Gowdy Cosmologies and Their Generalizations as a Theoretical Laboratory for Classical and Quantum Gravity

Beverly K. Berger

- Kasner example
- Polarized Gowdy
- 2 ways to quantize — graviton creation
- Exploring the BKL conjecture
- In the expanding direction

Bad Honnef
29 July 2014

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Note: This talk covers > 40 years. Signs and variable names may be inconsistent in places.

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The Kasner (Bianchi I) Cosmology

The Kasner Spacetime (vacuum, Bianchi Type I):

$$ds^2 = -\,dt^2 \;+\; t^{2p_1}dx^2 \;+\; t^{2p_2}dy^2 \;+\; t^{2p_3}dz^2$$

where $$\sum_{i=1}^{3} p_i = 1 = \sum_{i=1}^{3} p_i^2$$

The three Kasner indices may be parametrized by a single variable u (introduced by BKL) where $1 \leq u < \infty$:

$$p_1 = \frac{-u}{u^2+u+1} \quad ; \quad p_2 = \frac{u+1}{u^2+u+1} \quad ; \quad p_3 = \frac{u(u+1)}{u^2+u+1}$$

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Note that one Kasner index is always negative.

The Kasner Singularity:

In the collapse (expansion) direction, one Kasner axis is expanding (collapsing). However,

$$\sqrt{{}^3 g} = t$$

and the first non-zero curvature invariant blows up as $t \to 0$ unless $u = \infty$:

$$\kappa = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{16}{t^4} \frac{u^2 (u+1)^2}{(u^2+u+1)^3}$$

This is a spacelike, curvature blowup singularity just as for FRW.

The Kasner Spacetime is a “free particle” in **minisuperspace**:

In terms of $d\tau = e^{-3\Omega}\, dt$ and the momenta conjugate to the MSS variables, Einstein’s equations may be obtained by variation of the Hamiltonian constraint

$$H = -\,p_\Omega^2 + p_+^2 + p_-^2 = 0$$

to yield

$$\left(\frac{p_+}{p_\Omega}\right)^2 + \left(\frac{p_-}{p_\Omega}\right)^2 = v_+^2 + v_-^2 = 1$$

$$\beta_\pm = v_\pm \, |\Omega|$$

Note that the straight line trajectory in MSS may be described by a single angle θ which may be shown to be equivalent to *u*.

The Kasner singularity is “**velocity term dominated**” (VTD).

Minisuperspace quantization of Kasner (Wheeler-DeWitt equation):

(One of many methods) Promote the Hamiltonian constraint to be an operator in minisuperspace acting on the wavefunction

$$\psi(\Omega, \beta_+, \beta_-)$$

via the Klein-Gordon equation

$$-\frac{\partial^2\psi}{\partial\Omega^2} + \frac{\partial^2\psi}{\partial\beta_+^2} + \frac{\partial^2\psi}{\partial\beta_-^2} = 0$$

to yield a (massless) free K-G field in 2 + 1 dimensions.

Generic MSS issues: (1) approximation to what? (2) time? (3) interpretation? Wave function of the universe?
Generic K-G issues: (1) negative energy? (2) negative probability?

See M.P. Ryan, Jr., L.C. Shepley, *Homogeneous Relativistic Cosmologies*, (Princeton U Press, 1975)

Use a different set of variables to describe the Kasner cosmology:

$$ds^2 = e^{(\lambda+\tau)/2}\left(-e^{-2\tau}d\tau^2 + d\theta^2\right) + e^{-\tau}\left(e^{P}dx^2 + e^{-P}dy^2\right)$$

Einstein's equations yield solutions for P and λ in terms of a constant υ and time τ.

$$\frac{d^2P}{d\tau^2} = 0 \qquad\qquad P = -\upsilon\tau$$

$$\frac{d\lambda}{d\tau} + \left(\frac{dP}{d\tau}\right)^2 = 0 \quad\longrightarrow\quad \lambda = -\upsilon^2\tau$$

Hamiltonian constraint:

$$H = \pi_\lambda\pi_\tau + \tfrac{1}{2}\pi_P^2 = 0$$

ADM quantization in MSS in these variables can yield a Schrödinger equation after separating out the λ degree of freedom:

$$i\,\pi_\lambda \frac{\partial \psi}{\partial \tau} = -\tfrac{1}{2} \frac{\partial^2 \psi}{\partial P^2}$$

This is the equation for a free particle so that wavepackets which follow the classical MSS trajectory may be constructed.

Of course, the same MSS issues are found in this quantization.

More details and the generalization to polarized Gowdy models may be found in:

See B.K. Berger, Ann. Phys. **83**, 458 (1974) and Ph.D. Thesis, U. of Maryland, 1972;
C.W. Misner, Phys. Rev. D **8**, 3271 (1973); B.K. Berger, Phys. Rev. D 11, 2770 (1975)

Polarized Gowdy

R. H. Gowdy, Phys. Rev. Lett. 27, 826 (1971); Ann. Phys. (N. Y.) 83, 203 (1974).

Polarized Gowdy with 3-torus spatial topology:

$$ds^2 = e^{(\lambda+\tau)/2}\left(-e^{-2\tau} d\tau^2 + d\theta^2\right) + e^{-\tau}\left(e^P dx^2 + e^{-P} dy^2\right)$$

Now allow P and λ to depend on θ as well as τ. The equations become for $\dot{} \equiv \partial/\partial\tau$ and $' \equiv \partial/\partial\theta$:

$$\ddot{P} - e^{-2\tau} P'' = 0$$

$$\dot{\lambda} = -\left[(\dot{P})^2 + e^{-2\tau}(P')^2\right], \quad \lambda' = \dot{P}P'$$

Constraints:

$$\mathcal{H}^0 = \pi_\lambda \pi_\tau + \tfrac{1}{2}\pi_P^2 + \tfrac{1}{2} e^{-2\tau}(P')^2 = 0$$

$$\mathcal{H}^1 = \pi_\tau \tau' + \pi_\lambda \lambda' + \pi_P P' = 0$$

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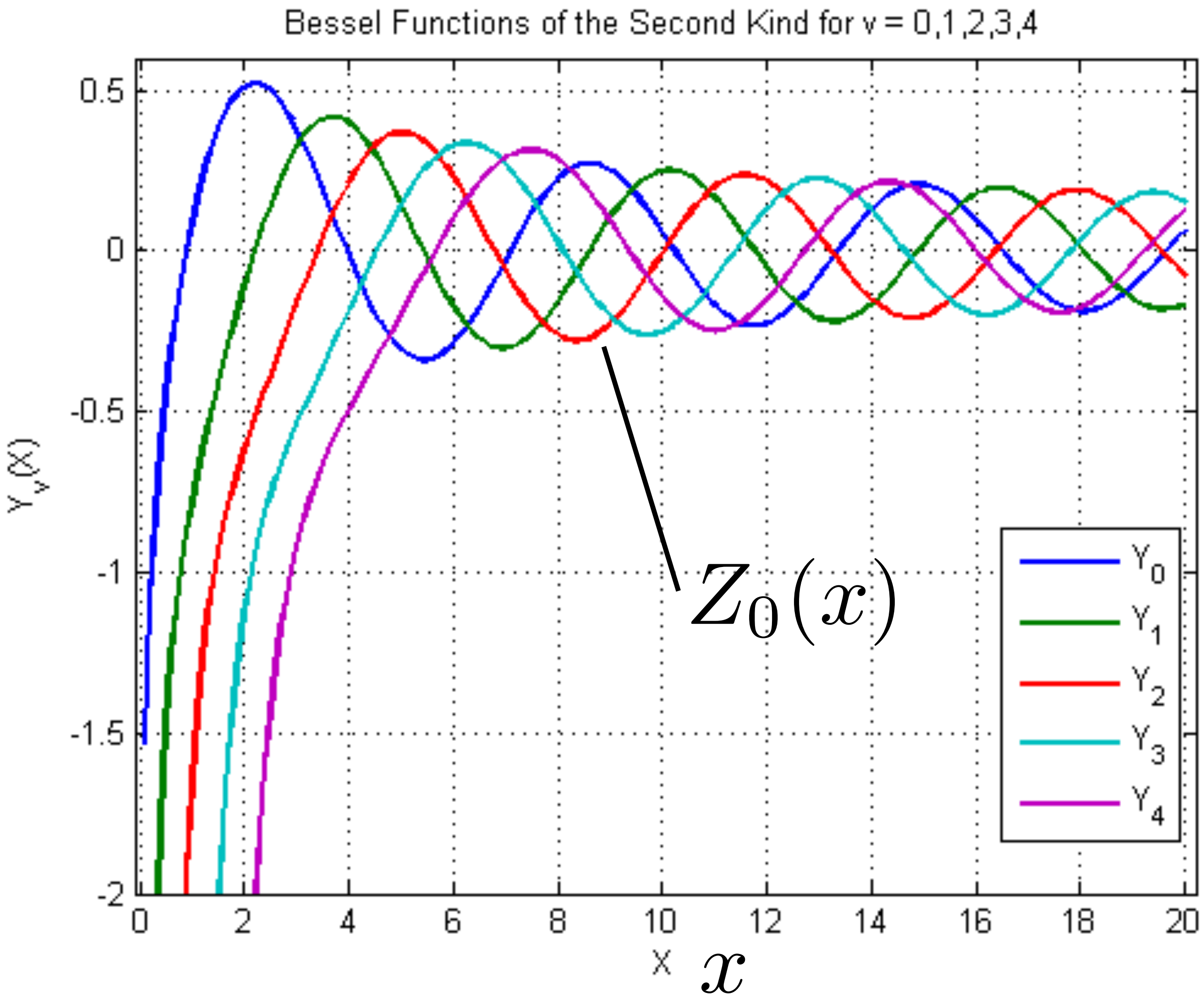
Explicit solution to wave equation and asymptotics for $t = e^{-\tau}$:

$$P(\theta, t) = \zeta \ln t + \sum_{n=1}^{\infty} Z_0(nt)\cos(n\theta + \phi_n)$$



$$\tau \to \infty$$

$$P(\theta, \tau) \to -\,v(\theta)\tau$$

$$\lambda(\theta, \tau) \to -\,[v(\theta)]^2\,\tau$$

Different Kasner at every spatial point as the singularity is approached.

Example of the BKL conjecture.

Argument of Bessel function: $nt = ne^{-\tau} = \dfrac{h(t)}{\lambda_n}$

Explicit solution to wave equation and asymptotics for $t = e^{-\tau}$:

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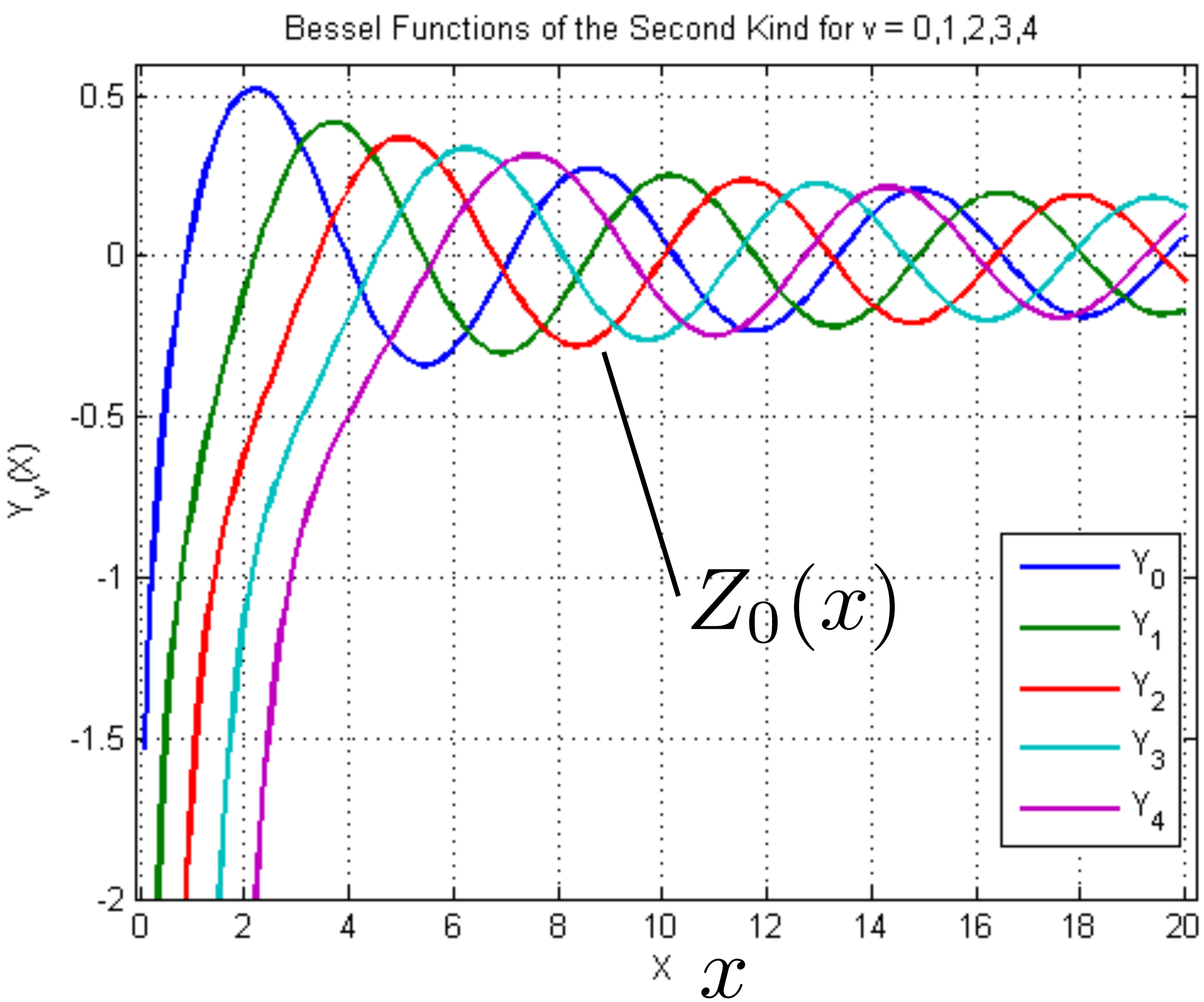

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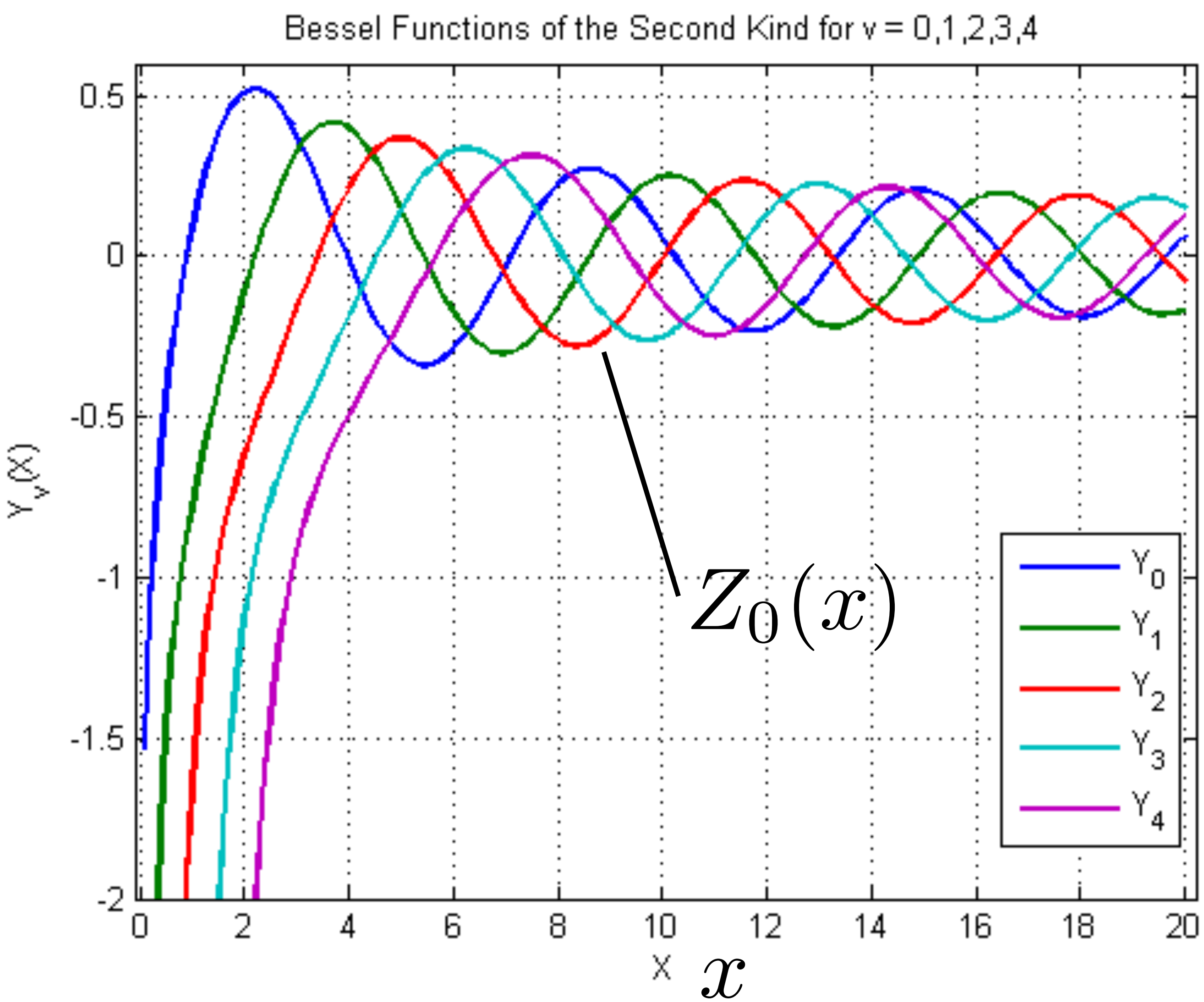


$$\lim_{t \to \infty} \{s(\theta, t),\ s_{,t}(\theta, t),\ s_{,\theta}(\theta, t)\} \approx \mathcal{O}(t^0)$$

$$t \to \infty$$

$$P \to \zeta \ln t + \frac{1}{\sqrt{t}} s(t, \theta)$$

$$0 = \lambda_{,t} - t\left(P_{,t}^2 + P_{,\theta}^2\right)$$

$$\bar{\lambda} \to c^2\, t$$

The waves control the dynamics of the background described by the spatial average of λ. This background spacetime contains a stiff fluid — the effective source created by the gravitational waves.

Interlude on “classical” particle creation

Classical analog of cosmological particle creation

Gowdy is a special case (but was done first).

Consider a scalar field $\phi(\vec{x}, t)$ in an anisotropic, spatially homogeneous, background. Perform a mode expansion and change time variable so that each mode satisfies a time-dependent-frequency harmonic oscillator equation

$$\ddot{\phi}_{\vec{k}} + \omega^2_{\vec{k}}(\tau)\,\phi_{\vec{k}} = 0$$

Suppressing the mode vector,

$$\phi(\tau) = A\,Z_1(\tau) + B\,Z_2(\tau)$$

with *A*, *B* chosen to yield the Wronskian

$$W(Z_1, Z_2) \equiv Z_1 \dot{Z}_2 - Z_2 \dot{Z}_1 = 1$$

B.K. Berger, Phys. Rev. D **18**, 4367 (1978)

Classical analog of cosmological particle creation

WKB (adiabatic) limit: wave mode period much shorter than time scale of frequency change (high frequency)

$$\phi_{WKB} \approx \omega^{-1/2} A \cos\left(\int^{\tau} \omega d\tau' + \xi\right) + \omega^{-1/2} B \sin\left(\int^{\tau} \omega d\tau' + \xi\right)$$

gives a (formal) energy in the mode

$$E = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\omega^2\phi^2$$

with the number of quanta

$$N = \frac{E}{\omega} = \tfrac{1}{2}(A^2 + B^2)$$

Classical analog of cosmological particle creation

Case 1: There is a singularity at τ_S where $\lim_{\tau \to \tau_S} \omega(\tau) = 0$.

ϕ behaves as a free particle: $\phi_S \approx q_0 + p_0 \tau$

To preserve the Wronskian and if, e.g., the singularity occurs at an infinite value of the time variable,

$$Z_1 \approx \beta + \alpha\tau \qquad , \qquad Z_2 \approx 1/\alpha$$

to yield

$$A = p_0/\alpha \qquad , \qquad B = \alpha\, q_0 - \beta\, p_0$$

and

$$N = \tfrac{1}{2}\left[\alpha^2\, q_0^2 + \left(\frac{1}{\alpha^2} + \beta^2\right) p_0^2 - 2\,\alpha\,\beta\, p_0\, q_0\right]$$

N is the number of WKB quanta given $q_0,\ p_0$ at the singularity. Add zero-point energy to recover full QM results.

Classical analog of cosmological particle creation

Case 2: Frozen cosmology

Choose $\tau_0 > \tau_S$ where $>$ means "after".

To preserve the Wronskian and if, e.g., the singularity occurs at an infinite value of the time variable, set $\omega(\tau) = \omega(\tau_0) = \omega_0$ for all earlier times.

Further assume that

$$\phi_0 = C \sin\left(\int^{\tau_0} \omega d\tau + \zeta\right)$$

holds at this time with

$$N(\tau_0) \equiv N_0 = \frac{\omega_0}{2}|C|^2$$

but that the solution is described by

$$Z_1 \approx \beta + \alpha\tau \qquad , \qquad Z_2 \approx 1/\alpha$$

This yields an arguably unphysical amplification factor

$$\frac{N(\tau)}{N_0} \approx \frac{\alpha^2}{2\omega_0} \rightarrow \infty \;\text{ as }\; \tau_0 \rightarrow \tau_S$$

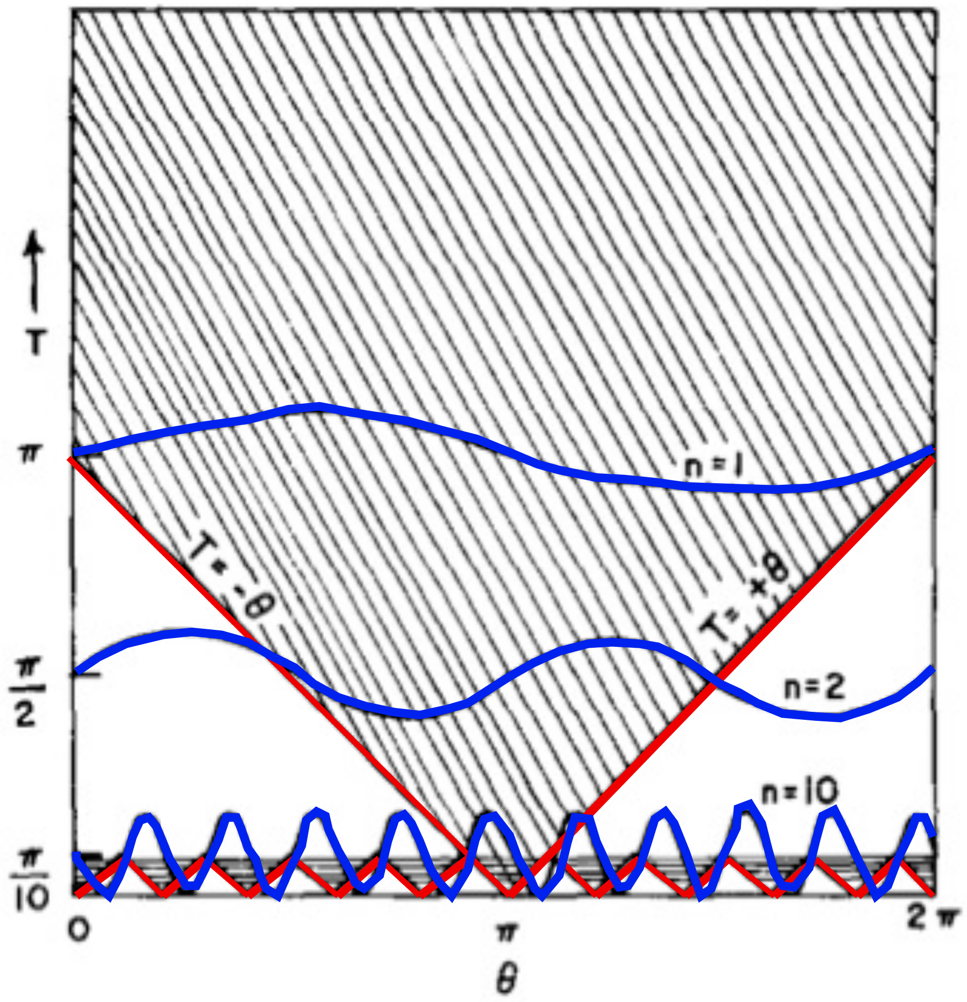
Polarized Gowdy: quantum graviton creation

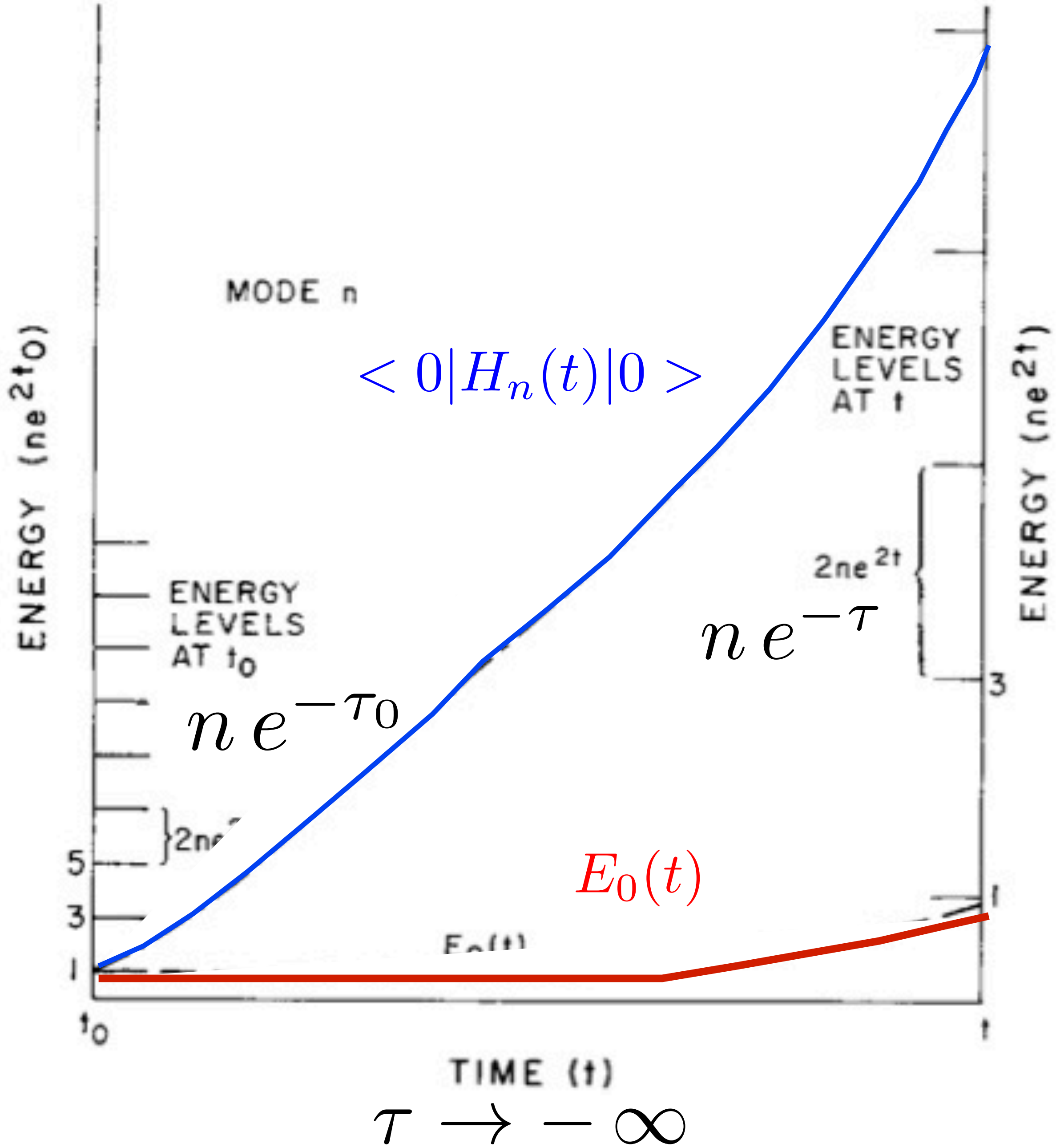
A Case 2 Example:

Quantization 1: Instantaneous diagonalization and Bogoliubov transformations. Treat *P* as a scalar field in a Kasner-like background.

Pick an initial time τ_0 . At that time, define a set of states of the scalar field *P*. Define creation and annihilation operators and a number operator. Find the transformation coefficients to write the number of particles in a state, i.e. the expectation value, at τ_0 with the comparable number of particles defined at the final time τ_f . This calculation proceeds mode by mode. The momentum constraint imposes overall zero momentum on the field *P*.

B.K. Berger, Ann. Phys. **83**, 458 (1974) and Ph.D. Thesis, U. of Maryland, 1972





Issues: Instantaneous diagonalization at the initial time fails to represent the Kasner like behavior at that time.

Success: Early example of graviton creation in a cosmology.

A Case 1 Example:

Quantization 2: Embrace the evolution from Kasner-like free particle behavior to that for a scalar field in a time dependent background.

Use established methods to solve the time-dependent-frequency harmonic oscillator. For each mode, the wavefunction is that for a minimum uncertainty wavepacket at early times (appropriate for the free-particle MSS behavior) and that for gravitons in a background at late times. The particle number is conserved and represents the number of gravitons at late times.

Issues: Neglecting the usual MSS issues, this quantization seems reasonable. It is in the spirit of the ADM quantization because the constraints are solved *a priori*.

C.W. Misner, Phys. Rev. D **8**, 3271 (1973); B.K. Berger, Phys. Rev. D 11, 2770 (1975)

Ground state wavefunction for mode n of field P:

$$ip_\lambda \frac{\partial \psi_n}{\partial \tau} = \left(-\frac{1}{2}\frac{\partial^2}{\partial q_n{}^2} + \tfrac{1}{2} n^2 e^{4\tau} q_n{}^2\right)\psi_n .$$

$$\lim_{\tau\to+\infty} \psi_n = \sum_n a_N \exp\left[-\frac{i}{p_\lambda}(N+\tfrac{1}{2})\int^\tau d\tau' \omega_n\right] \times \phi_n(q_n;\omega_n) ,$$

WKB limit

$$\psi_{n,0} = \mathfrak{N}\left[Z_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{-1/2} \times \exp\left[-\tfrac{1}{2}|n|\, e^{2\tau} Z_1\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right) q_n^2 \Big/ Z_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right.$$

Valid at all times, min. uncert. wp at early times

$$A_n = i|n|\, e^{2\tau} Z_1\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\hat{q}_n - Z_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\frac{\partial}{\partial q_n}$$

Annihilation operator

$$A_n \psi_{n,0} \equiv 0$$

Ground state wavefunction for mode n of field P:

$\tau \to -2\tau \; ; \; P_n \to q_n$

$$ip_\lambda \frac{\partial \psi_n}{\partial \tau} = \left(-\frac{1}{2} \frac{\partial^2}{\partial {q_n}^2} + \tfrac{1}{2} n^2 e^{4\tau} {q_n}^2 \right) \psi_n \, .$$

$$\lim_{\tau \to +\infty} \psi_n = \sum_n a_N \exp\left[-\frac{i}{p_\lambda} (N + \tfrac{1}{2}) \int^\tau d\tau' \omega_n \right] \times \phi_n(q_n; \omega_n) \, ,$$

WKB limit

$$\psi_{n,0} = \mathfrak{N} \left[Z_0 \left(\frac{|n|}{2p_\lambda} e^{2\tau} \right) \right]^{-1/2} \times \exp\left[-\tfrac{1}{2} |n| e^{2\tau} Z_1 \left(\frac{|n|}{2p_\lambda} e^{2\tau} \right) q_n^2 \Big/ Z_0 \left(\frac{|n|}{2p_\lambda} e^{2\tau} \right) \right]$$

Valid at all times, min. uncert. wp at early times

$$A_n = i |n| e^{2\tau} Z_1 \left(\frac{|n|}{2p_\lambda} e^{2\tau} \right) \hat{q}_n - Z_0 \left(\frac{|n|}{2p_\lambda} e^{2\tau} \right) \frac{\partial}{\partial q_n}$$

Annihilation operator

$$A_n \psi_{n,0} \equiv 0$$

These models have been used extensively by others to explore a number of formalisms for quantum gravity or quantum field theory in curved spacetime.

For an extensive review, see J.F. Barbero G., E.J.S. Villaseñor, Living Rev. Relativity **13**, 6 (2010).

C.P. Winlove, D. Raine, "Pair Creation and the Gowdy Model," Ann. Phys. 93, 116 (1975)

Husain, V., “Quantum effects on the singularity of the Gowdy cosmology”, Class. Quantum Grav., 4, 1587–1591, (1987).

Husain, V. and Smolin, L., “Exactly solvable quantum cosmologies from two Killing field reductions of general relativity”, Nucl. Phys. B, 327, 205, (1989).

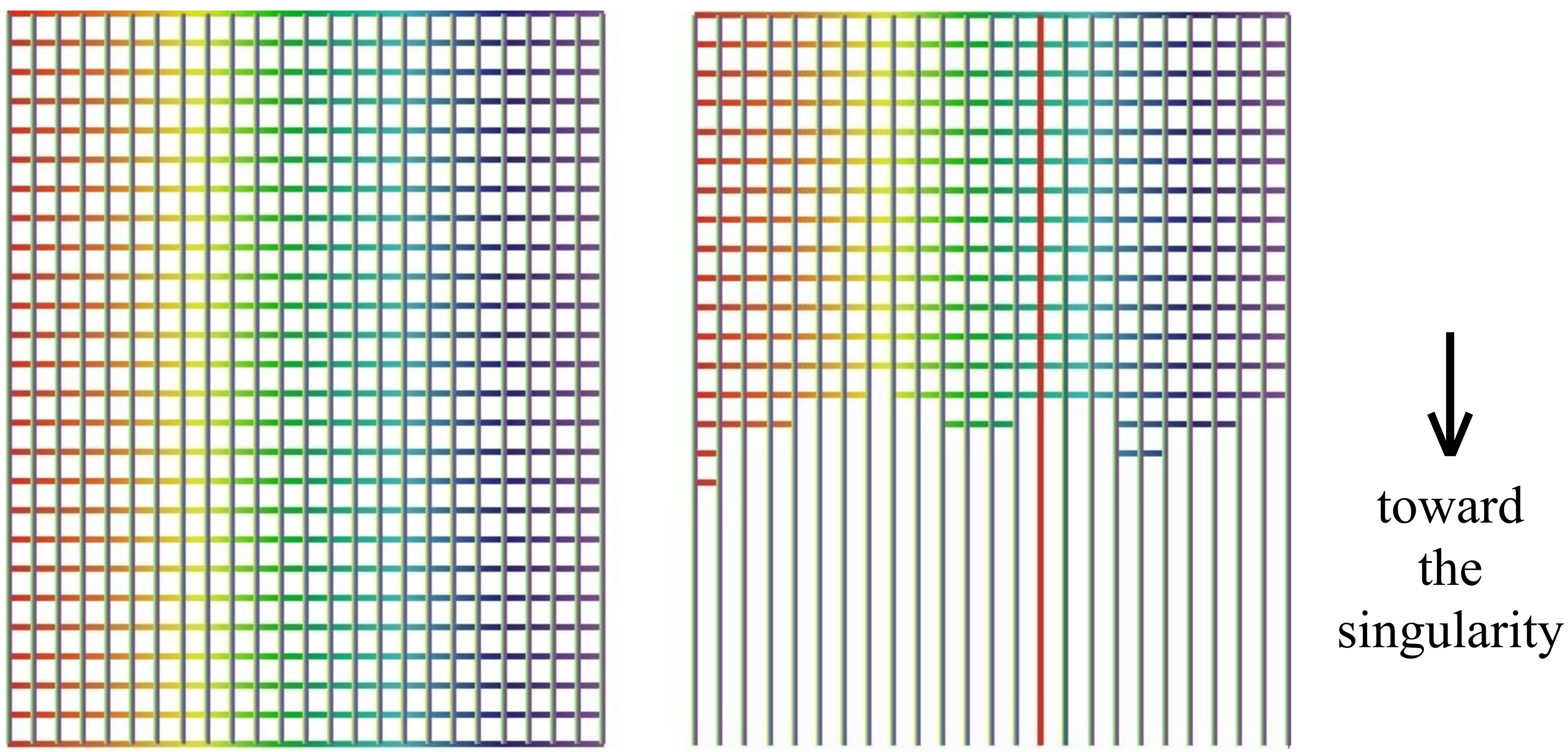
Mena Marugán, G.A., “Canonical quantization of the Gowdy model”, Phys. Rev. D, 56, 908–919, (1997).

Torre, C.G., “Schrödinger representation for the polarized Gowdy model”, Class. Quantum Grav., 24, 1–13, (2007).

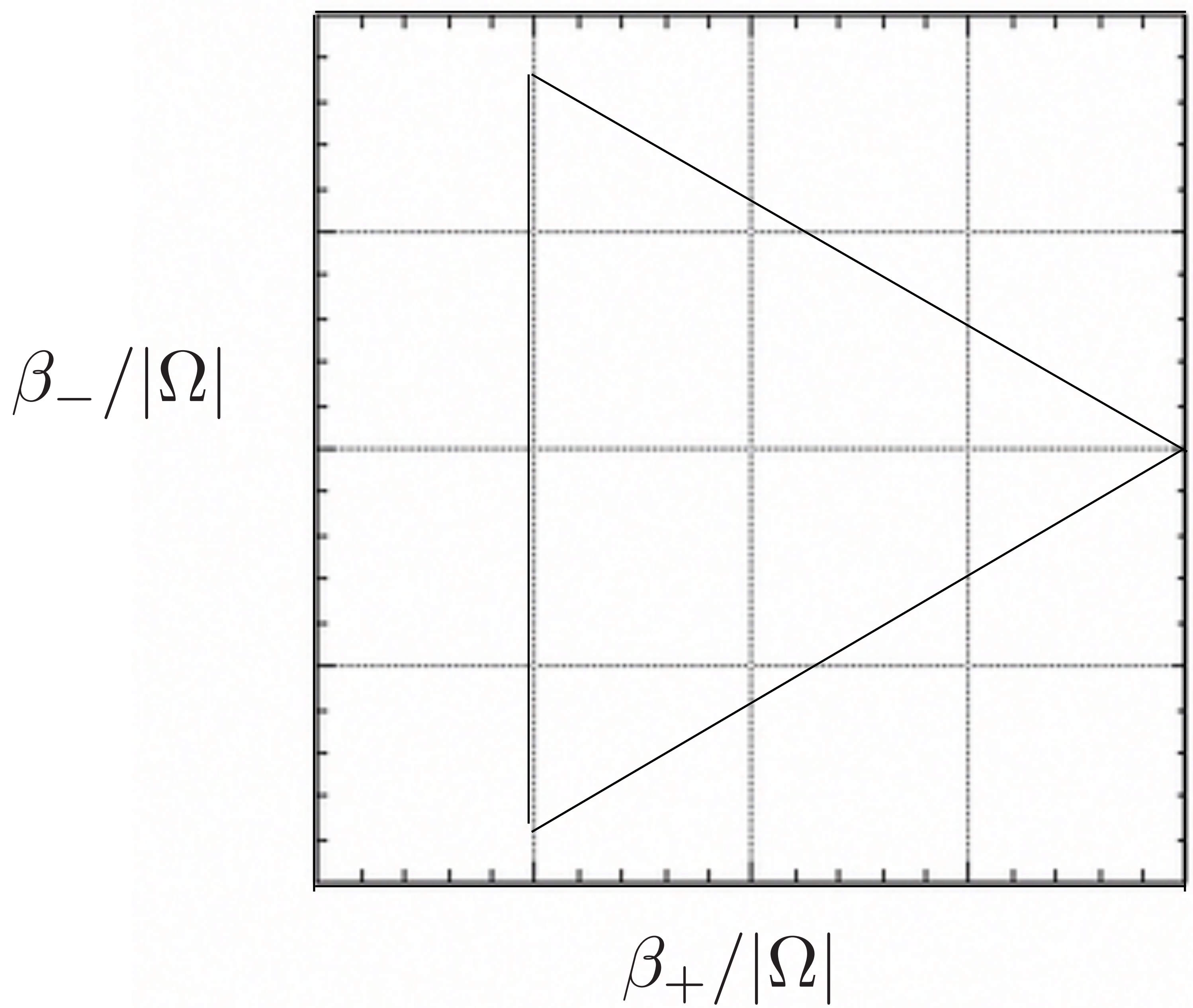
The BKL Conjecture

Spatially **inhomogeneous** cosmological spacetimes:

BKL claim that sufficiently close to the singularity, spatial derivatives become dynamically irrelevant compared to time derivatives so that each spatial point evolves as a separate universe with either an AVTD or Mixmaster singularity.



A Mixmaster simulation with > 250 bounces:



Ringström has proven that the Mixmaster singularity for non-Taub initial data is of the curvature blow-up type.

H. Ringström, Class.Quant.Grav. 17 (2000) 713-731.

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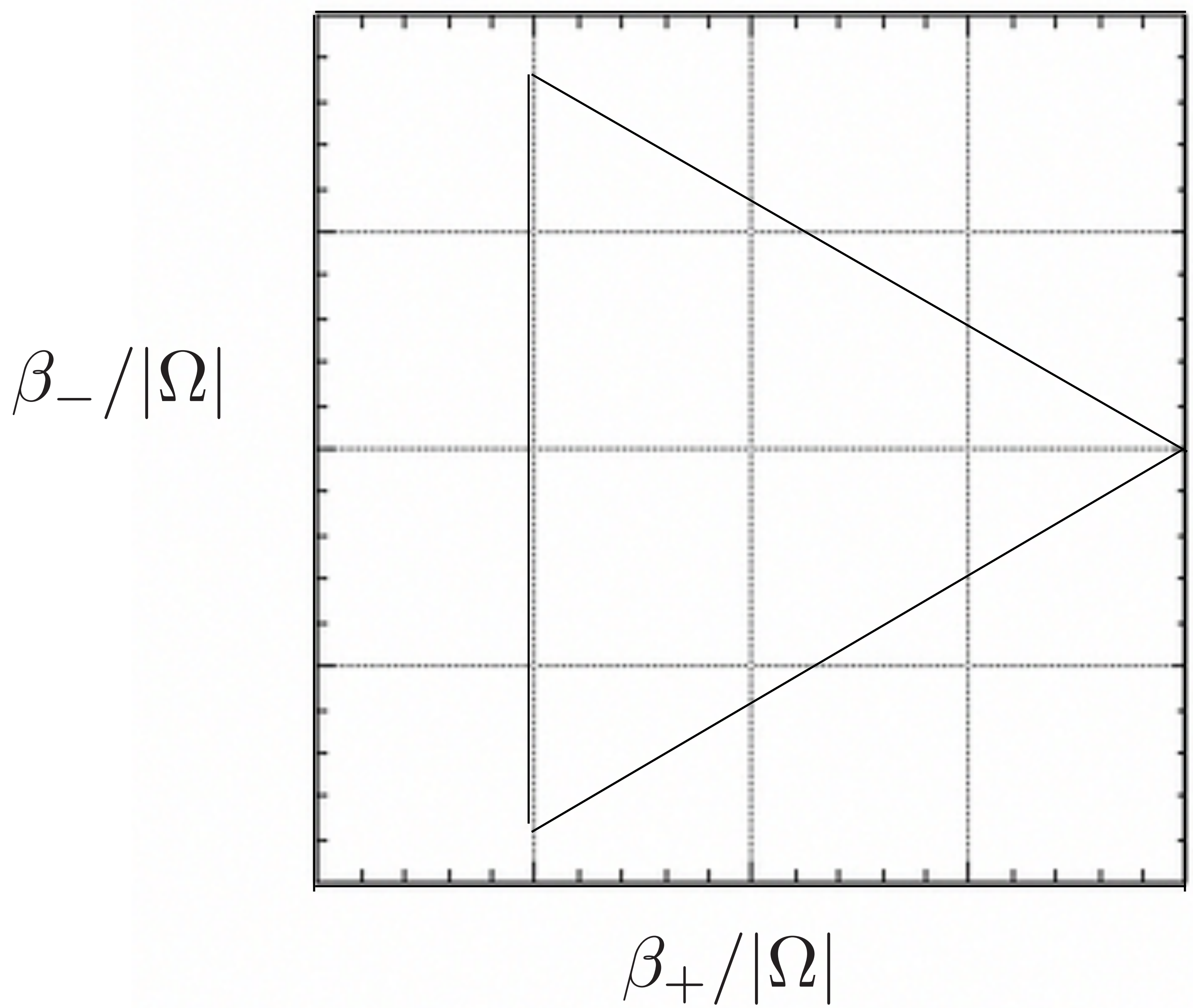

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Kasner metric in Gowdy coordinates:

$$g = e^{\lambda/2} t^{-1/2}(-dt^2 + d\theta^2) + t(e^P dx^2 + e^{-P} dy^2)$$

Add θ-dependence for P and λ to obtain polarized Gowdy.

Rotate the axes in the x-y plane to obtain generic Gowdy:

$$g = e^{\lambda/2} t^{-1/2}(-dt^2 + d\theta^2) + e^P t\,(dx + Q\,dy)^2 + e^{-P} t\,dy^2$$

Rotate in the x- θ and y- θ planes to obtain generic T^2-symmetric:

$$g = e^{\lambda/2}\, t^{-1/2}(-dt^2 + e^{\mu/2} d\theta^2)$$

$$+ e^P t\,(dx + Q\,dy)^2 + e^{-P} t\,(dy + G\,d\theta)^2$$

Generic Gowdy models:

$$ds^2 = e^{(\lambda+\tau)/2}\left(-e^{-2\tau}d\tau^2 + d\theta^2\right) + e^{P-\tau}\left(d\sigma + Qd\delta\right)^2 + e^{-P-\tau}d\delta^2$$

Einstein's equations consist of wave equations for P and Q and constraints which may be solved for λ. The wave equations may be obtained by variation of

$$2\mathcal{H} = \pi_P^2 + e^{-2P}\pi_Q^2 + e^{-2\tau}P,_\theta^2 + e^{2(P-\tau)}Q,_\theta^2$$

where $\mathcal{H} \neq 0$.

As $\tau \to \infty$, the VTD solution (neglect spatial derivatives) is

$$P(\theta,\tau) \to v(\theta)\,\tau \quad , \quad \pi_P(\theta,\tau) \to v(\theta)$$

$$Q(\theta,\tau) \to Q^0(\theta) \quad , \quad \pi_Q(\theta,\tau) \to \pi_Q^0(\theta)$$

Terms in the Hamiltonian act as potentials. For AVTD behavior of the model, these potentials must decay exponentially.

$$V_1 = e^{-2P}\,\pi_Q^2 \to e^{-2v\tau}\left(\pi_Q^0\right)^2$$

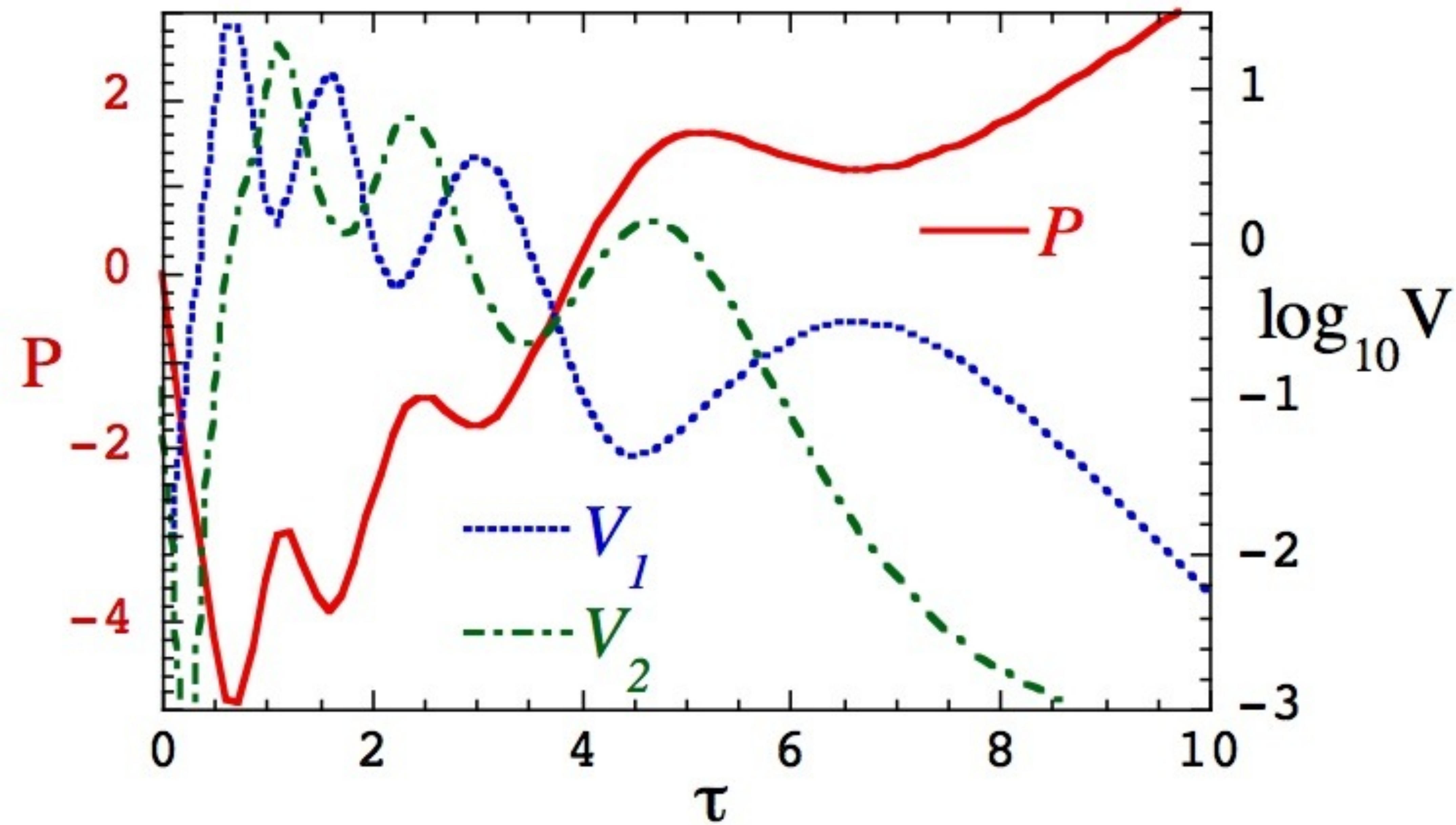
requires $v > 0$ for consistency.

$$V_2 = e^{2(P-\tau)}\,(Q_{,\theta})^2 \to e^{2(v-1)\tau}\,(Q_{0,\theta})^2$$

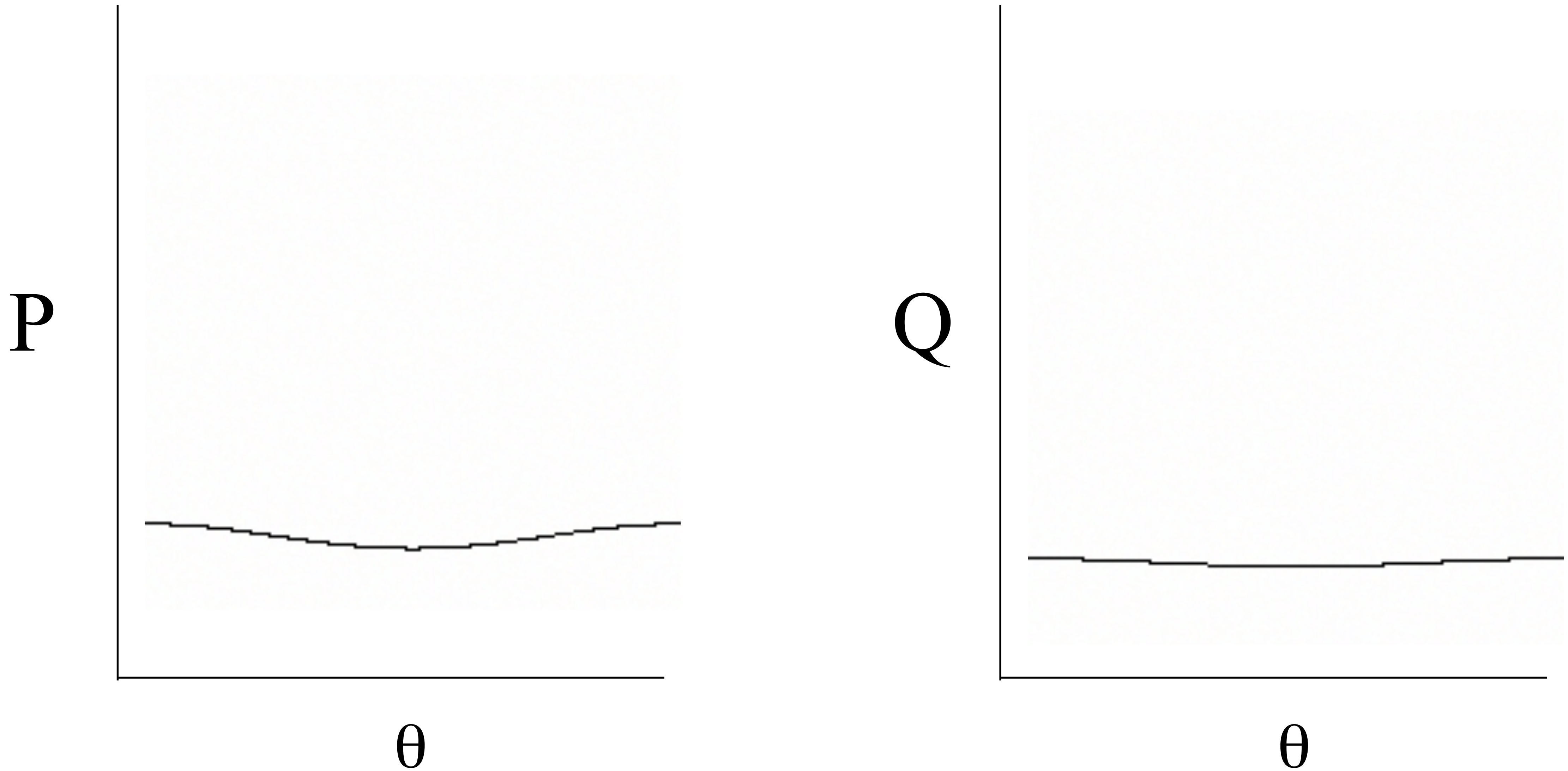
requires $v < 1$ for consistency.

This means that the singularity is AVTD (at any spatial point) only if $0 \leq v < 1$.

Numerical simulations show how v is driven into the range (0,1) by bounces off the potentials. A typical single spatial point is shown.



B.K. Berger, D. Garfinkle, Phys. Rev. D **57**, 4767 (1998).


P
θ
Q
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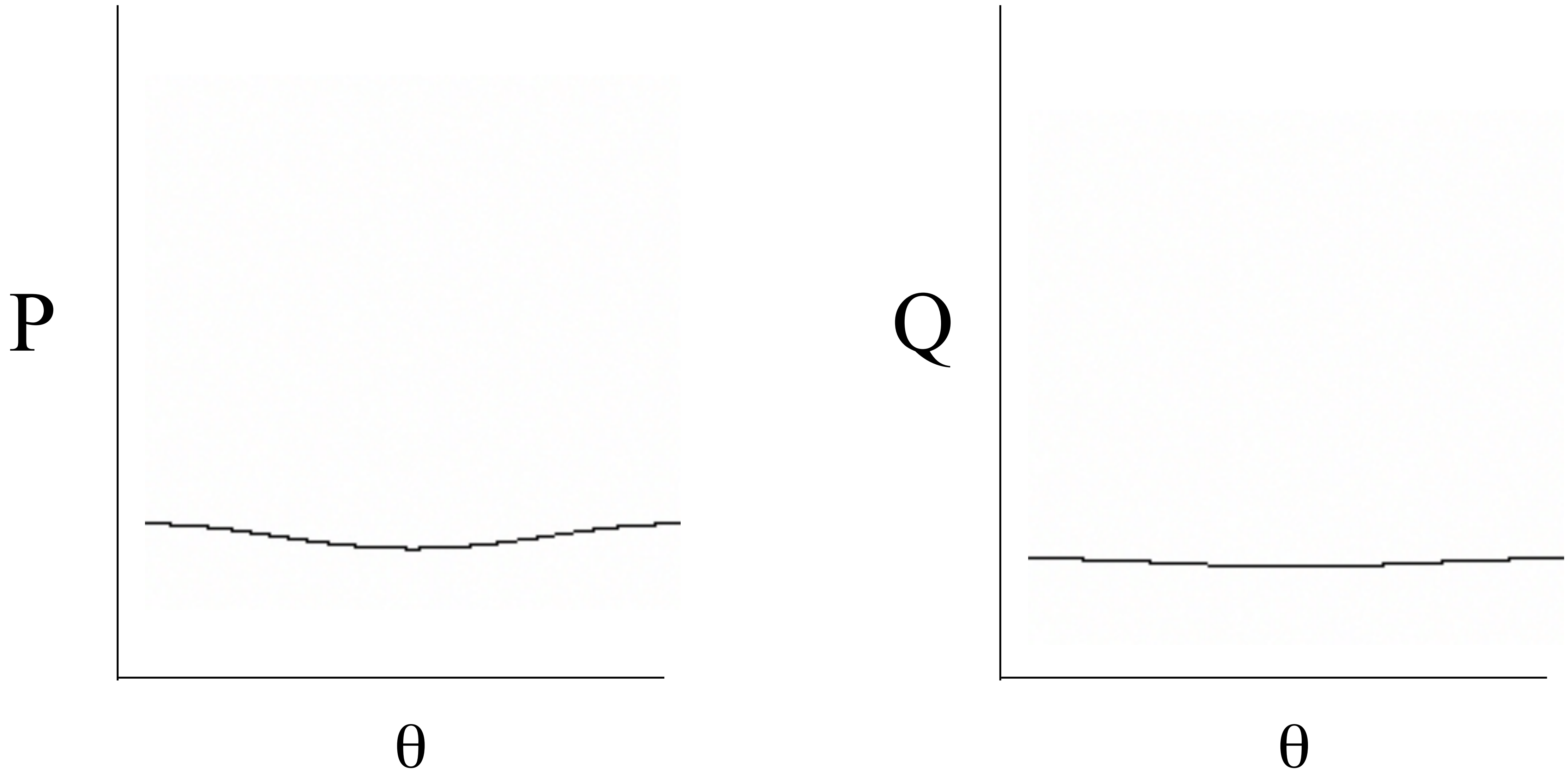
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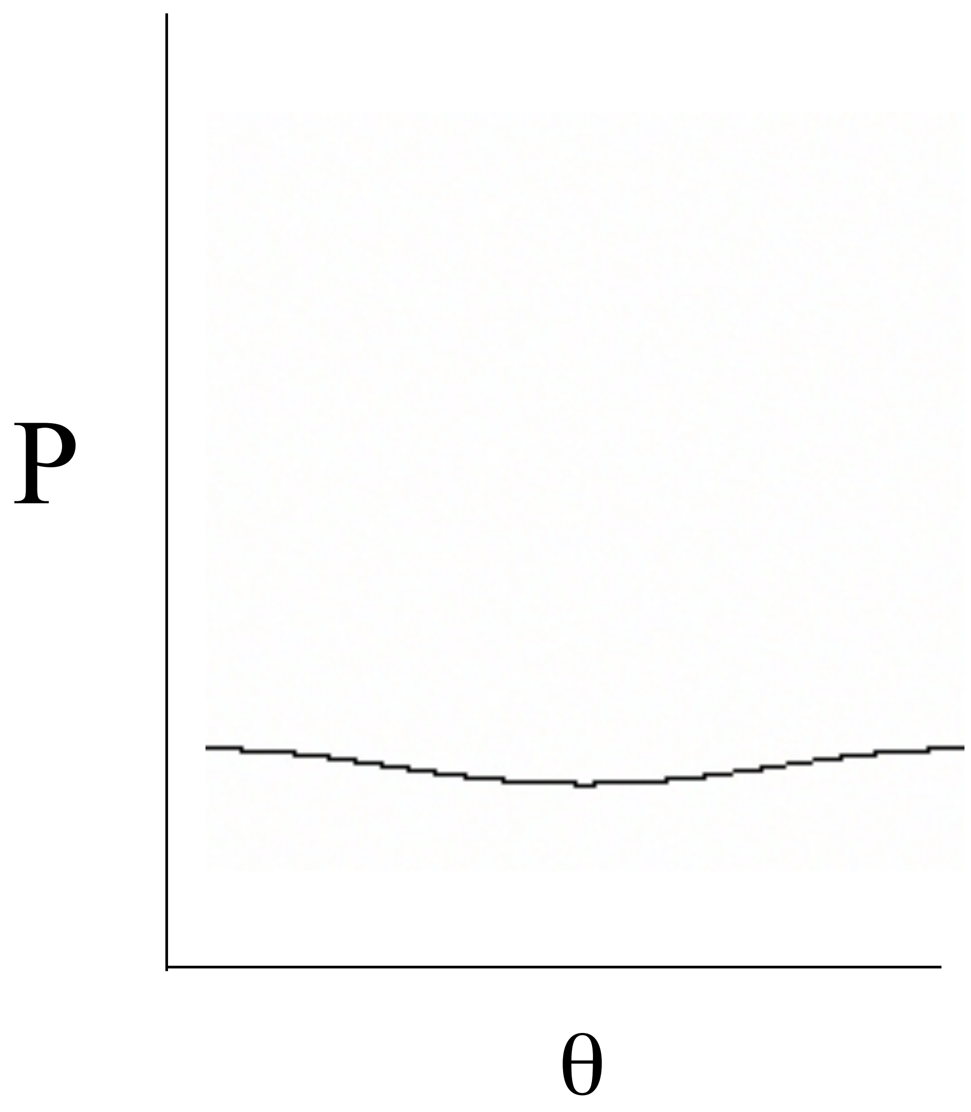
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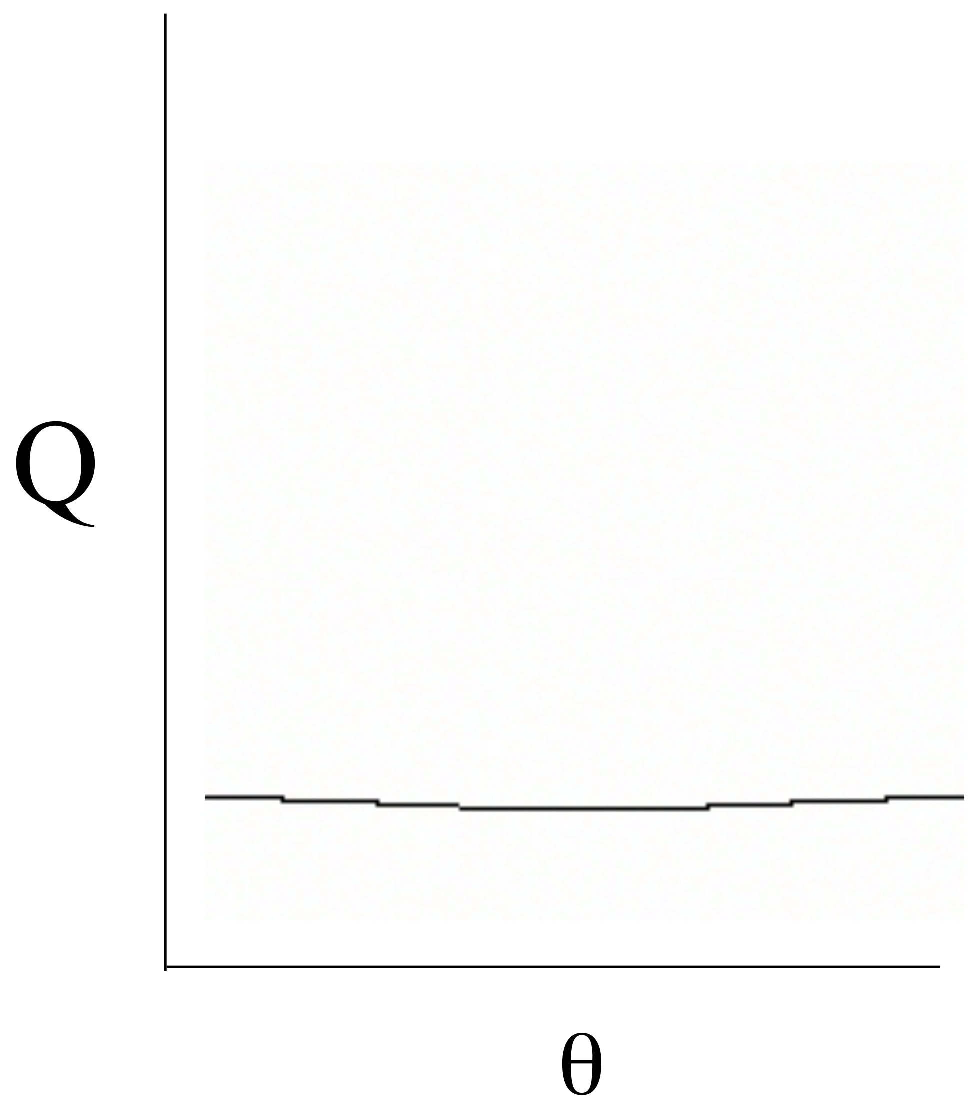

Q
θ

General T^2 symmetric spacetime adds twist potential:

$$ds^2 = -e^{(\lambda-3\tau)/2}d\tau^2 + e^{(\lambda+\mu+\tau)/2}d\theta^2$$
$$+e^{P-\tau}\left[d\sigma + Qd\delta + \left(\int^\tau (Q\Theta) - Q\int^\tau \Theta\right) d\theta\right]^2$$
$$+e^{-P-\tau}\left[d\delta - \left(\int^\tau \Theta\right) d\theta\right]^2$$

where $\Theta = e^{(\lambda+2P+3\tau)/2} e^{\mu/4}\kappa$

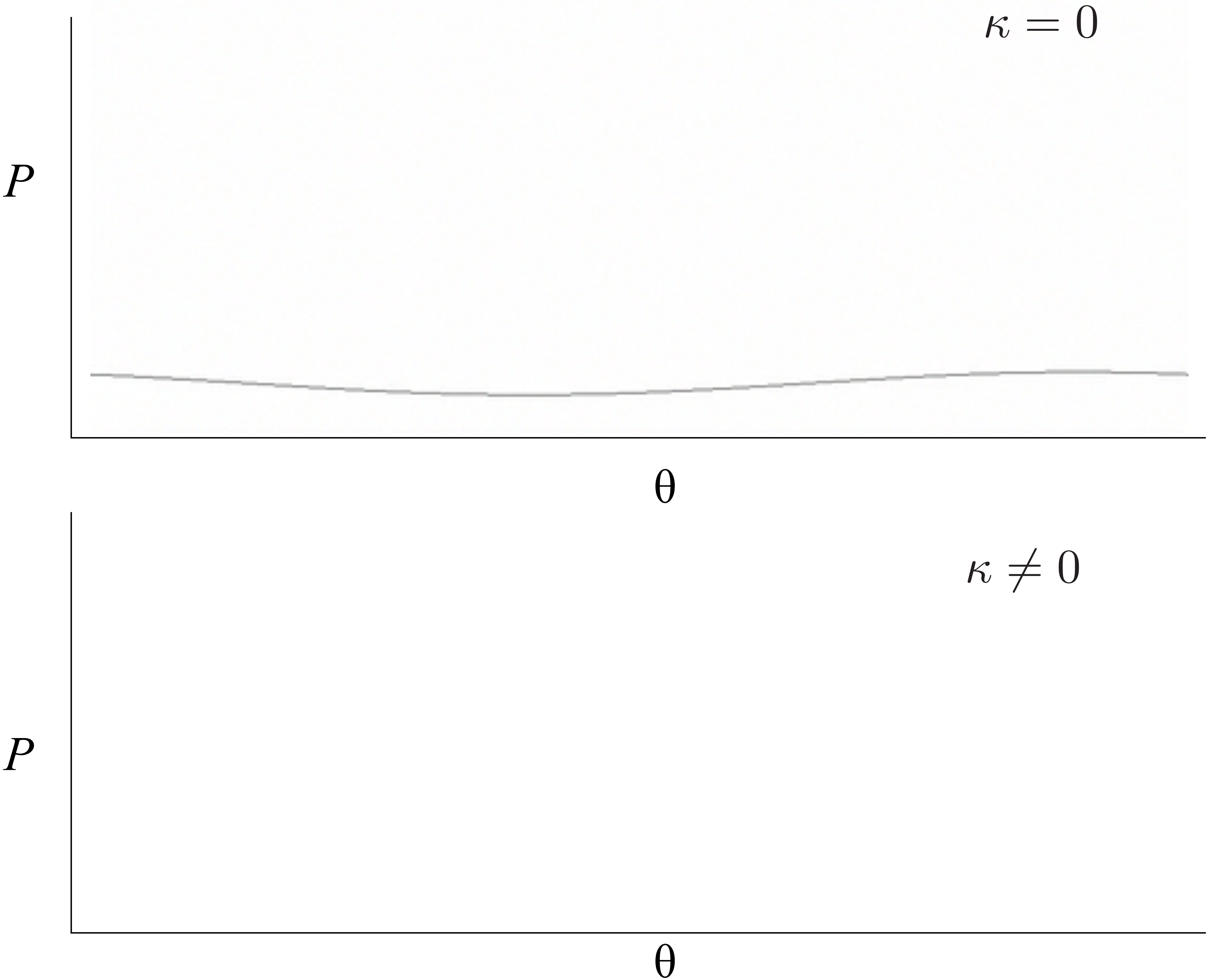
Hamiltonian formulation:

$$H = H_0 + H_{small} + H_{kin} + H_{curv} + H_{twist}$$

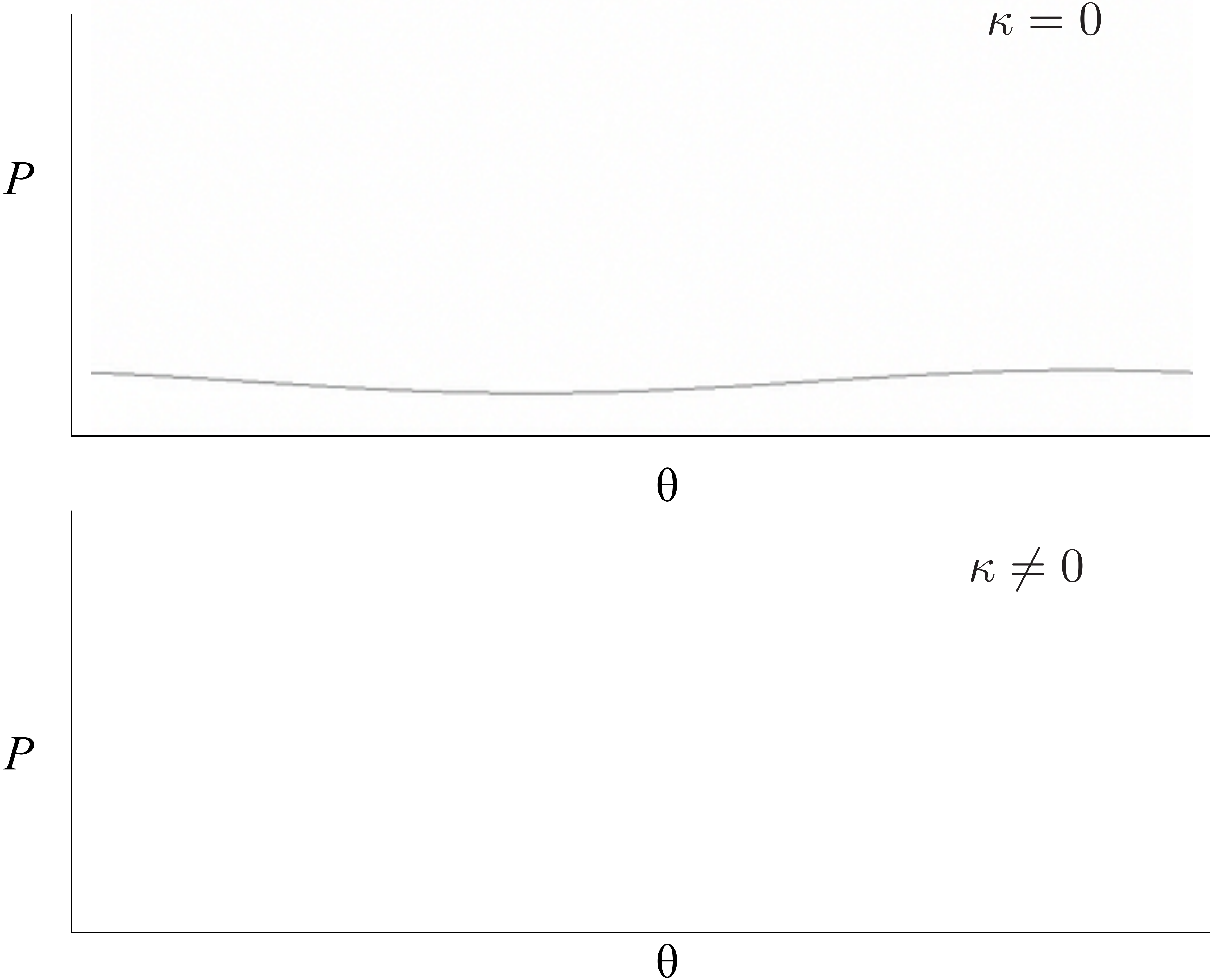
$$H = \frac{\pi_P^2}{4\pi_\lambda} + \frac{P_{,\theta}^2 e^{-2\tau}}{4\pi_\lambda} + \frac{\pi_Q^2 e^{-2P}}{4\pi_\lambda} + \frac{Q_{,\theta}^2 e^{2(P-\tau)}}{4\pi_\lambda} + \sigma\kappa^2\pi_\lambda e^{(\lambda+2P+3\tau)/2}$$

Gowdy: $\kappa = 0, \quad \pi_\lambda = \frac{1}{2}$

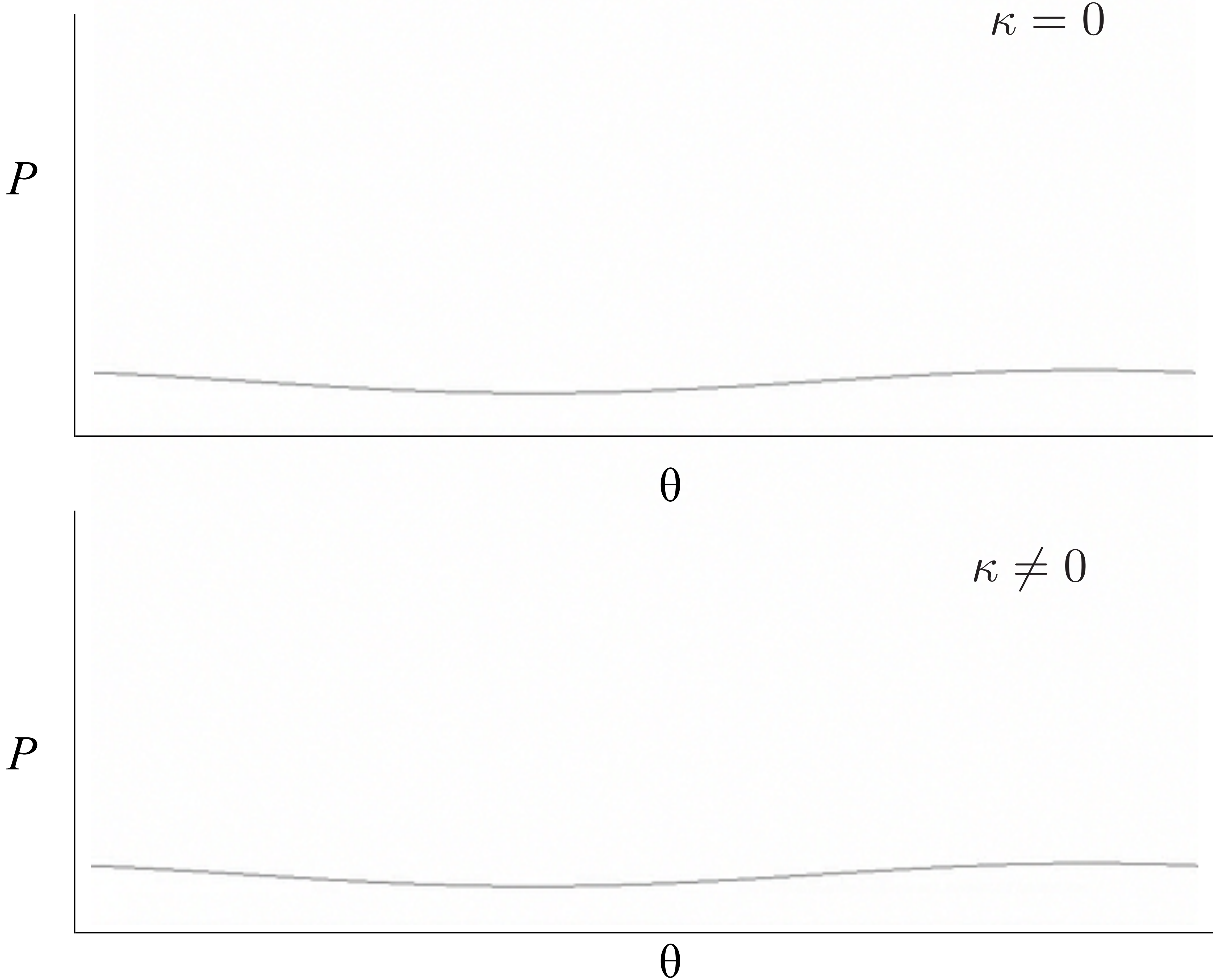
B.K. Berger, J. Isenberg, M. Weaver, PRD **64**, 084006 (2001)



$\kappa = 0$
P
θ
$\kappa \neq 0$
P
θ

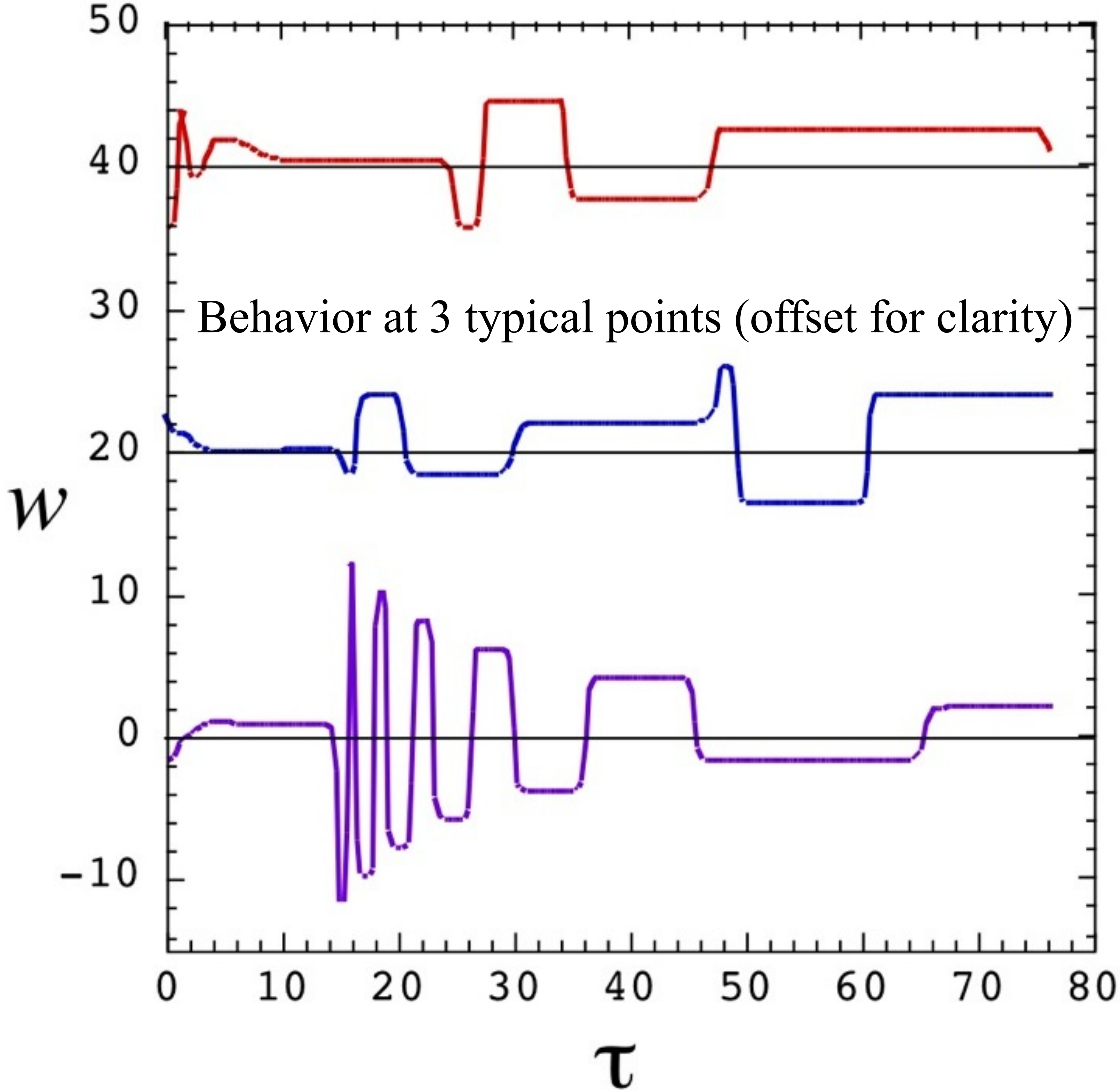


κ = 0
P
θ
κ ≠ 0
P
θ


$\kappa = 0$
P
θ
$\kappa \neq 0$
P
θ

Bounce laws relate *w* before and after bounce.
Quantitative agreement is found.

$$w = \frac{\pi_P}{2\pi_\lambda}$$

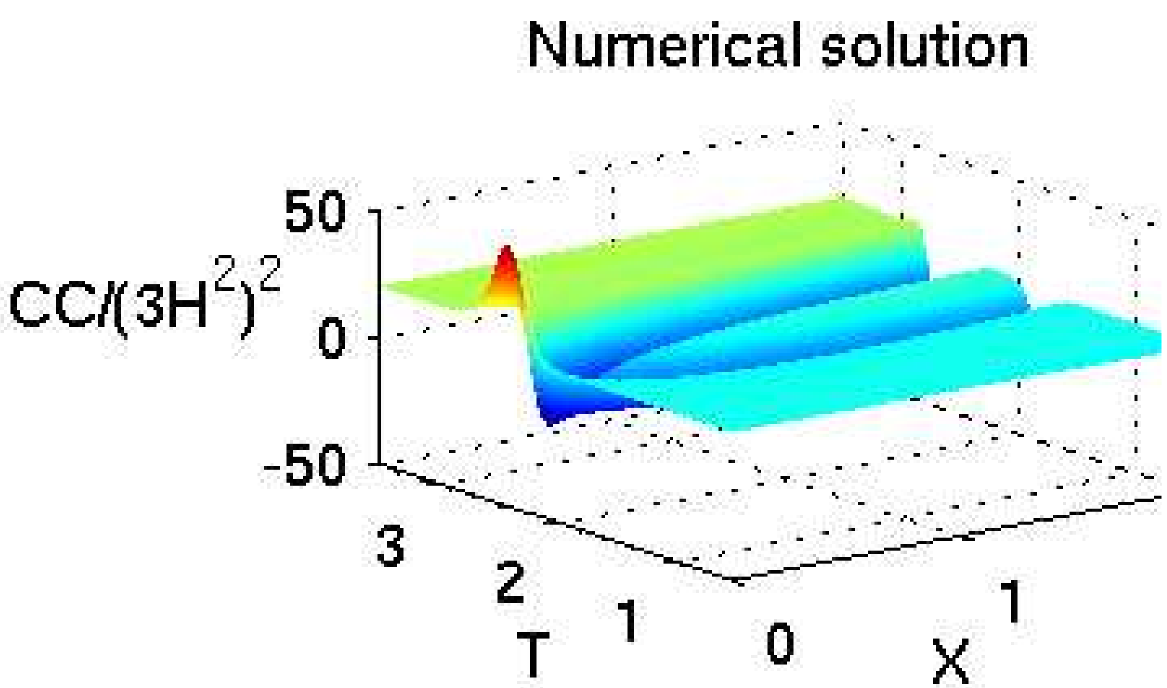


Recurrent spikes: a non-local feature of G_2 evolution

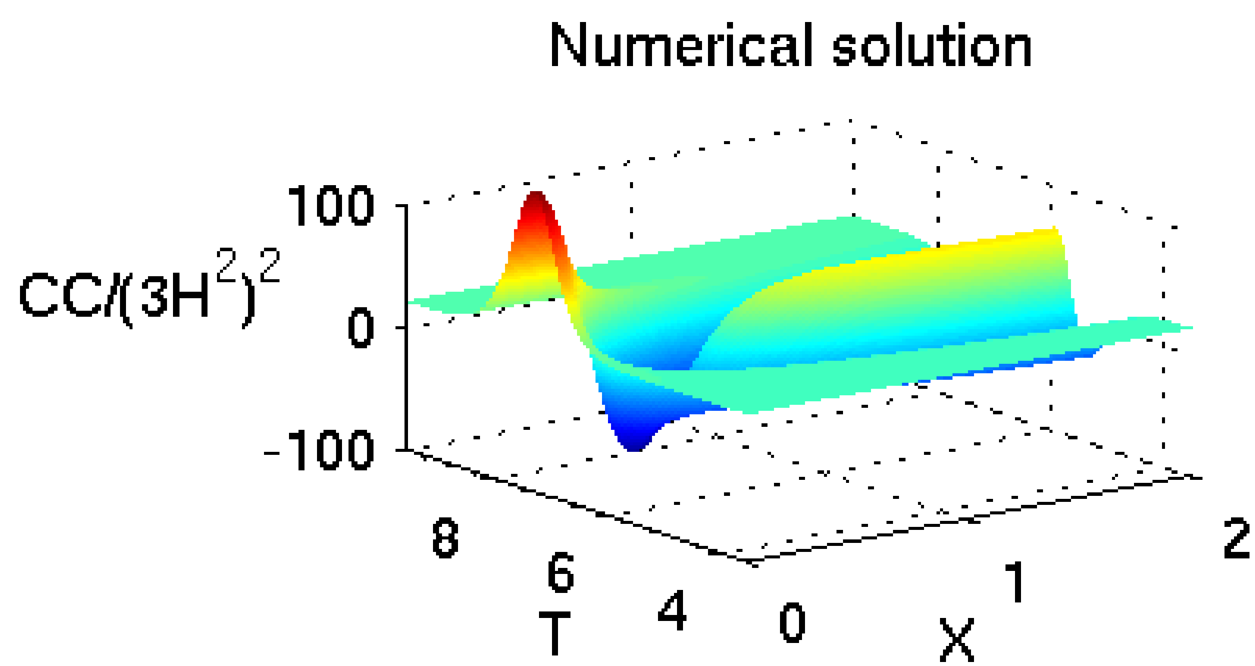
W.C. Lim, L. Andersson, D. Garfinkle, F. Pretorius: Spikes in the Mixmaster Regime of G2 Cosmologies, Phys. Rev. D 79, 123526 (2009)

J.M. Heinzle, C. Uggla, W.C. Lim: Spike Oscillations, arXiv: 1206.0932, preprint

Advanced numerical methods with high spatial resolution reveal recurrent spikes.



Original spike

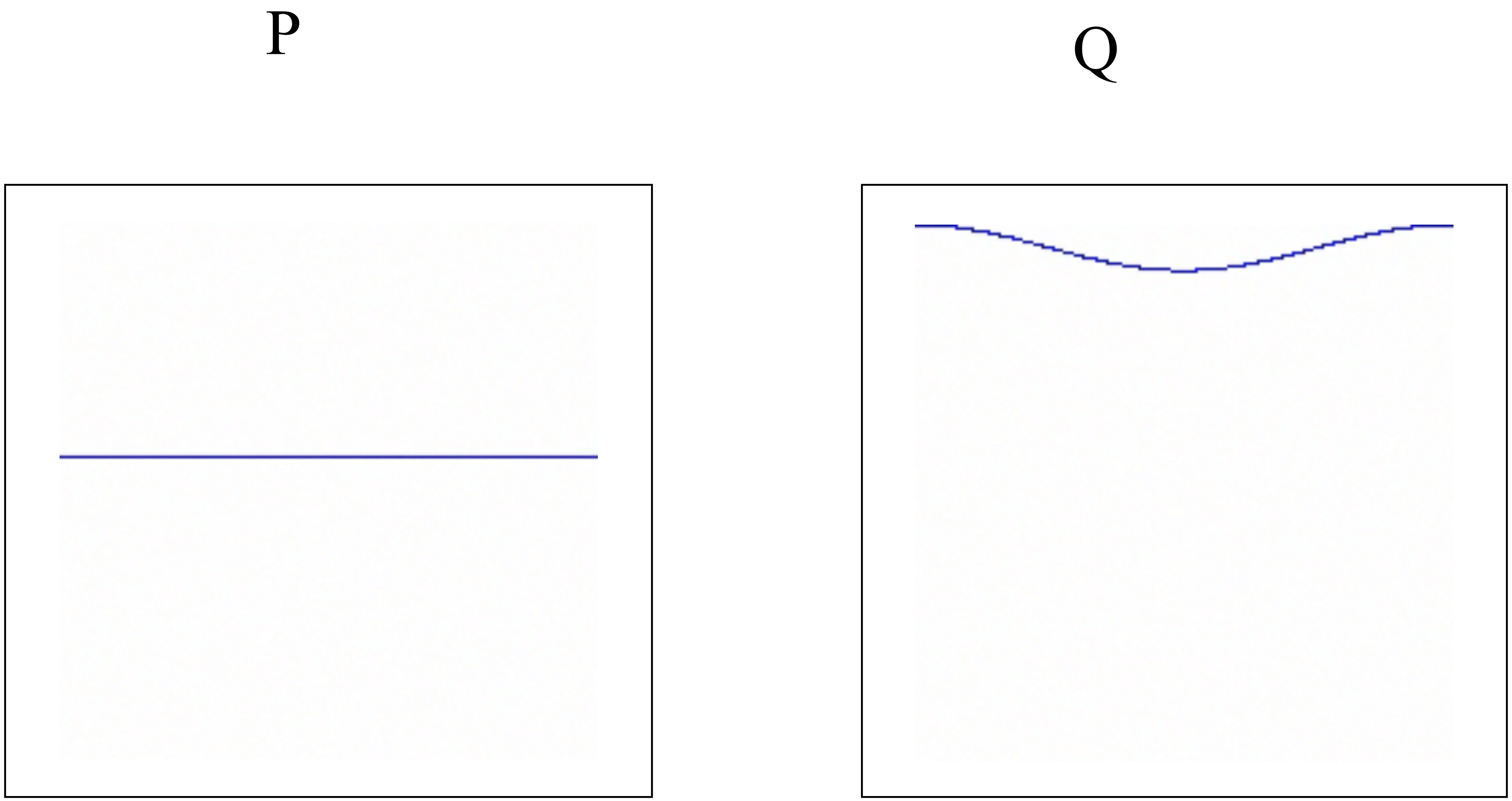


First recurrence

Numerical simulations match exact spike solutions with increasing accuracy as singularity is approached.

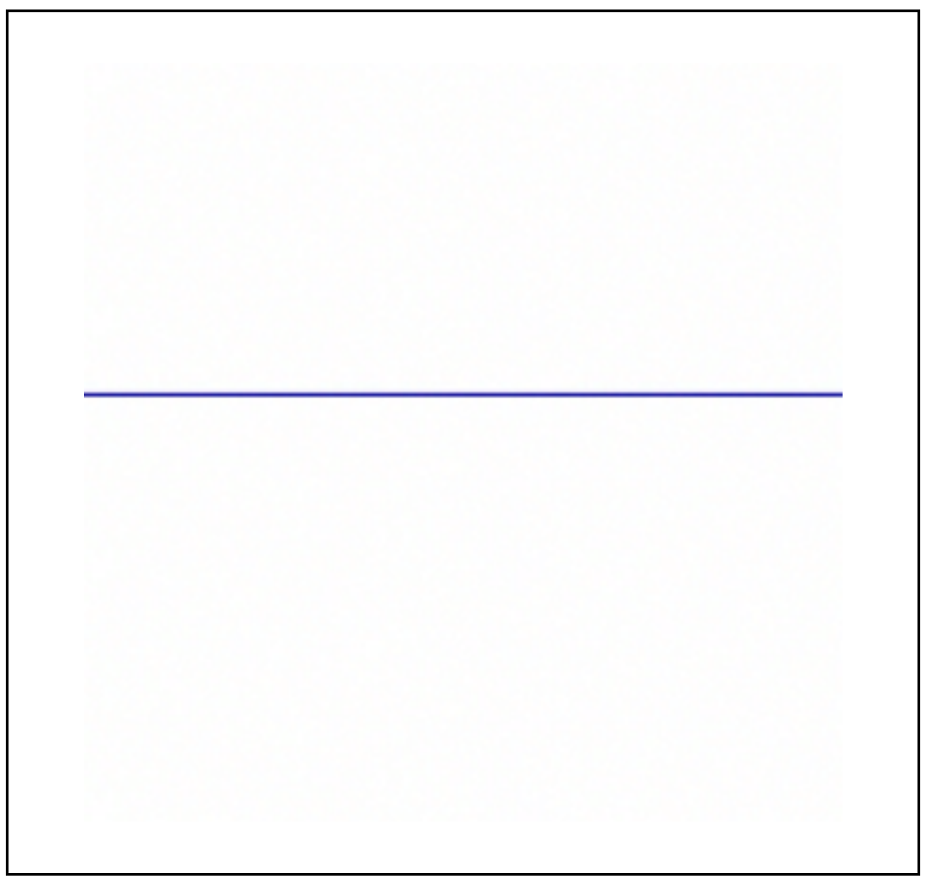
Expanding cosmologies

Generic Gowdy cosmologies: nonlinear terms decay. Each polarization propagates as a linear wave.

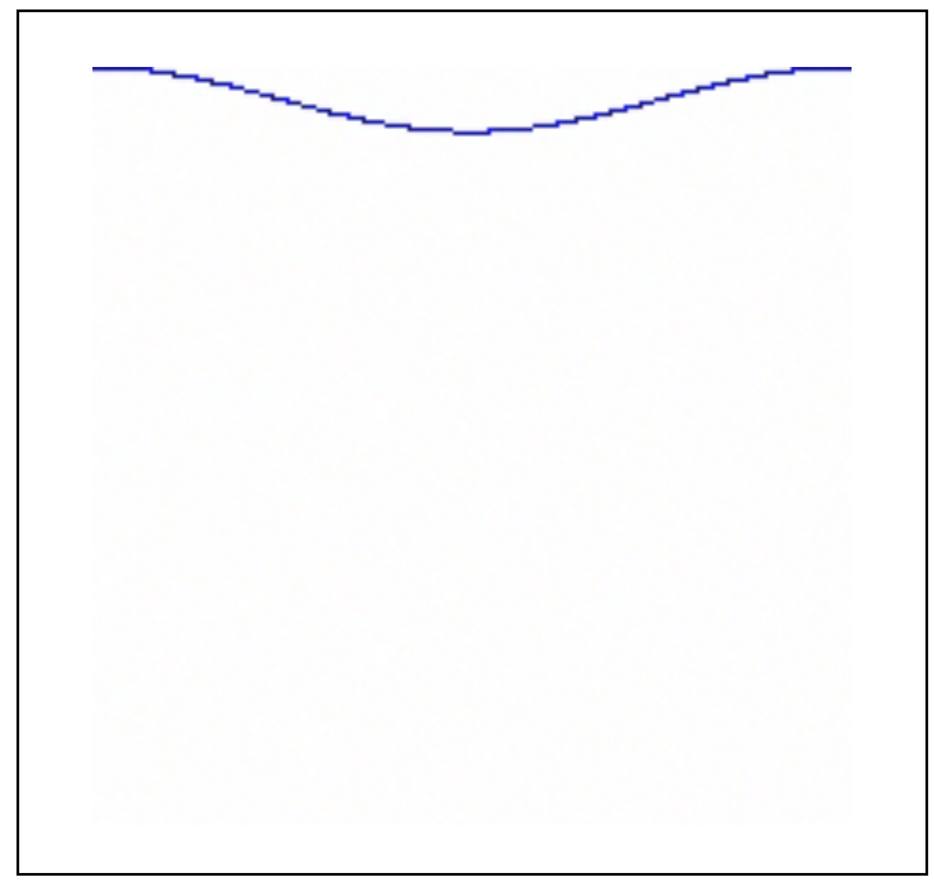


Generic Gowdy cosmologies: nonlinear terms decay. Each polarization propagates as a linear wave.

P

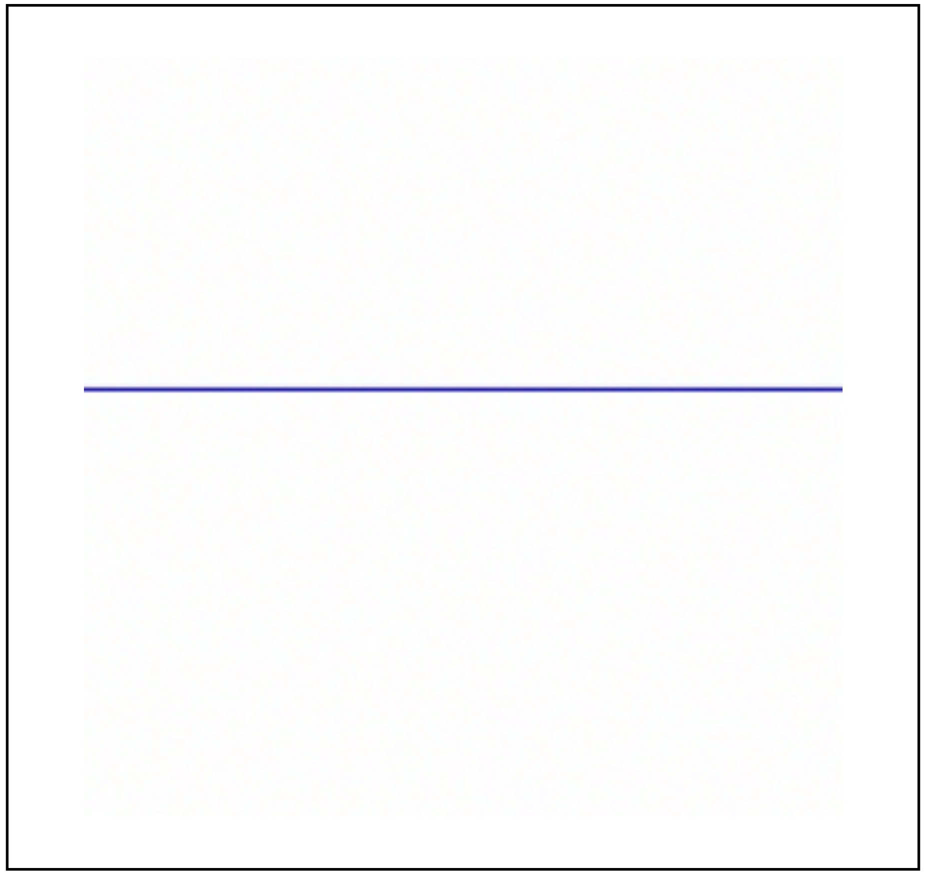


Q

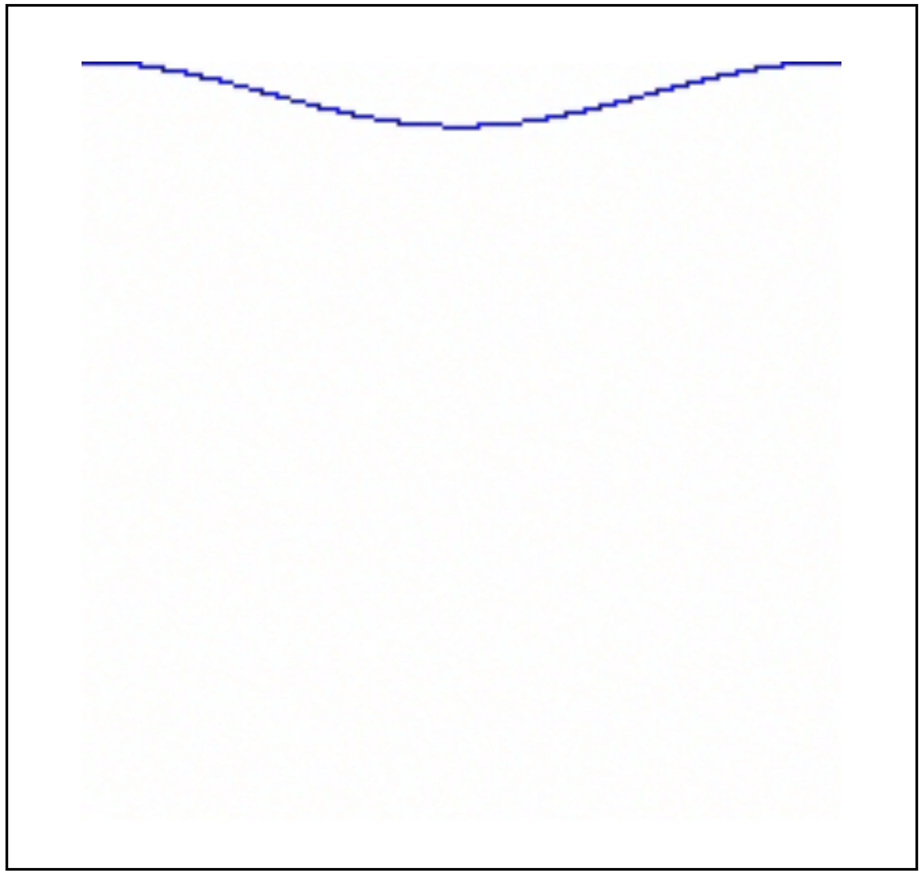


Generic Gowdy cosmologies: nonlinear terms decay. Each polarization propagates as a linear wave.

P



Q



Ringström's solutions for $\bar{\zeta} \leq 0$ $\quad \bar{\zeta} = \frac{\bar{\alpha}^2 + 4\bar{\beta}\bar{\gamma}}{4}$

Ringström has proven that the asymptotics of expanding Gowdy spacetimes depends on the sign of ζ. The behavior described so far only holds for ζ > 0. The waves decay as $t^{-1/2}$ as before but the spatial averages of P and Q are oscillatory in ln t rather than respectively logarithmic and constant in t as $t \rightarrow \infty$. λ behaves as before. He speculates that the solutions are asymptotically

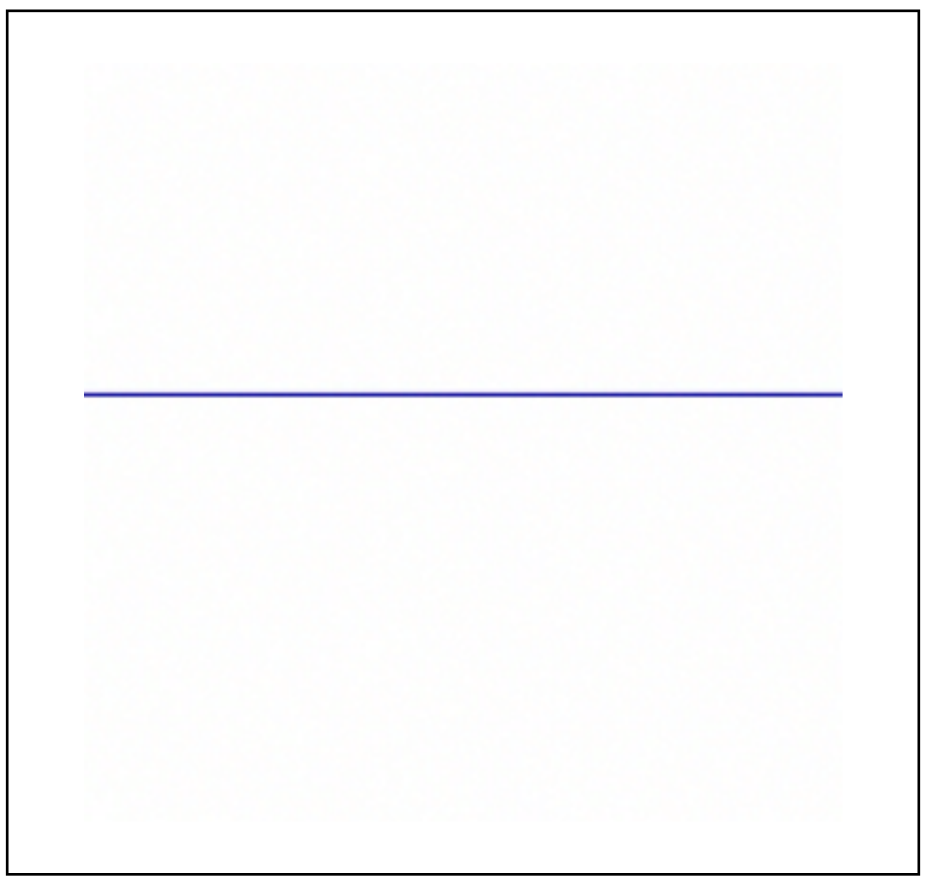
$$e^P = \frac{|\beta|}{\sqrt{-\zeta}} \left[c_1 - c_2 \cos\left(\sqrt{-\zeta} \ln \frac{t}{t_0} \right) \right]$$

$$Q = \frac{\alpha}{2\beta} + \frac{\sqrt{-\zeta}}{|\beta|} \frac{\left[c_2 \sin\left(\sqrt{-\zeta} \ln \frac{t}{t_0} \right) \right]}{\left[c_1 - c_2 \cos\left(\sqrt{-\zeta} \ln \frac{t}{t_0} \right) \right]}$$

H. Ringström, Comm. Pure Appl. Math. **57**, 657 (2004).

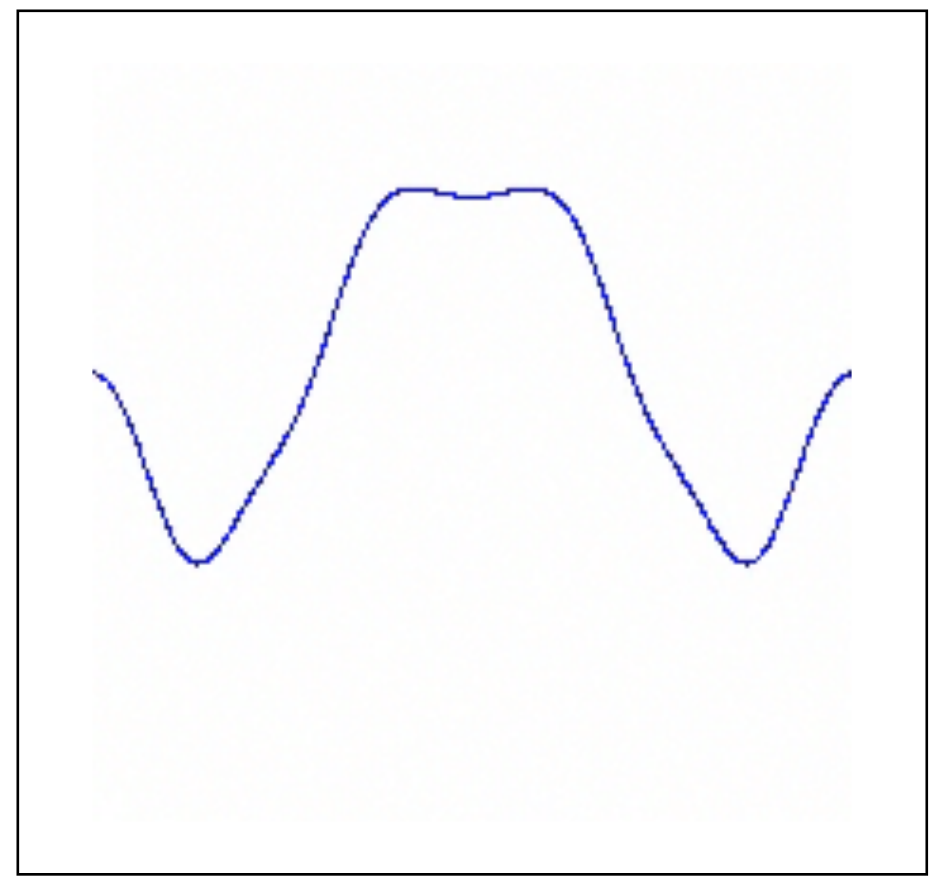
$\zeta > 0$

P



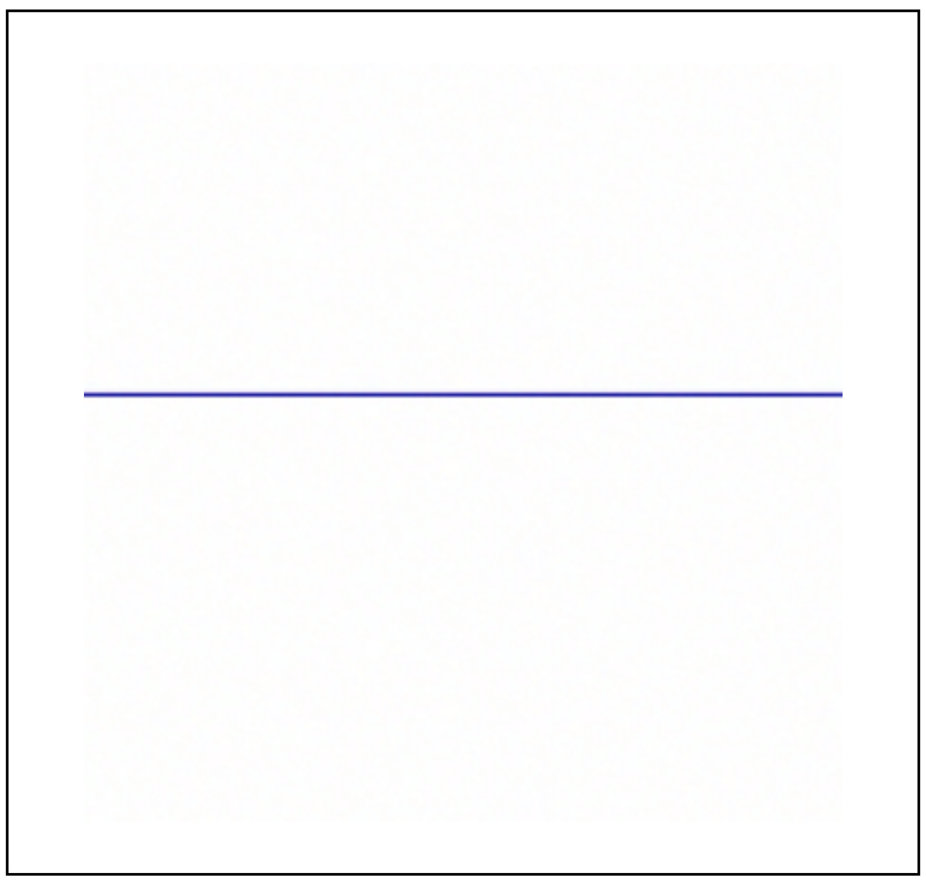
$\zeta < 0$

P



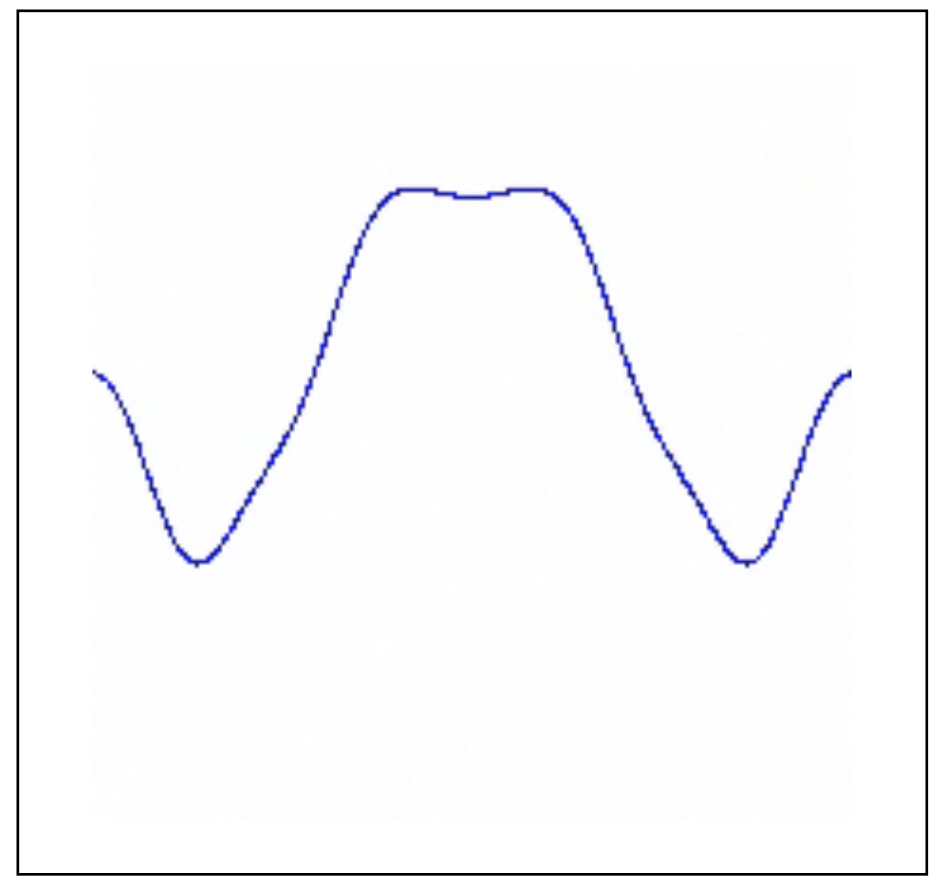
$\zeta > 0$

P



$\zeta < 0$

P



$\zeta > 0$

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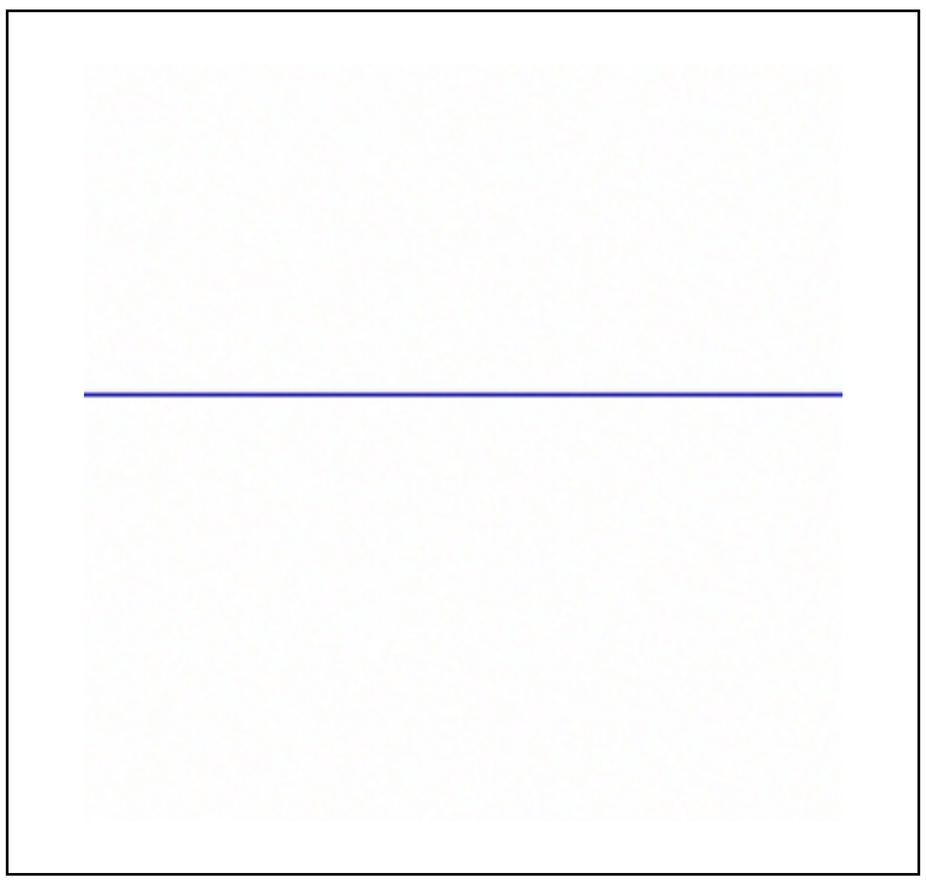
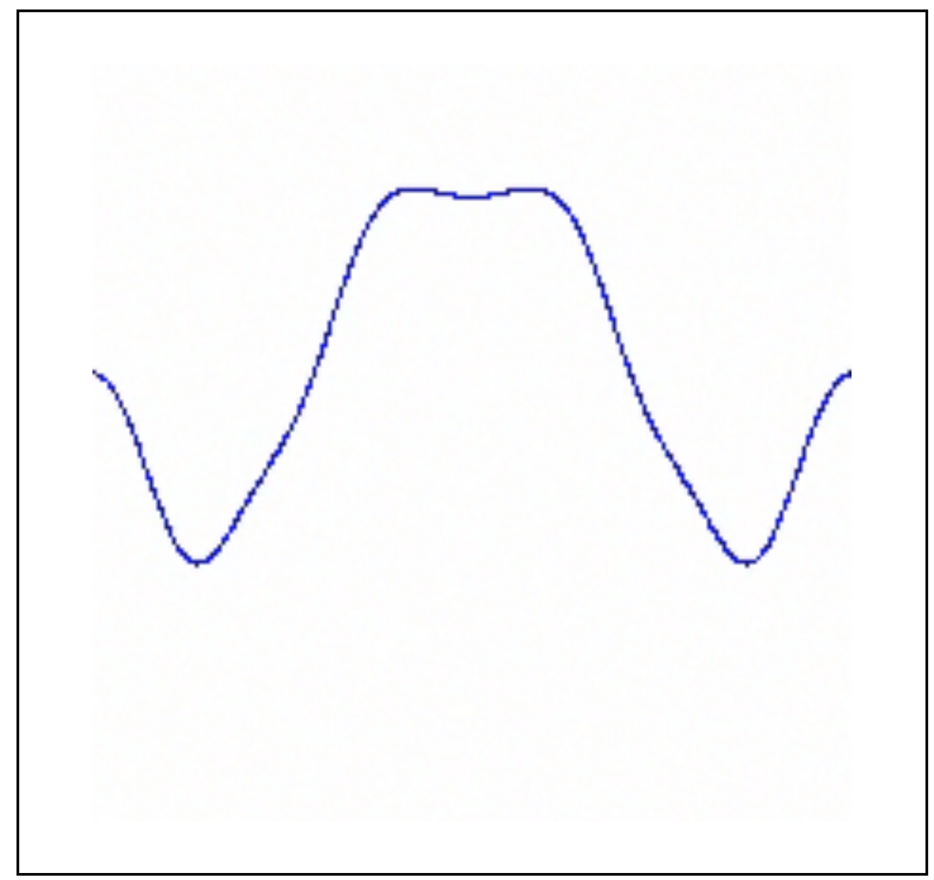


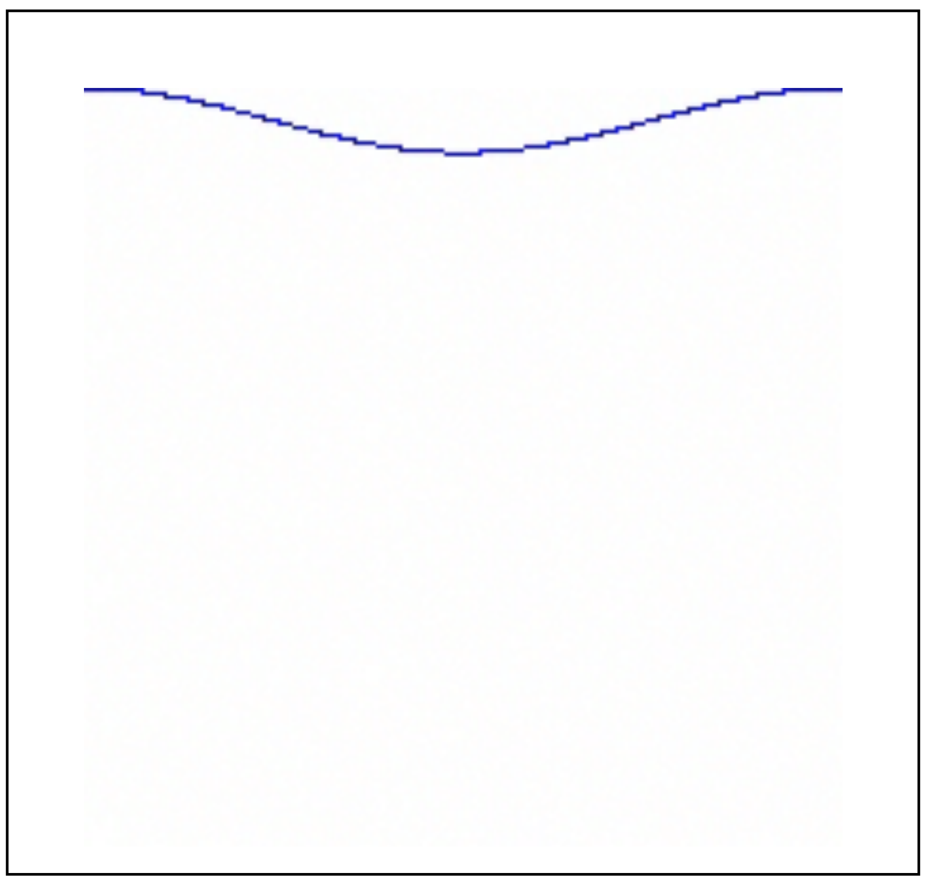
$\zeta < 0$

P



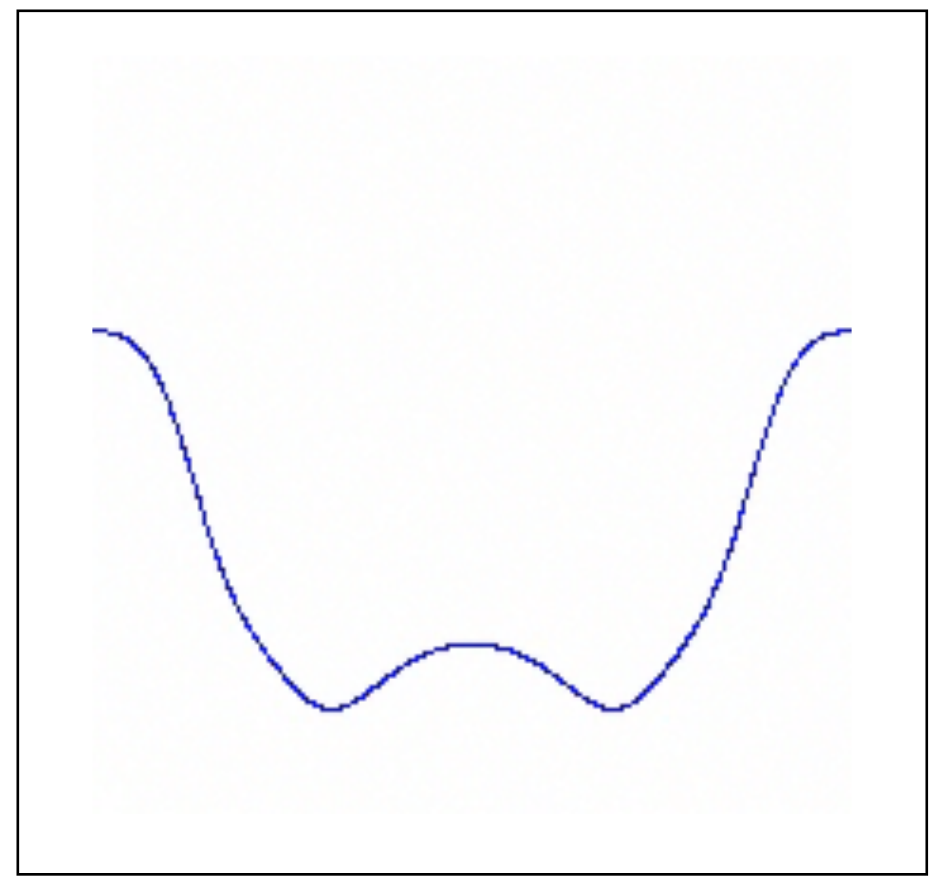
$\zeta > 0$

Q



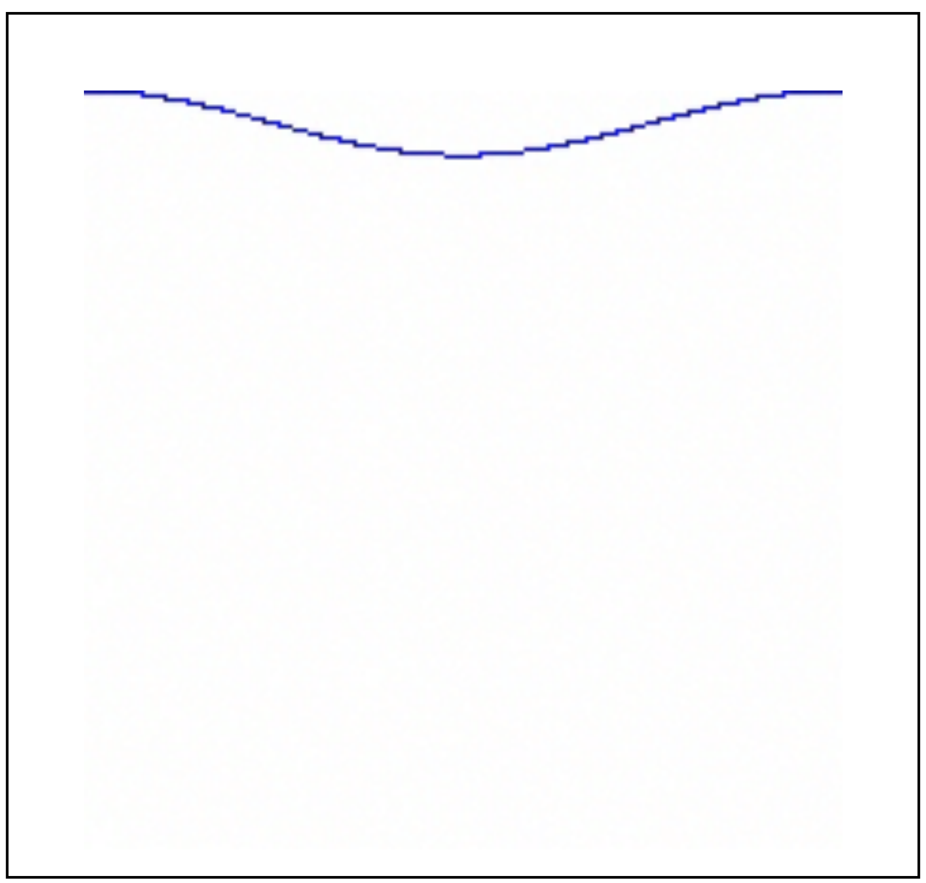
$\zeta < 0$

Q



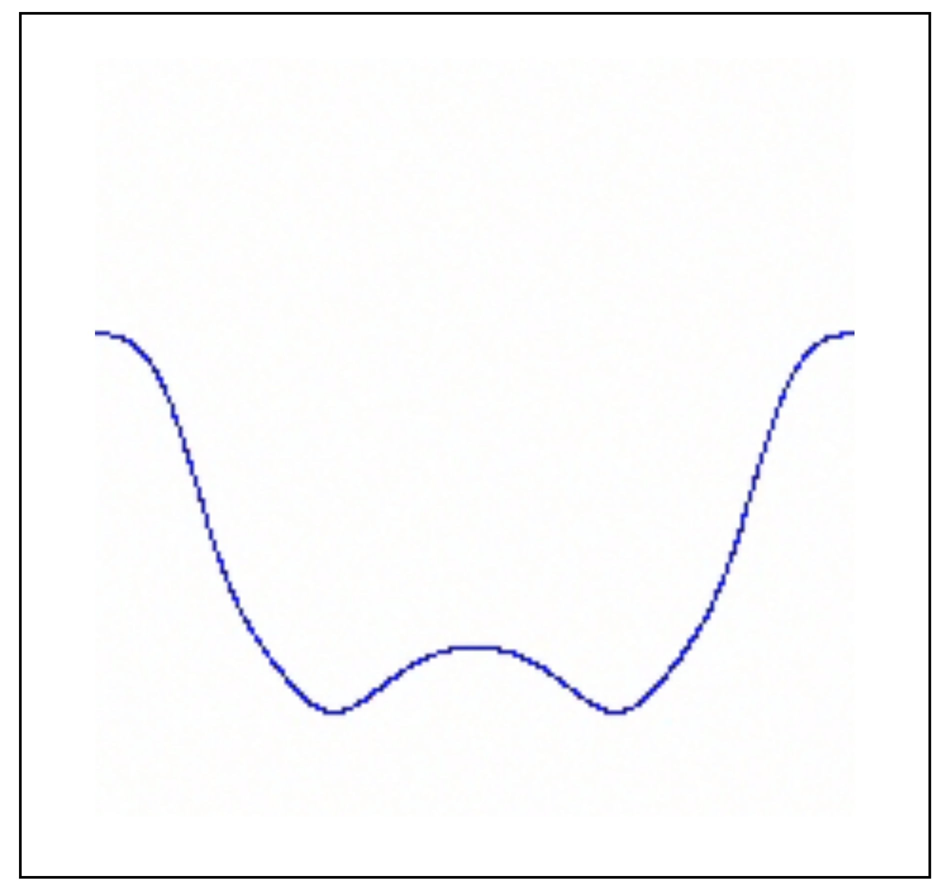
$\zeta > 0$

Q



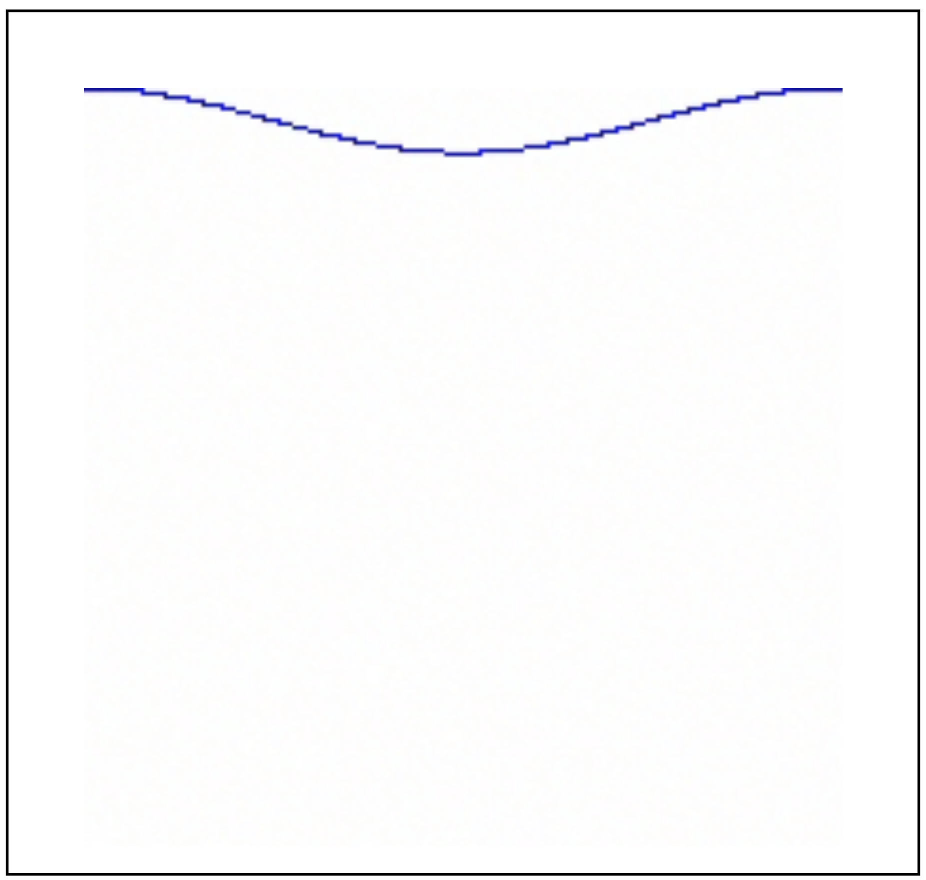
$\zeta < 0$

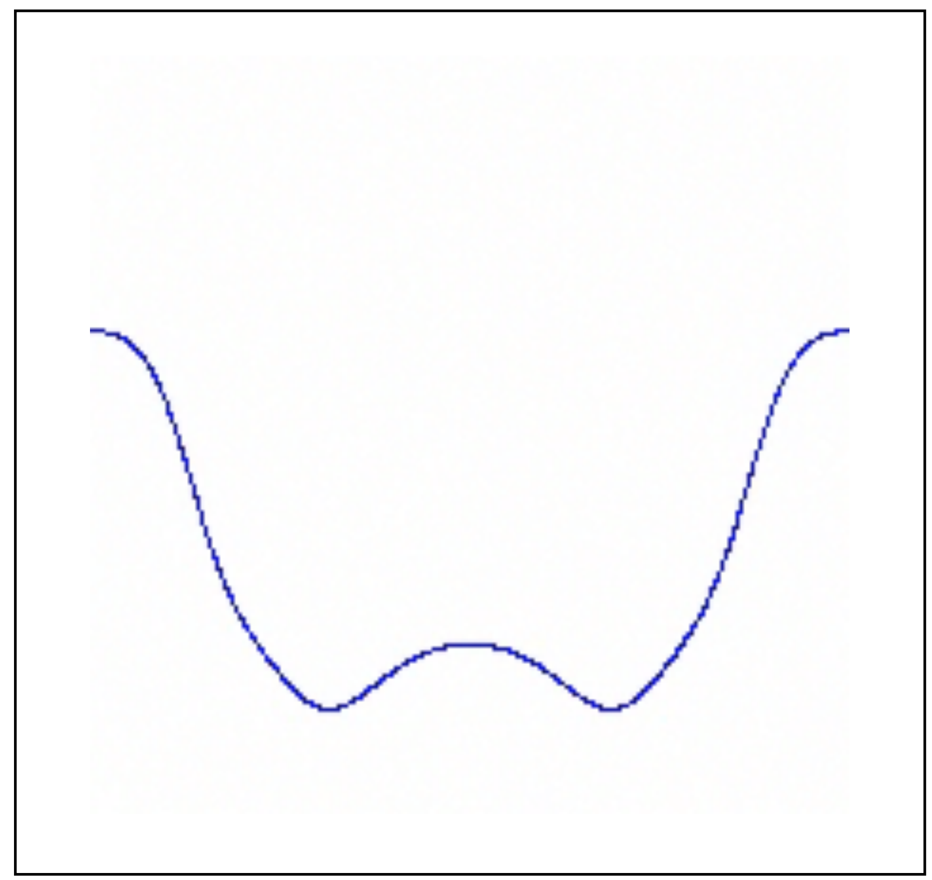
Q



$\zeta > 0$ $\zeta < 0$

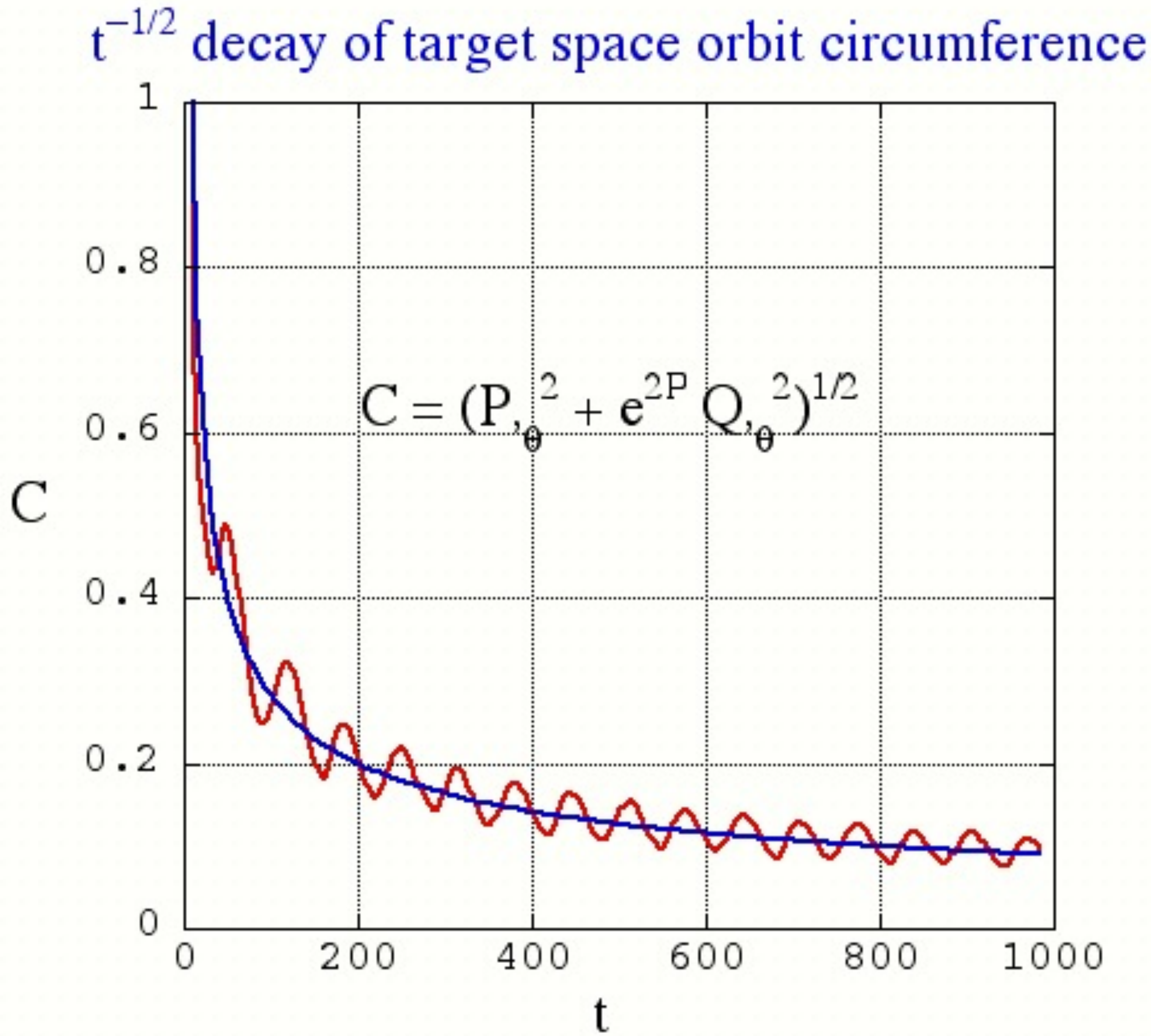
Q Q

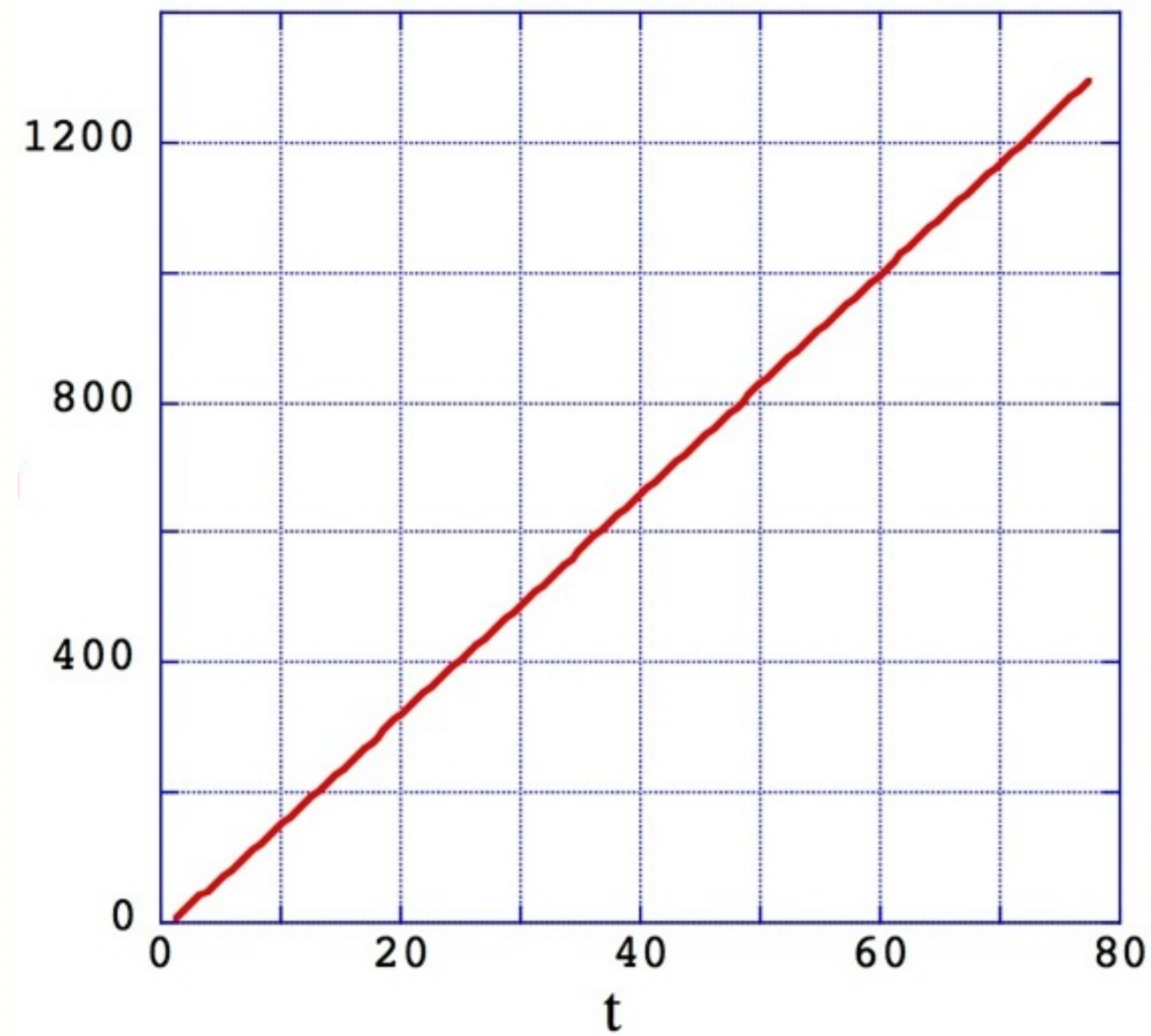




For all Gowdy models: Gravitational wave amplitudes decay

Linear growth of spatial average of λ





Generic T^2-symmetric metric (polarized case):

Replaces ν

$$
\begin{aligned}
g \;=\;& e^{\lambda/2} t^{-1/2}(-dt^2 + 4\pi_\lambda^2 d\theta^2) + e^{2P} t\, dx^2 \\
&+ e^{-P} t \left[dy + \left(\int^t dt' \kappa\, e^{\lambda+2P} \frac{2\pi_\lambda}{(t')^{5/2}} \right) d\theta \right]^2
\end{aligned}
$$

Replaces G, twist constant

$$
0 = P_{,tt} + \frac{1}{t} P_{,t} - \frac{P_{,\theta\theta}}{4\pi_\lambda^2} + \frac{\kappa^2 e^{\lambda/2+P}}{2t^{7/2}} (1 + tP_{,t}) + \frac{P_{,\theta}\, \pi_{\lambda,\theta}}{4\pi_\lambda^3}
$$

$$
\lambda_{,t} = t \left(P_{,t}^2 + \frac{P_{,\theta}^2}{4\pi_\lambda^2} \right) - \kappa^2 \frac{e^{\lambda/2+P}}{t^{5/2}}
$$

$$
\pi_{\lambda,t} = \kappa^2 \frac{e^{\lambda/2+P}}{2t^{5/2}} \pi_\lambda
$$

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Replaces ν

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Exponentials change the Gowdy behavior.

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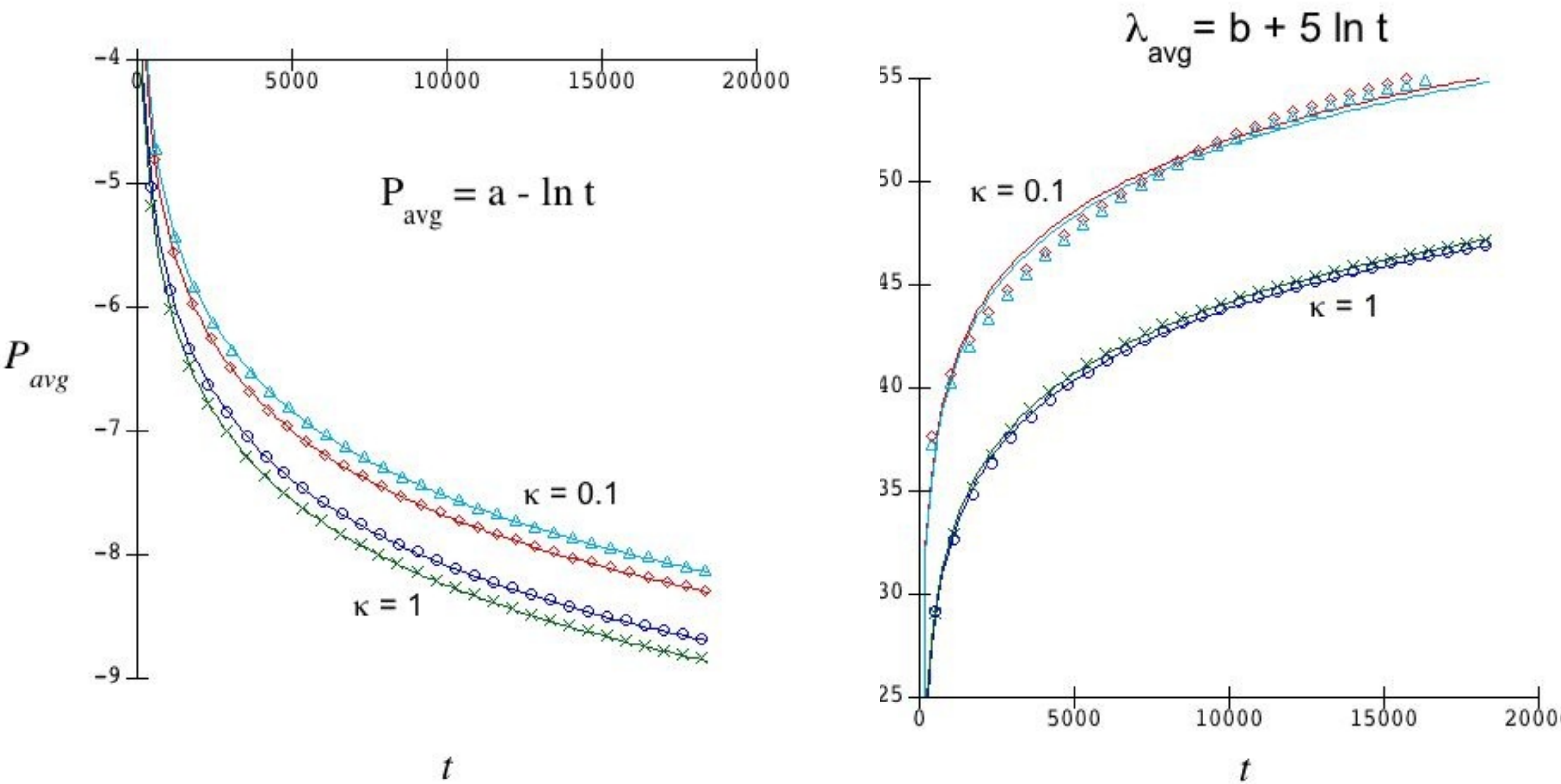
$$\pi_{\lambda,t} = \kappa^2 \frac{e^{\lambda/2 + P}}{2t^{5/2}} \pi_\lambda$$

Exponentials change the Gowdy behavior.

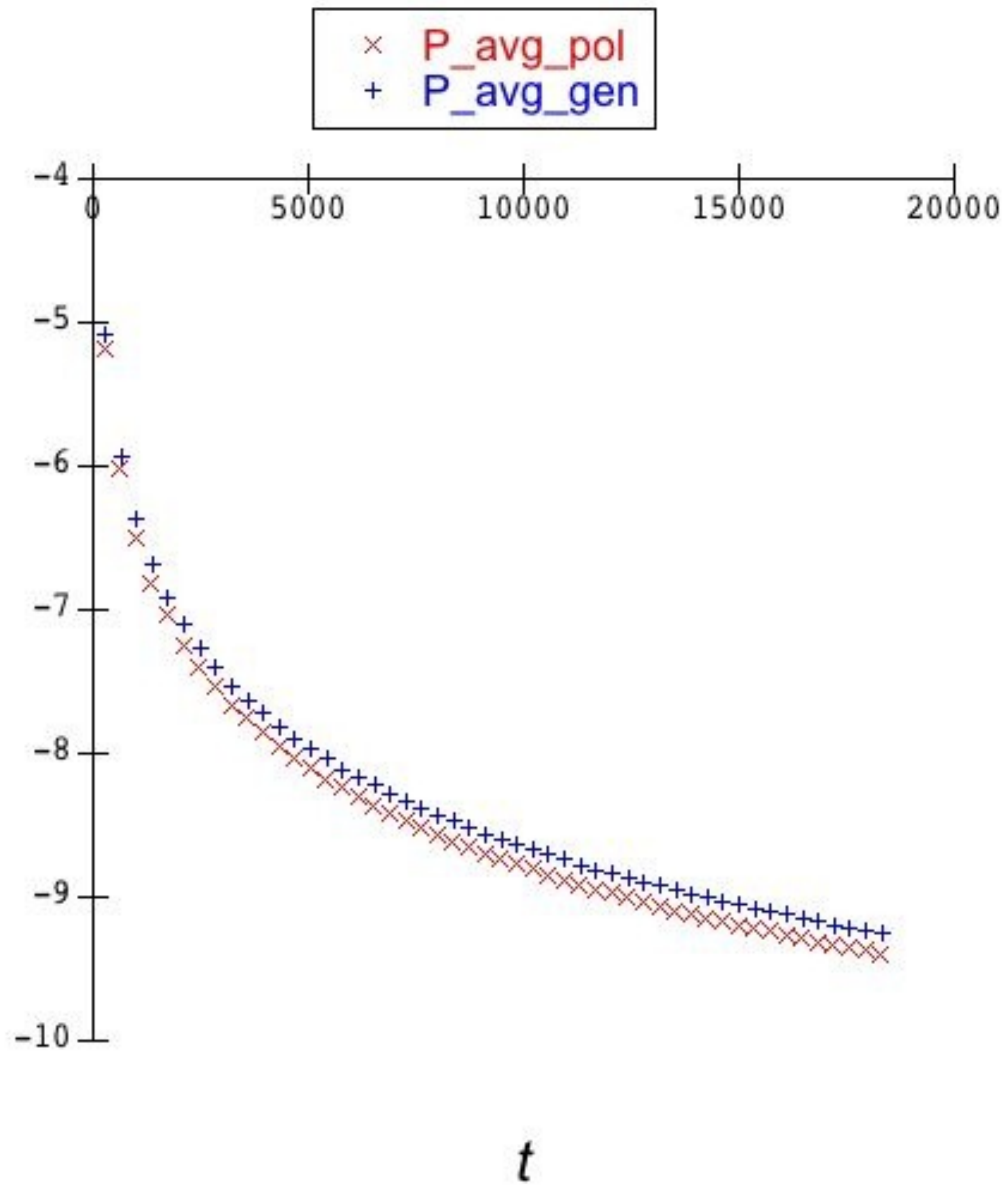
Make exponential terms as small as possible:

$$\bar{P} = -\ln t \quad , \quad \bar{\lambda} = 5 \ln t$$

Triangles, etc. are simulation data for different initial conditions; solid lines are fits using conjectured form.



Generic T^2-symmetric spacetimes — compared to polarized:



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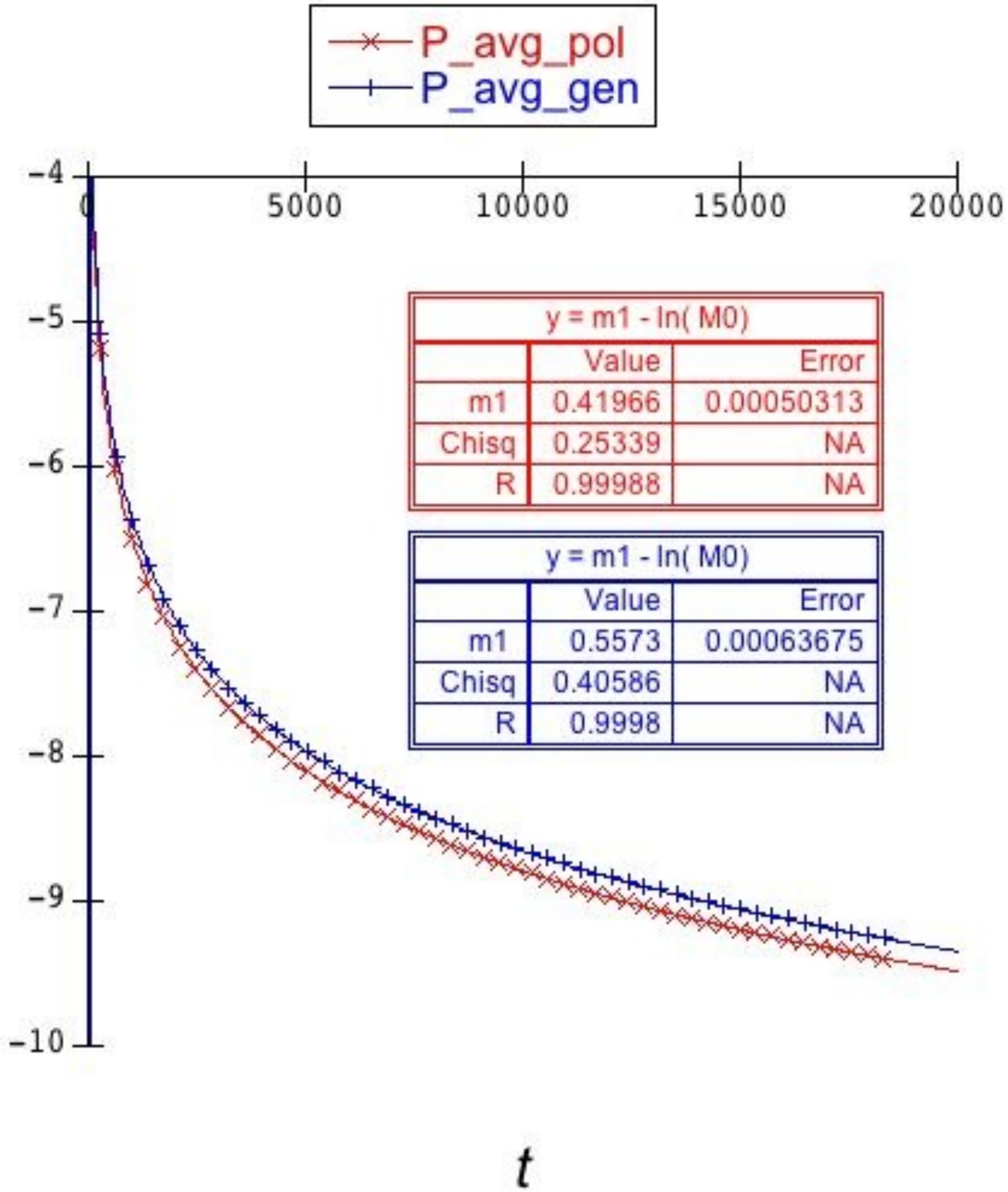

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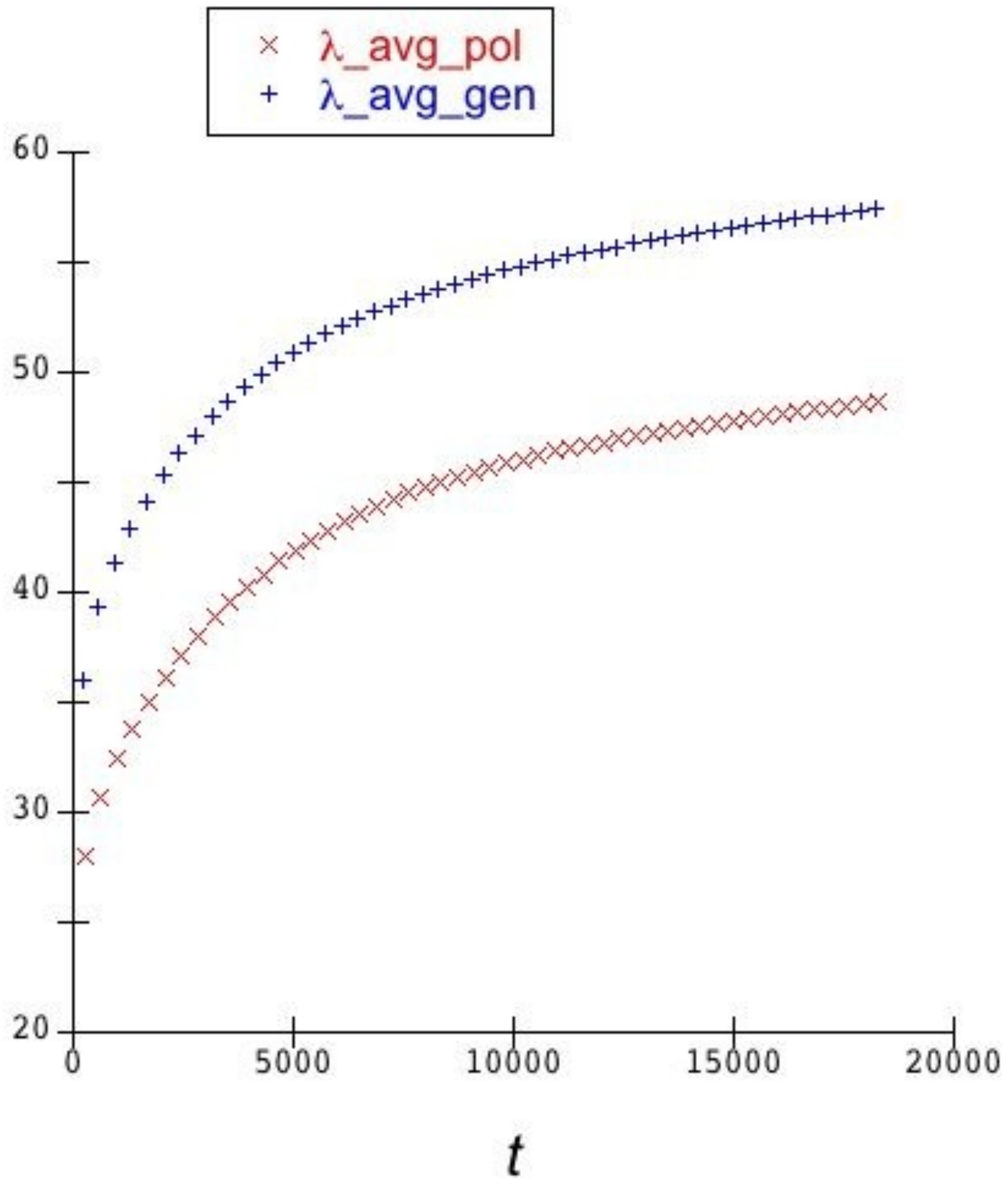

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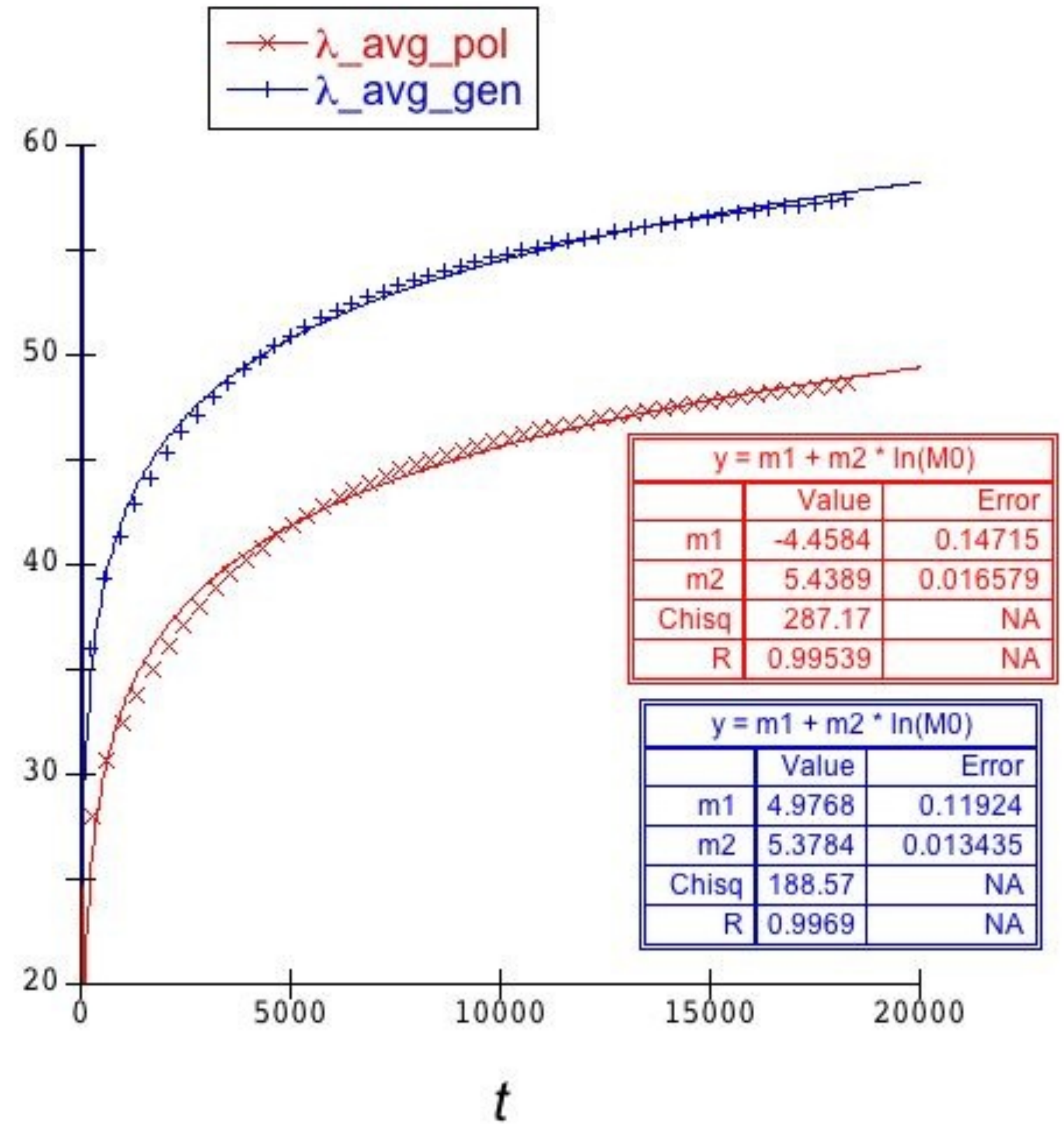
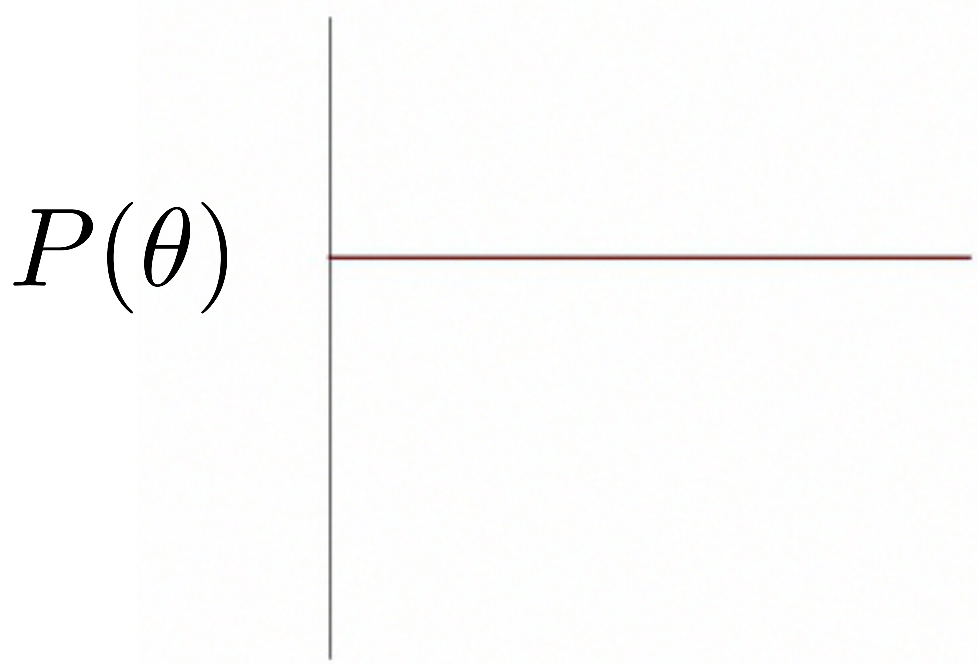
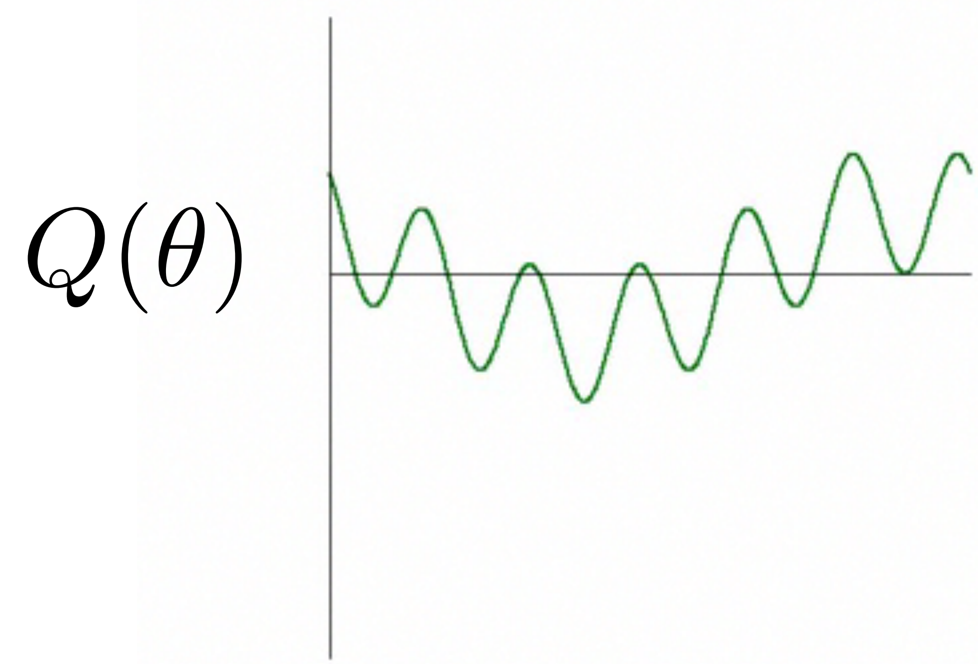


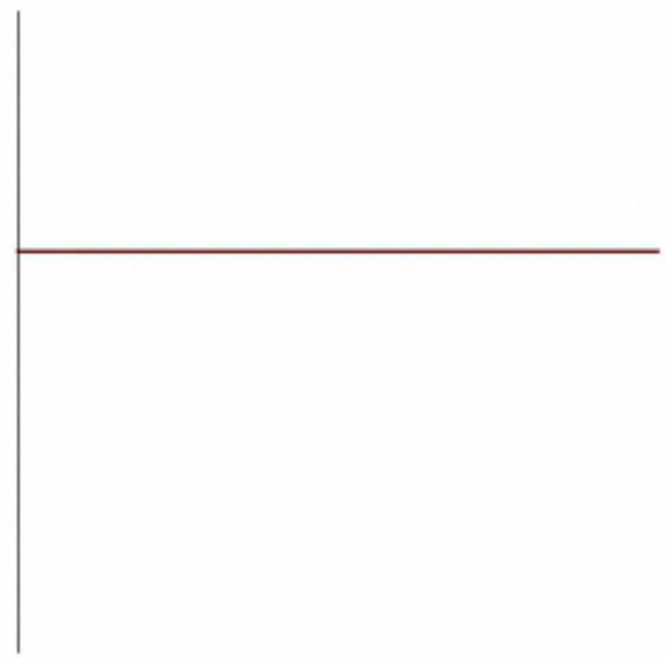
Movies from the early part of a typical generic simulation: how the exponential terms drive the solution into the required form for P and Q.



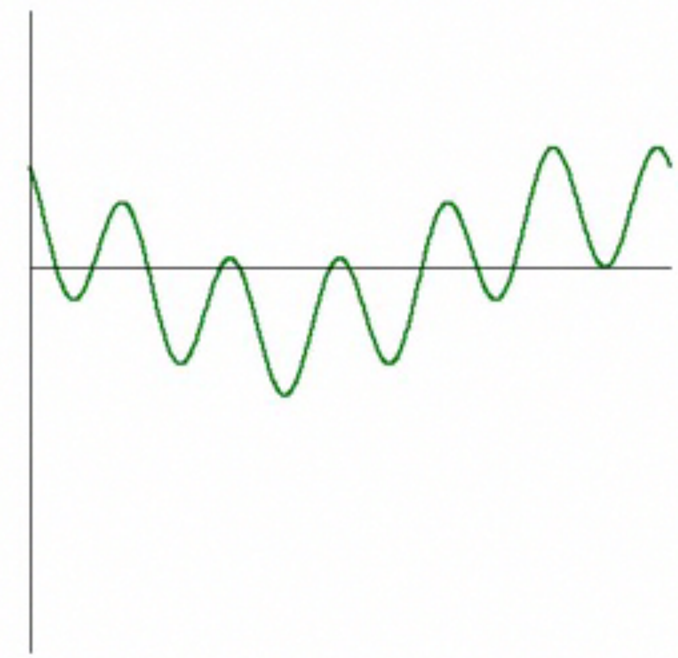


Movies from the early part of a typical generic simulation: how the exponential terms drive the solution into the required form for P and Q.

$P(\theta)$



$Q(\theta)$



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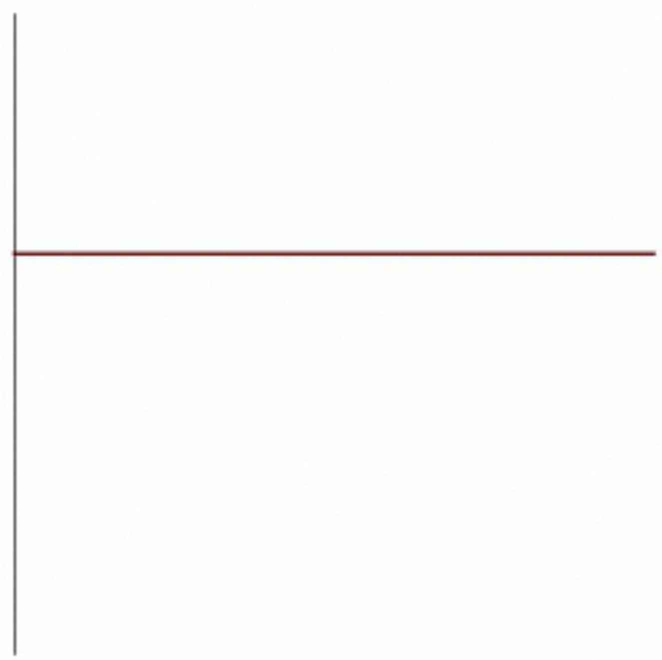

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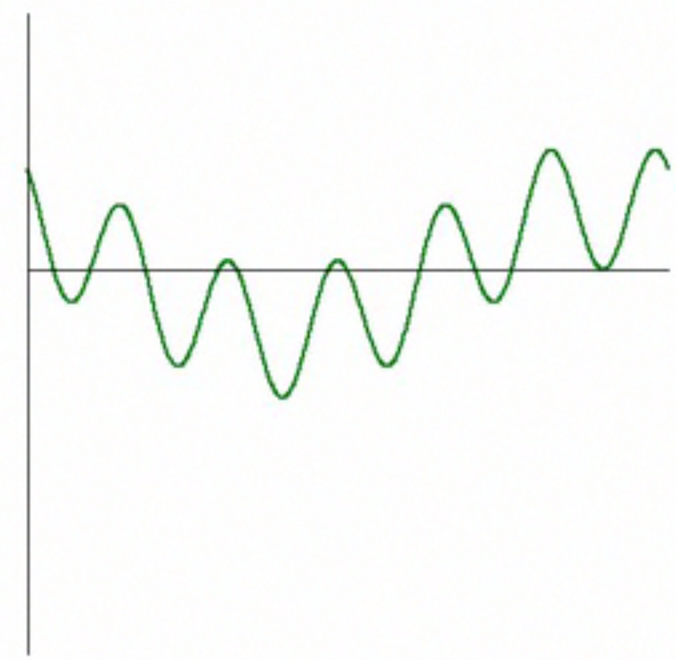

The behavior found in the simulations is not completely crazy:

Instability of spatially homogeneous solutions in the class of $\mathbb{T}^2$-symmetric solutions to Einstein's vacuum equations

Hans Ringström

December 4, 2013

Proposition 1. *Let $(P_{\rm bg}, \alpha_{\rm bg}, \lambda_{\rm bg})$ be a pseudo-homogeneous polarised solution to (2) and (4)–(6) with $K \neq 0$. Assuming that the relevant existence interval is (t_0, ∞) for some $t_0 \geq 0$, fix $t_a \in (t_0, \infty)$. Then there is an $\epsilon > 0$ such that if (P, α, λ) is a non-pseudo-homogeneous polarised solution to (2) and (4)–(6) with the same $K \neq 0$, the property that t_a belongs to its existence interval and such that*

$$\|(P - P_{\rm bg})(t_a, \cdot)\|_{C^1} + \|\partial_t(P - P_{\rm bg})(t_a, \cdot)\|_{C^0} + \|(\alpha - \alpha_{\rm bg})(t_a, \cdot)\|_{C^1} + \|(\lambda - \lambda_{\rm bg})(t_a, \cdot)\|_{C^0} \leq \epsilon,$$

then there is a time sequence $t_k \to \infty$, $k = 1, 2, \dots$, such that

$$\lim_{t\to\infty} \|\alpha(t, \cdot)\|_{C^0} = 0, \tag{12}$$

$$\lim_{t\to\infty} \left\| \frac{P(t, \cdot)}{\ln t} + 1 \right\|_{C^0} = 0, \tag{13}$$

$$\lim_{k\to\infty} \left\| \frac{\lambda(t_k, \cdot)}{\ln t_k} - 5 \right\|_{C^0} = 0. \tag{14}$$

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$\|(P - P_{bg})$ ... $|_{C^0} \leq \epsilon,$

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$$\lim_{t \to \infty} \left\| \frac{P(t, \cdot)}{\ln t} + 1 \right\|_{C^0} = 0, \quad (12)$$

$$\lim_{k \to \infty} \left\| \frac{\lambda(t_k, \cdot)}{\ln t_k} - 5 \right\|_{C^0} = 0. \quad (13)$$

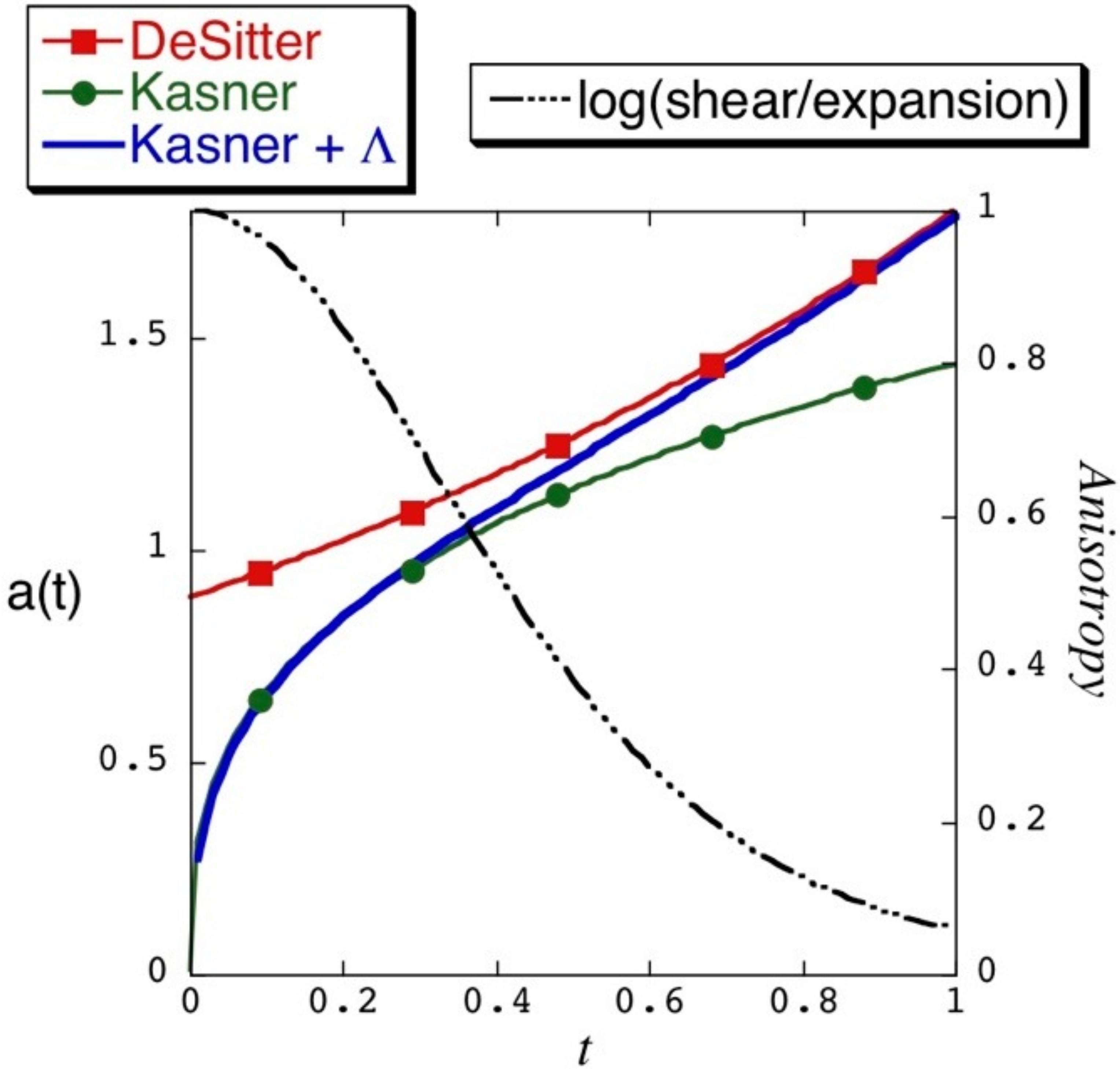
(14)

Final remarks:

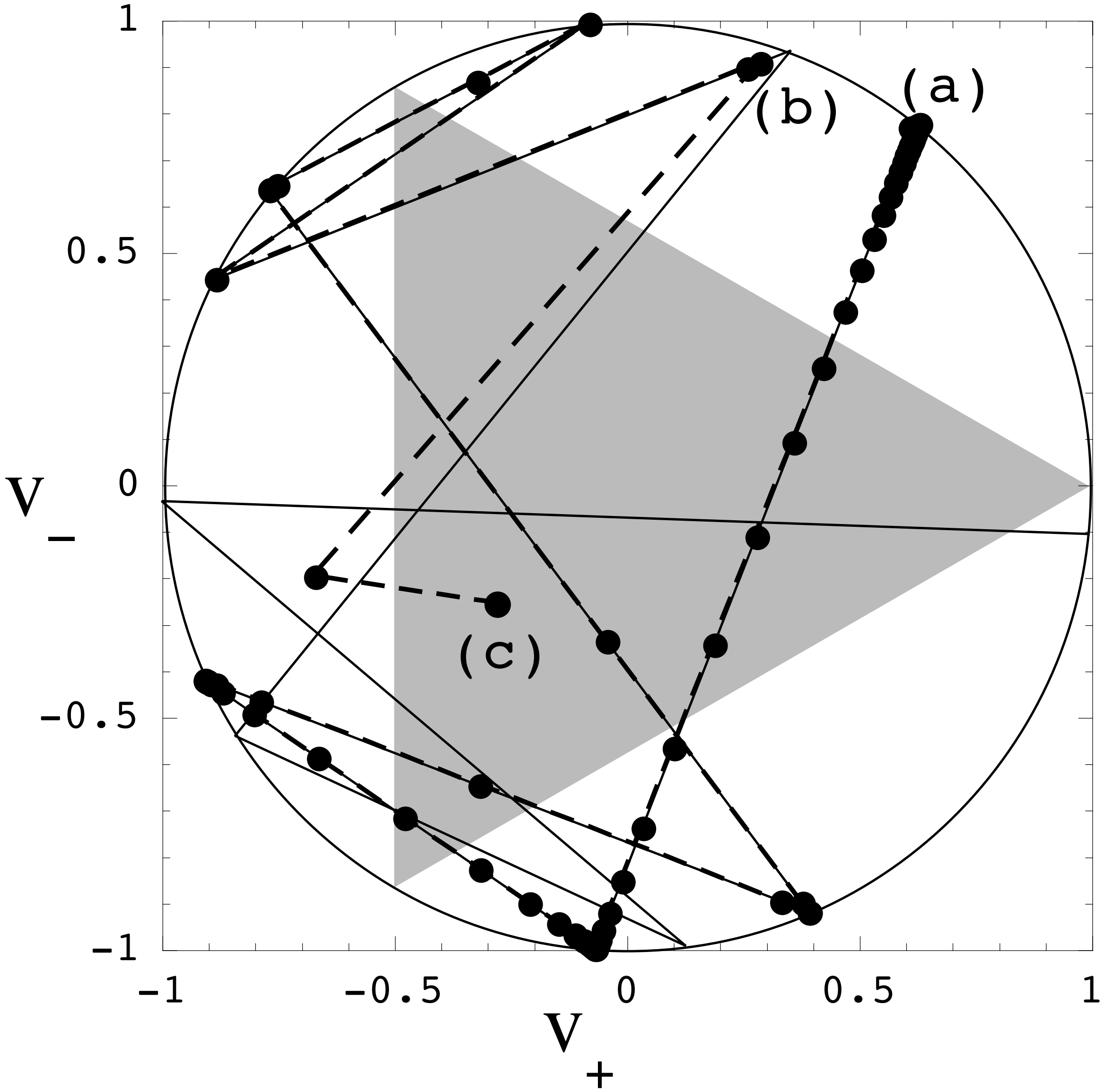
- Gowdy models and their generalizations are tractable both mathematically and numerically.
- Yet, they are sufficiently interesting to offer serious testing grounds for classical and quantum cosmology.
- While they cannot represent the actual universe, generic and generic + twist models exhibit interesting phenomenology that may be present in physically relevant spacetimes.
- I recommend that you consider testing your favorite formalisms on these models!

the end

Aside on the role of matter (or effective matter):



Is the Mixmaster singularity generic?



The vacuum trajectory always returns to the Kasner circle.

Minimally coupled scalar field destroys Mixmaster oscillations because $v_+^2 + v_-^2 < 1$.

$$H = -p_\Omega^2 + p_+^2 + p_-^2 + p_\varphi^2 + e^{4\Omega} V(\beta_+, \beta_-) + e^{6\Omega} V(\varphi) = 0$$

Scalar fields and extra dimensions can cause a final "bounce." Additional fields (e.g. magnetic) can restore Mixmaster dynamics by adding walls.

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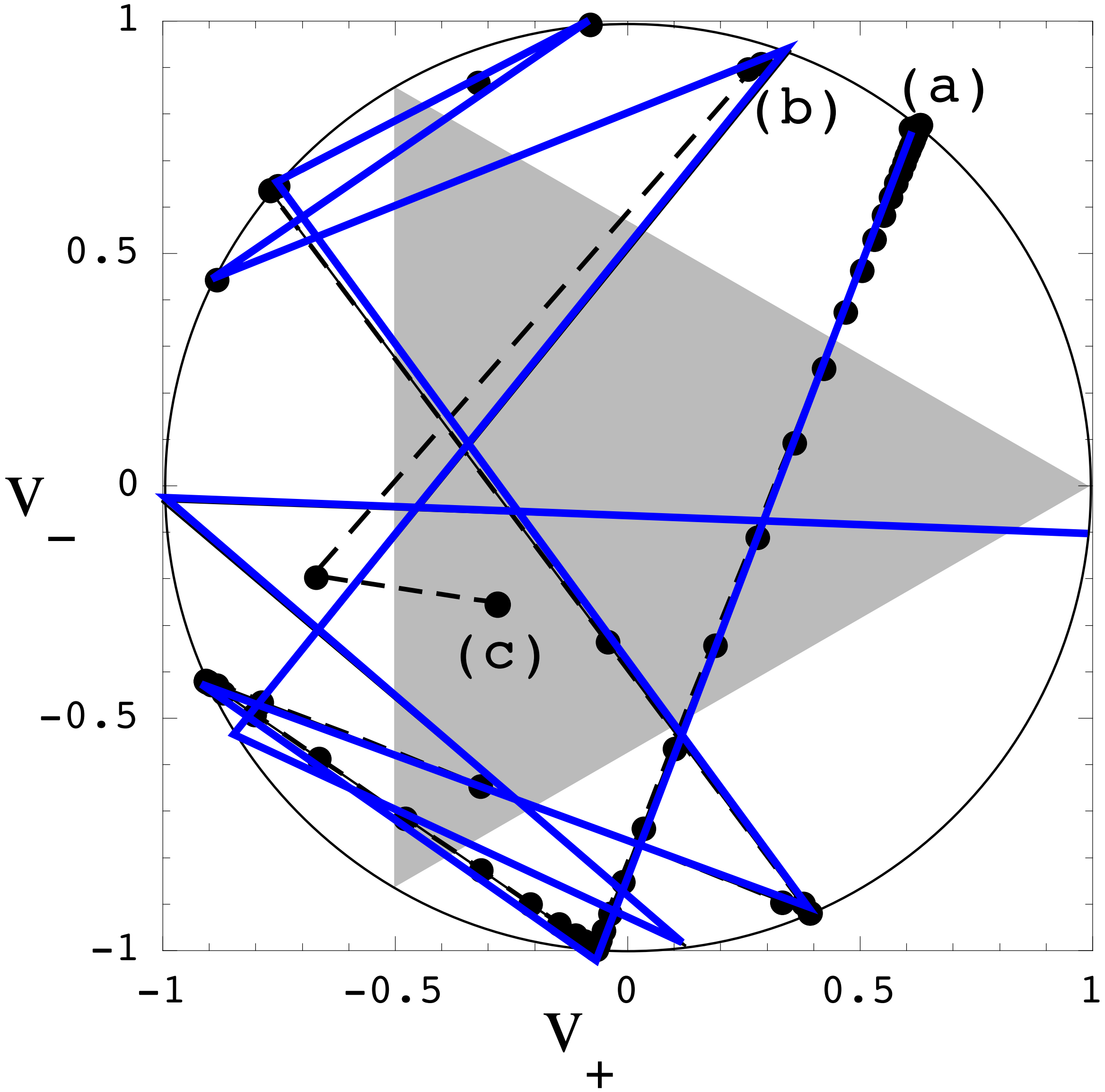

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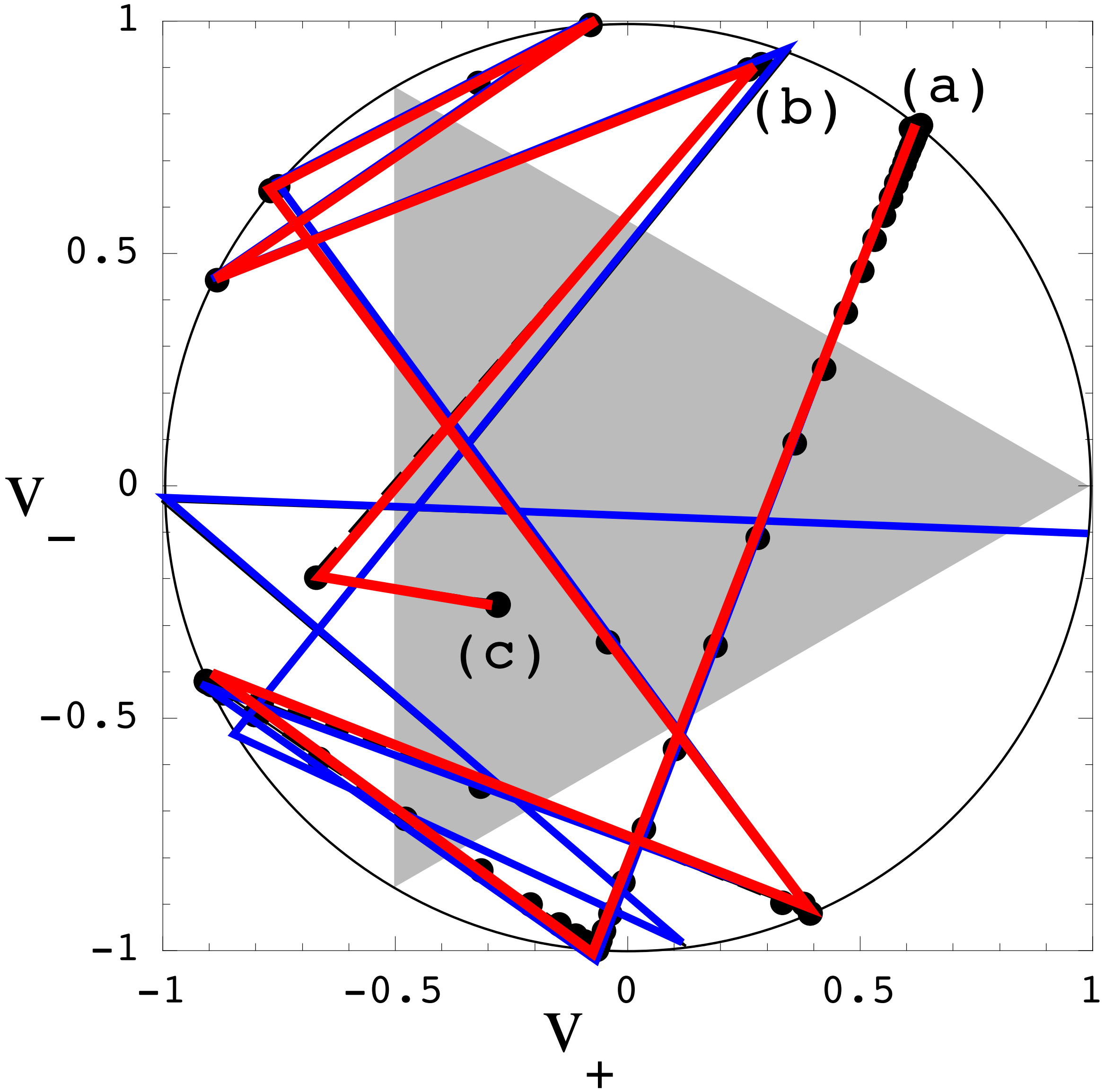

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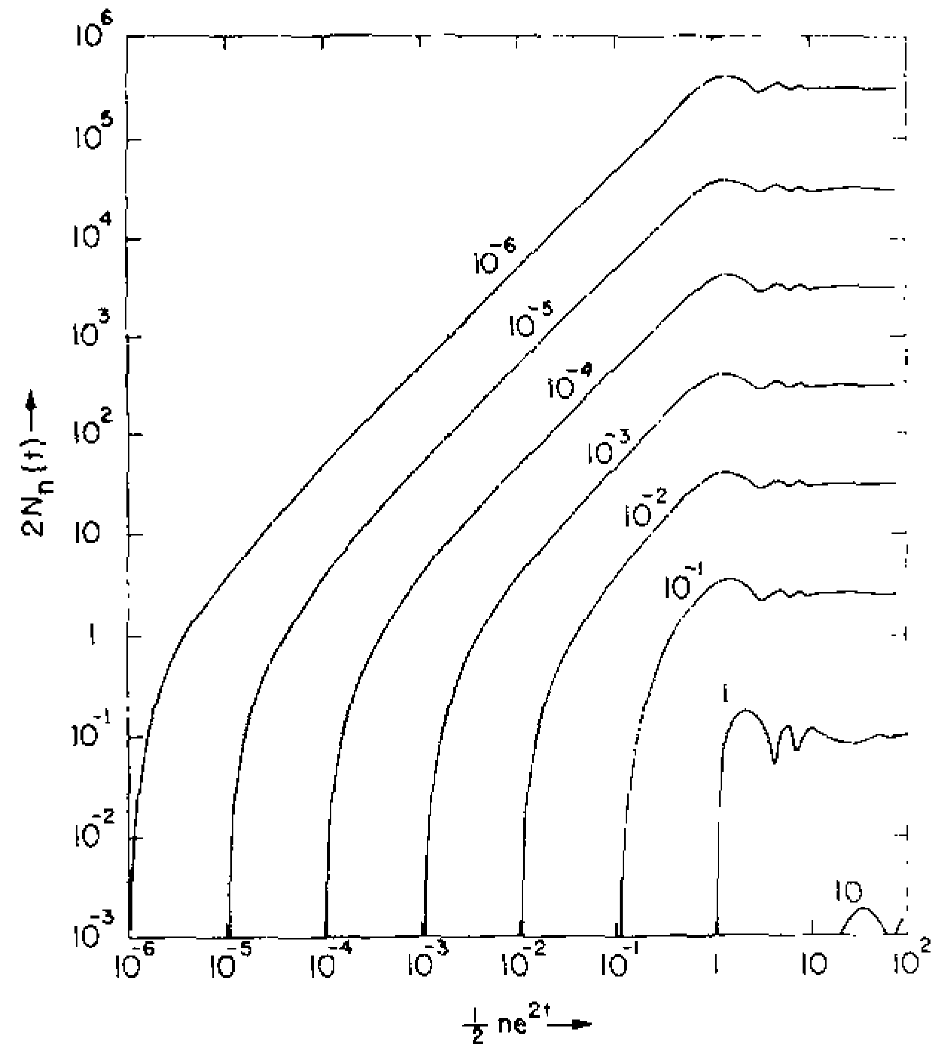


FIG. 3. Particle creation from a vacuum at t_0. The number of particles at time t in mode n, $2N_n(t)$, is plotted as a function of the final time parameter $\frac{1}{2}ne^{2t}$. The curves are labeled by the initial time parameter $\frac{1}{2}ne^{2t_0}$. Equation (110) is used to calculate $2N_n(t)$.

Quantization 3: Use this model as a laboratory to explore some issues of quantum field theory in curved spacetime.

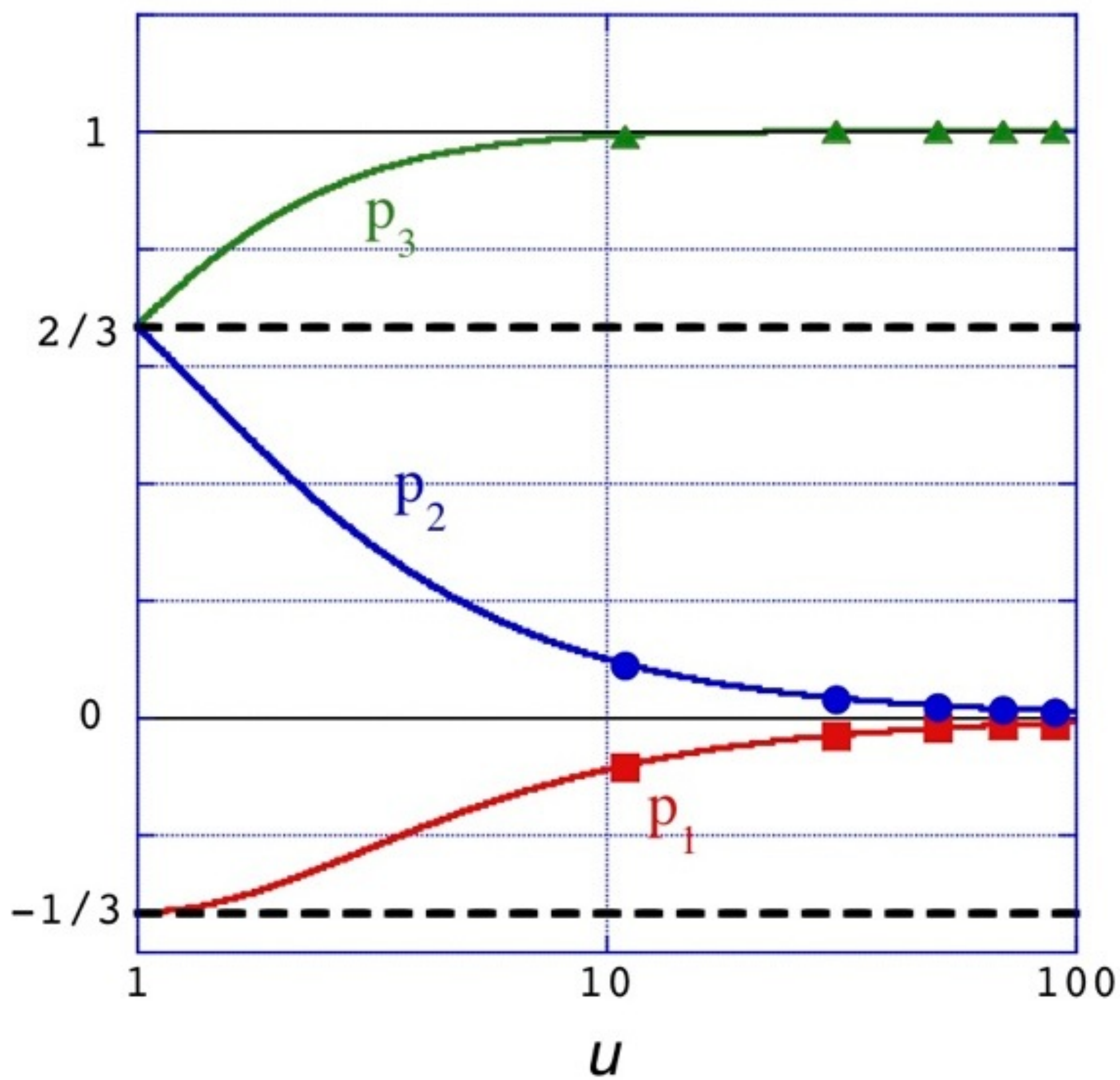
The scalar field *P* can be analyzed to show the Kasimir energy from discrete modes associated with 3-torus topology and an overall violation of the energy conditions leading to singularity avoidance via a bounce.

B.K. Berger, Ann. Phys. (N.Y.) **156**, 155 (1984)

Horizon size vs mode wavelength:

$$h(\tau) = \int_{\infty}^{\tau} d\tau' \left(-\frac{g_{00}}{g_{11}} \right)^{1/2} = e^{-\tau}$$

The Kasner Spacetime (vacuum, Bianchi Type I):



Each u-value in $[1, \infty]$ indicates a distinct Kasner evolution.

A set of measure zero: The (1,0,0) Kasner $u = \infty$ is the Minkowski spacetime in different coordinates.

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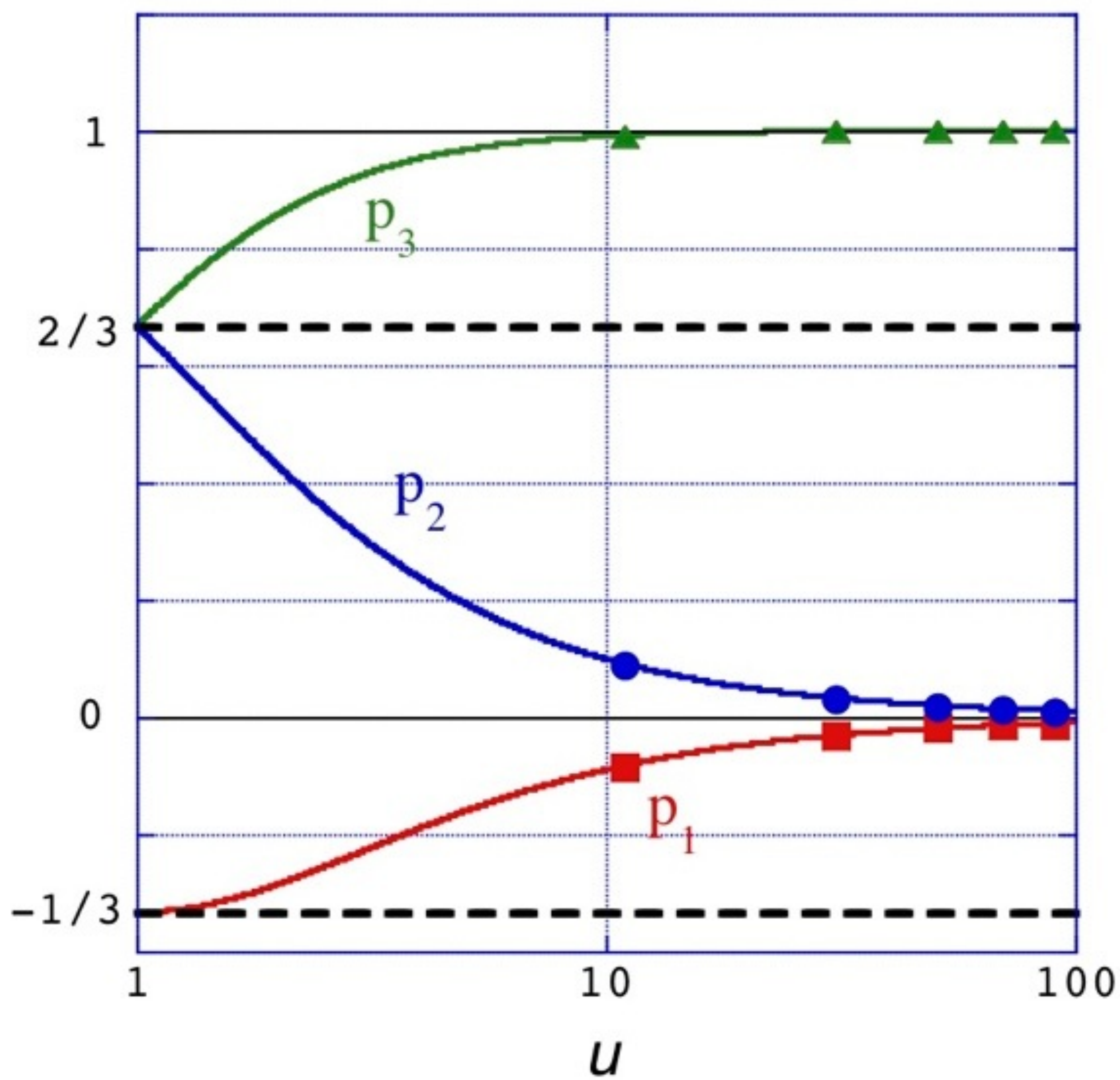

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