der Elektrodynamik, Steiner, Stuttgart, 2005.

Soli Deo Gloria.